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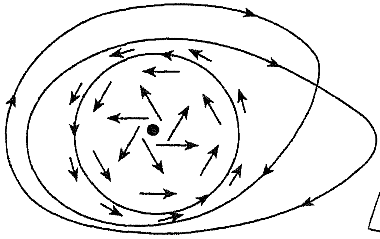
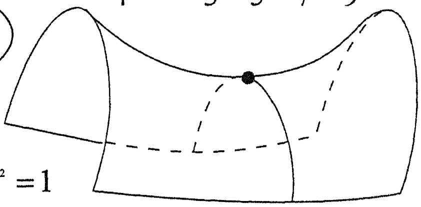
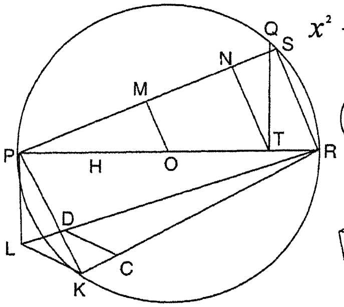
$$\frac{\partial^2 u(x,t)}{\partial t^2} = \frac{\partial^2 u(x,t)}{\partial x^2}$$

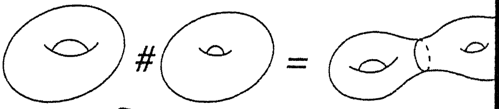


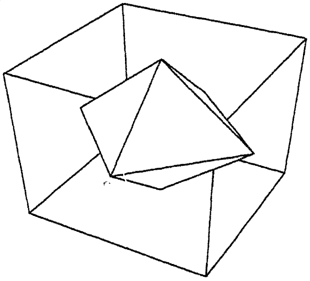
$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots$$





$$x^2 - dy^2 = 1$$





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Algebraic stacks

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Abstract. This is an expository article on the theory of algebraic stacks. After introducing the general theory, we concentrate in the example of the moduli stack of vector bundles, giving a detailed comparison with the moduli scheme obtained via geometric invariant theory.

Keywords. 2 categories; algebraic stacks; moduli spaces; vector bundles.

1. Introduction

The concept of algebraic stack is a generalization of the concept of scheme, in the same sense that the concept of scheme is a generalization of the concept of projective variety. In many moduli problems, the functor that we want to study is not representable by a scheme. In other words, there is no fine moduli space. Usually this is because the objects that we want to parametrize have automorphisms. But if we enlarge the category of schemes (following ideas that go back to Grothendieck and Giraud [Gi], and were developed by Deligne, Mumford and Artin [DM, Ar2]) and consider algebraic stacks, then we can construct the 'moduli stack', that captures all the information that we would like in a fine moduli space. For other sources on stacks, see [E, La, LaM, Vi].

The idea of enlarging the category of algebraic varieties to study moduli problems is not new. In fact Weil invented the concept of abstract variety to give an algebraic construction of the Jacobian of a curve.

These notes are an introduction to the theory of algebraic stacks. I have tried to emphasize ideas and concepts through examples instead of detailed proofs (I give references where these can be found). In particular, §3 is a detailed comparison between the moduli *scheme* and the moduli *stack* of vector bundles.

First I will give a quick introduction in subsection 1.1, just to give some motivations and get a flavor of the theory of algebraic stacks.

Section 2 has a more detailed exposition. There are mainly two ways of introducing stacks. We can think of them as 2-functors (I learnt this approach from Nitsure and Sorger, cf. subsection 2.1), or as categories fibered on groupoids. (This is the approach used in the references, cf. subsection 2.2.) From the first point of view it is easier to see in which sense stacks are generalizations of schemes, and the definition looks more natural, so conceptually it seems more satisfactory. But since the references use categories fibered on groupoids, after we present both points of view, we will mainly use the second.

The concept of stack is merely a categorical concept. To do geometry we have to add some conditions, and then we get the concept of algebraic stack. This is done in subsection 2.3.

In subsection 2.4 we introduce a third point of view to understand stacks: as groupoid spaces.

In subsection 2.5 we define for algebraic stacks many of the geometric properties that are defined for schemes (smoothness, irreducibility, separatedness, properness, etc...). In subsection 2.6 we introduce the concept of point and dimension of an algebraic stack, and in subsection 2.7 we define sheaves on algebraic stacks.

In §3 we study in detail the example of the moduli of vector bundles on a scheme X, comparing the moduli stack with the moduli scheme.

Prerequisites. In the examples, I assume that the reader has some familiarity with the theory of moduli spaces of vector bundles. A good source for this material is [HL]. The necessary background on Grothendieck topologies, sheaves and algebraic spaces is in Appendix A, and the notions related to the theory of 2-categories are explained in Appendix B.

1.1 *Quick introduction to algebraic stacks*

We will start with an example: vector bundles (with fixed prescribed Chern classes and rank) on a projective scheme X over an algebraically closed field k. What is the moduli stack $\mathcal{M}_X$ of vector bundles on X? I do not know a short answer to this, but instead it is easy to define what is a morphism from a scheme B to the moduli stack $\mathcal{M}_X$. It is just a family of vector bundles parametrized by B. More precisely, it is a vector bundle V on $B \times X$ (hence flat over B) such that the restriction to the slices $b \times X$ have prescribed Chern classes and rank. In other words, $\mathcal{M}_X$ has the property that we expect from a fine moduli space: the set of morphisms $\mathrm{Hom}(B, \mathcal{M}_X)$ is equal to the set of families parametrized by B.

We will say that a diagram

$$\begin{tikzcd} B \arrow[r, "f"] \arrow[dr, "g"'] & B' \arrow[d, "g'"] \\ & \mathcal{M}_X \end{tikzcd} \qquad (1)$$

is commutative if the vector bundle V on $B \times X$ corresponding to g is isomorphic to the vector bundle $(f \times \mathrm{id}_X)^* V'$, where V' is the vector bundle corresponding to g'. Note that in general, if L is a line bundle on B, then V and $V \otimes p_B^* L$ won't be isomorphic, and then the corresponding morphisms from B to $\mathcal{M}_X$ will be different, as opposed to what happens with moduli schemes.

A k-point in the stack $\mathcal{M}_X$ is a morphism $u : \mathrm{Spec}\, k \to \mathcal{M}_X$, in other words, it is a vector bundle V on X, and we say that two points are isomorphic if they correspond to isomorphic vector bundles. But we should not think of $\mathcal{M}_X$ just as a set of points, it should be thought of as a-category. The objects of $\mathcal{M}_X$ are points[1], i.e. vector bundles on X, and a morphism in $\mathcal{M}_X$ is an isomorphism of vector bundles. This is the main difference between a scheme and an algebraic stack: the points of a scheme form a *set*, whereas the points of a stack form a *category*, in fact a *groupoid* (i.e. a category in which all morphisms are isomorphisms). Each point comes with a group of automorphisms. Roughly speaking, a scheme (or more generally, an algebraic space [Ar1, K]) can be thought of as an algebraic stack in which these groups of automorphisms are all trivial. If p is the k-point in $\mathcal{M}_X$ corresponding to a vector bundle V on X, then the group of automorphisms associated to p is the group of vector bundle automorphisms of V. This is why algebraic stacks are well suited to serve as moduli of objects that have automorphisms.

---

[1] To be precise, we should consider also B-valued points, for any scheme B, but we will only consider k-valued points for the moment.

An algebraic stack has an atlas. This is a scheme U and a (representable) surjective morphism $u : U \longrightarrow \mathcal{M}_X$ (with some other properties). As we have seen, such a morphism u is equivalent to a family of vector bundles parametrized by U. The precise definition of representable surjective morphism of stacks will be given in §2. In this situation it implies that for every vector bundle V over X there is at least one point in U whose corresponding vector bundle is isomorphic to V. The existence of an atlas for an algebraic stack is the analog of the fact that for a scheme B there is always an *affine* scheme U and a surjective morphism $U \longrightarrow B$ (if $\{U_i \longrightarrow B\}$ is a covering of B by affine subschemes, take U to be the disjoint union $\coprod U_i$). Many local properties (smooth, normal, reduced...) can be studied by looking at the atlas U. It is true that in some sense an algebraic stack looks, locally, like a scheme, but we shouldn't take this too far. For instance the atlas of the classifying stack BG (parametrizing principal G-bundles, cf. Example 2.18) is just a single point. The dimension of an algebraic stack $\mathcal{M}_X$ will be defined as the dimension of U minus the relative dimension of the morphism u. The dimension of an algebraic stack can be negative (for instance, $\dim(BG) = -\dim(G)$).

We will see that many geometric concepts that appear in the theory of schemes have an analog in the theory of algebraic stacks. For instance, one can define coherent sheaves on them. We will give a precise definition in §2, but the idea is that a coherent sheaf L on an algebraic stack $\mathcal{M}_X$ is a functor that, for each morphism $g : B \longrightarrow \mathcal{M}_X$, gives a coherent sheaf L_B on B, and for each commutative diagram like (1), gives an isomorphism between $f^*L_{B'}$ and L_B. The coherent sheaf L_B should be thought of as the pullback 'g^*L' of L under g (the compatibility condition for commutative diagrams is just the condition that $(g' \circ f)^*L$ should be isomorphic to $f^*g'^*L$).

Let's look at another example: the moduli quotient (Example 2.18). Let G be an affine algebraic group acting on X. For simplicity, assume that there is a normal subgroup H of G that acts trivially on X, and that $\overline{G} = G/H$ is an affine group acting freely on X and furthermore there is a quotient by this action $X \longrightarrow B$ and this quotient is a principal $\overline{G}$-bundle. We call $B = X/G$ the *quotient scheme*. Each point corresponds to a G-orbit of the action. But note that B is also equal to the quotient $X/\overline{G}$, because H acts trivially and then G-orbits are the same thing as $\overline{G}$-orbits. We can say that the quotient scheme 'forgets' H.

One can also define the *quotient stack* $[X/G]$. Roughly speaking, a point p of $[X/G]$ again corresponds to a G-orbit of the action, but now each point comes with an automorphism group: given a point p in $[X/G]$, choose a point $x \in X$ in the orbit corresponding to p. The automorphism group attached to p is the stabilizer G_x of x. With the assumptions that we have made on the action of G, the automorphism group of any point is always H. Then the quotient stack $[X/G]$ is not a scheme, since the automorphism groups are not trivial. The action of H is trivial, but the moduli stack still 'remembers' that there was an action by H. Observe that the stack $[X/\overline{G}]$ is not isomorphic to the stack $[X/G]$ (as opposed to what happens with the quotient schemes). Since the action of $\overline{G}$ is free on X, the automorphism group corresponding to each point of $[X/\overline{G}]$ is trivial, and it can be shown that, with the assumptions that we made, $[X/\overline{G}]$ is represented by the scheme B (this terminology will be made precise in §2).

2. Stacks

2.1 *Stacks as 2-functors: Sheaves of sets*

Given a scheme M over a base scheme S, we define its (contravariant) functor of points $\mathrm{Hom}_S(-,M)$

$$\begin{array}{rccc} \mathrm{Hom}_S(-,M): & (\mathrm{Sch}/S) & \longrightarrow & (\mathrm{Sets}) \\ & B & \longmapsto & \mathrm{Hom}_S(B,M) \end{array}$$

where (Sch/S) is the category of S-schemes, B is an S-scheme, and $\mathrm{Hom}_S(B,M)$ is the set of S-scheme morphisms. If we give (Sch/S) the Zariski (or étale, or fppf) topology, $\tilde{M} = \mathrm{Hom}_S(-,M)$ is a sheaf (see Appendix A for the definition of topologies and sheaves on categories). Furthermore, given schemes M and N there is a bijection (given by Yoneda Lemma) between the set of morphisms of schemes $\mathrm{Hom}_S(M,N)$ and the set of natural transformations between the associated functors $\tilde{M}$ and $\tilde{N}$, hence the category of schemes is a full subcategory of the category of sheaves on (Sch/S).

A sheaf of sets on (Sch/S) with a given topology is called a space[2] with respect to that topology (this is the definition given in ([La], 0)).

Then schemes can be thought of as sheaves of sets. Moduli problems can usually be described by functors. We say that a sheaf of sets F is representable by a scheme M if F is isomorphic to the functor of points $\mathrm{Hom}_S(-,M)$. The scheme M is then called the fine moduli scheme. Roughly speaking, this means that there is a one to one correspondence between families of objects parametrized by a scheme B and morphisms from B to M.

Example 2.1 (*Vector bundles*). Let X be a projective scheme over an algebraically closed field k. We define the moduli functor $\underline{M}'_X$ of vector bundles of fixed rank r and Chern classes c_i by sending the scheme B to the set $\underline{M}'_X(B)$ of isomorphism classes of vector bundles on $B \times X$ (hence flat over B) with rank r and whose restriction to the slices $\{b\} \times X$ have Chern classes c_i. These vector bundles should be thought of as families of vector bundles parametrized by B. A morphism $f: B' \to B$ is sent to $\underline{M}'_X(f) = f^*: \underline{M}'_X(B) \to \underline{M}'_X(B')$, the map of sets induced by the pullback. Usually we will also fix a polarization H in X and restrict our attention to stable or semistable vector bundles with respect to this polarization (see [HL] for definitions), and then we consider the corresponding functors $\underline{M}'^{s}_X$ and $\underline{M}'^{ss}_X$.

Example 2.2 (*Curves*). The moduli functor M_g of smooth curves of genus g over a Noetherian base S is the functor that sends each scheme B to the set $M_g(B)$ of isomorphism classes of smooth and proper morphisms $C \to B$ (where C is an S-scheme) whose fibers are geometrically connected curves of genus g. Each morphism $f: B' \to B$ is sent to the map of sets induced by the pullback f^*.

None of these examples are sheaves (then none of these are representable), because of the presence of automorphisms. They are just presheaves (= functors). For instance, given a curve C over S with nontrivial automorphisms, it is possible to construct a family $f: C \to B$ such that every fiber of f is isomorphic to C, but C is not isomorphic to $B \times C$ (see [E]). This implies that M_g does not satisfy the monopresheaf axiom.

---

[2] Note that the concept of space is just a categorical concept. To do geometry we need to add some algebraic and technical conditions (existence of an atlas, quasi-separatedness,...). After we add these conditions (see Definitions 4.3 or 4.4), we have an algebraic space.

This can be solved by taking the sheaf associated to the presheaf (sheafification). In the examples, this amounts to change isomorphism classes of families to equivalence classes of families, declaring two families to be equivalent if they are locally (using the étale topology over the parametrizing scheme B) isomorphic. In the case of vector bundles, this is the reason why one usually considers two vector bundles V and V' on $X \times B$ equivalent if $V \cong V' \otimes p_B^* L$ for some line bundle L on B. The functor obtained with this equivalence relation is denoted $\underline{M}_X$ (and analogously for $\underline{M}^s_X$ and $\underline{M}^{ss}_X$).

Note that if two families V and V' are equivalent in this sense, then they are locally isomorphic. The converse is only true if the vector bundles are simple (only automorphisms are scalar multiplications). This will happen, for instance, if we are considering the functor $\underline{M}'^s_X$ of stable vector bundles, since stable vector bundles are simple. In general, if we want the functor to be a sheaf, we have to use a weaker notion of equivalence, but this is not done because for other reasons there is only hope of obtaining a fine moduli space if we restrict our attention to stable vector bundles.

Once this modification is made, there are some situations in which these examples are representable (for instance, stable vector bundles on curves with coprime rank and degree), but in general they will still not be representable, because in general we do not have a universal family:

DEFINITION 2.3 (Universal family)

Let F be a representable functor, and let $\phi : F \longrightarrow \mathrm{Hom}_S(-,X)$ be the isomorphism. The object of $F(X)$ corresponding to the element id_X of $\mathrm{Hom}_S(X,X)$ is called the universal family.

Example 2.4 (*Vector bundles*). If V is a universal vector bundle (over $M \times X$, where M is the fine moduli space), it has the property that for any family W of vector bundles (i.e. W is a vector bundle over $B \times X$ for some parameter scheme B) there exists a morphism $f : B \longrightarrow M$ such that $(f \times \mathrm{id}_X)^* V$ is equivalent to W.

In other words, the functor $\underline{M}_X$ is represented by the scheme M iff there exists a universal vector bundle on $M \times X$.

When a moduli functor F is not representable and then there is no scheme X whose functor of points is isomorphic to F, one can still try to find a scheme X whose functor of points is an approximation to F in some sense. There are two different notions:

DEFINITION 2.5 (Corepresents) ([S], p. 60), ([HL], Definition 2.2.1)

We say that a scheme M corepresents the functor F if there is a natural transformation of functors $\phi : F \longrightarrow \mathrm{Hom}_S(-,M)$ such that

- Given another scheme N and a natural transformation $\psi : F \longrightarrow \mathrm{Hom}_S(-,N)$, there is a unique natural transformation $\eta : \mathrm{Hom}_S(-,M) \longrightarrow \mathrm{Hom}_S(-,N)$ with $\psi = \eta \circ \phi$.

$$\begin{tikzcd}
F \arrow[d, "\phi"] \arrow[dr, "\psi"] & \\
\mathrm{Hom}_S(-,M) \arrow[r, "\eta"] & \mathrm{Hom}_S(-,N)
\end{tikzcd}$$

This characterizes M up to unique isomorphism. Let $(\mathrm{Sch}/S)'$ be the functor category, whose objects are contravariant functors from (Sch/S) to (Sets) and whose morphisms are natural transformation of functors. Then M represents F iff $\mathrm{Hom}_S(Y,M) = \mathrm{Hom}_{(\mathrm{Sch}/S)'}(\mathcal{Y},F)$ for all schemes Y, where $\mathcal{Y}$ is the functor represented by Y. On the other hand, one can check that M corepresents F iff $\mathrm{Hom}_S(M,Y) = \mathrm{Hom}_{(\mathrm{Sch}/S)'}(F,\mathcal{Y})$ for all schemes Y. If M represents F, then it corepresents it, but the converse is not true. From now on we will denote a scheme and the functor that it represents by the same letter.

DEFINITION 2.6 (Coarse moduli)

A scheme M is called a coarse moduli scheme if it corepresents F and furthermore

- for any algebraically closed field k, the map $\phi(k) : F(\mathrm{Spec}k) \longrightarrow \mathrm{Hom}_S(\mathrm{Spec}k, M)$ is bijective.

If M corepresents F (in particular, if M is a coarse moduli space), given a family of objects parametrized by B we get a morphism from B to M, but we don't require the converse to be true, i.e. not all morphisms are induced by families.

Example 2.7 (*Vector bundles*). There is a scheme M_X^{ss} that corepresents $\underline{M}_X^{ss}$ (see [HL]). It fails to be a coarse moduli scheme because its closed points are in one to one correspondence with S-equivalence classes of vector bundles, and not with isomorphism classes of vector bundles. Of course, this can be solved 'by hand' by modifying the functor and considering two vector bundles equivalent if they are S-equivalent. Once this modification is done, M_X^{ss} is a coarse moduli space.

But in general M_X^{ss} doesn't represent the moduli functor $\underline{M}_X^{ss}$. The reason for this is that vector bundles have always nontrivial automorphisms (multiplication by scalar), but the moduli functor does not record information about automorphisms: recall that to a scheme B it associates just the set of equivalence classes of vector bundles. To record the automorphisms of these vector bundles, we define

$$\begin{array}{rccc} \mathcal{M}_X : & (\mathrm{Sch}/S) & \longrightarrow & (\text{groupoids}) \\ & B & \longmapsto & \mathcal{M}_X(B), \end{array}$$

where $\mathcal{M}_X(B)$ is the category whose objects are vector bundles V on $B \times X$ of rank r and with fixed Chern classes (note that the objects are vector bundles, not isomorphism classes of vector bundles), and whose morphisms are vector bundle isomorphisms (note that we use isomorphisms of vector bundles, not S-equivalence nor equivalence classes as before). This defines a 2-functor between the 2-category associated to (Sch/S) and the 2-category (groupoids) (for the definition of 2-categories and 2-functors, see Appendix B).

DEFINITION 2.8

Let (groupoids) be the 2-category whose objects are groupoids, 1-morphisms are functors between groupoids, and 2-morphisms are natural transformation between these functors. A presheaf in groupoids (also called quasi-functor) is a contravariant 2-functor $\mathcal{F}$ from (Sch/S) to (groupoids). For each scheme B we have a groupoid $\mathcal{F}(B)$ and for each morphism $f : B' \longrightarrow B$ we have a functor $\mathcal{F}(f) : \mathcal{F}(B) \longrightarrow \mathcal{F}(B')$ that is denoted by f^* (usually it is actually defined by a pull-back).

Example 2.9 (*Vector bundles*) ([La], 1.3.4). $\mathcal{M}_X$ is a presheaf. For each object B of (Sch/S) it gives the groupoid $\mathcal{M}_X(B)$ defined in Example 2.7. For each 1-morphism

$f : B' \to B$ it gives the functor $\mathcal{F}(f) = f^* : \mathcal{M}_X(B) \to \mathcal{M}_X(B')$ given by pull-back, and for every diagram

$$B'' \xrightarrow{g} B' \xrightarrow{f} B \tag{2}$$

it gives a natural transformation of functors (a 2-isomorphism) $\epsilon_{g,f} : g^* \circ f^* \to (f \circ g)^*$. This is the only subtle point. First recall that the pullback f^*V of a vector bundle (or more generally, any fiber product) is not uniquely defined: it is only defined up to unique isomorphism. First choose once and for all a pullback f^*V for each f and V. Then, given a diagram like 2, in principle $g^*(f^*V)$ and $(f \circ g)^*V$ are not the same, but (because both solve the same universal problem) there is a canonical isomorphism (the unique isomorphism of the universal problem) $g^*(f^*V) \to (f \circ g)^*V$ between them, and this defines the natural transformation of functors $\epsilon_{g,f} : g^* \circ f^* \to (f \circ g)^*$. By a slight abuse of language, usually we will not write explicitly these isomorphisms $\epsilon_{g,f}$, and we will write $g^* \circ f^* = (f \circ g)^*$. Since they are uniquely defined this will cause no ambiguity.

Example 2.10 (*Stable curves*) ([DM], Definition 1.1). Let B be an S-scheme. Let $g \geq 2$. A stable curve of genus g over B is a proper and flat morphism $\pi : C \to B$ whose geometric fibers are reduced, connected and one-dimensional schemes C_b such that

1. The only singularities of C_b are ordinary double points.
2. If E is a non-singular rational component of C_b, then E meets the other components of C_b in at least 3 points.
3. dim $H^1(\mathcal{O}_{C_b}) = g$.

Condition 2 is imposed so that the automorphism group of C_b is finite. A stable curve over B should be thought of as a family of stable curves (over S) parametrized by B.

For each object B of (Sch/S), let $\overline{\mathcal{M}}_g(B)$ be the groupoid whose objects are stable curves over B and whose (iso)morphisms are Cartesian diagrams

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & X \arrow[d] \\ B \arrow[r, equal] & B \end{tikzcd}.$$

For each morphism $f : B' \to B$ of (Sch/S), we define the pullback functor $f^* : \overline{\mathcal{M}}_g(B) \to \overline{\mathcal{M}}_g(B')$, sending an object $X \to B$ to $f^*X \to B'$ (and a morphism $\varphi : X_1 \to X_2$ of curves over B to $f^*\varphi : f^*X_1 \to f^*X_2$). And finally, for each diagram

$$B'' \xrightarrow{g} B' \xrightarrow{f} B$$

we have to give a natural transformation of functors (i.e. a 2-isomorphism in (groupoids)) $\epsilon_{g,f} : g^* \circ f^* \to (f \circ g)^*$. As in the case of vector bundles, this is defined by first choosing once an for all a pullback f^*X for each curve X and morphism f, and then $\epsilon_{g,f}$ is given by the canonical isomorphism between $g^*(f^*X)$ and $(f \circ g)^*X$. Since this isomorphism is canonical, by a slight abuse of language we usually write $g^* \circ f^* = (f \circ g)^*$.

Now we will define the concept of stack. First we have to choose a Grothendieck topology on (Sch/S), either the étale or the fppf topology. Later on, when we define algebraic stack, the étale topology will lead to the definition of a Deligne-Mumford stack ([DM, Vi, E]), and the fppf to an Artin stack ([La]). For the moment we will give a unified description.

In the following definition, to simplify notation we denote by $X|_i$ the pullback f_i^*X where $f_i : U_i \to U$ and X is an object of $\mathcal{F}(U)$, and by $X_i|_{ij}$ the pullback $f_{ij,i}^* X_i$ where $f_{ij,i} : U_i \times_U U_j \to U_i$ and X_i is an object of $\mathcal{F}(U_i)$. We will also use the obvious variations of this convention, and will simplify the notation using Remark 5.3.

DEFINITION 2.11 (Stack)

A stack is a sheaf of groupoids, i.e. a 2-functor (=presheaf) that satisfies the following sheaf axioms. Let $\{U_i \to U\}_{i\in I}$ be a covering of U in the site (Sch/S). Then

1. *Glueing of morphisms.* If X and Y are two objects of $\mathcal{F}(U)$, and $\varphi_i : X|_i \to Y|_i$ are morphisms such that $\varphi_i|_{ij} = \varphi_j|_{ij}$, then there exists a morphism $\eta : X \to Y$ such that $\eta|_i = \varphi_i$.
2. *Monopresheaf.* If X and Y are two objects of $\mathcal{F}(U)$, and $\varphi : X \to Y$, $\psi : X \to Y$ are morphisms such that $\varphi|_i = \psi|_i$, then $\varphi = \psi$.
3. *Glueing of objects.* If X_i are objects of $\mathcal{F}(U_i)$ and $\varphi_{ij} : X_j|_{ij} \to X_i|_{ij}$ are morphisms satisfying the cocycle condition $\varphi_{ij}|_{ijk} \circ \varphi_{jk}|_{ijk} = \varphi_{ik}|_{ijk}$, then there exists an object X of $\mathcal{F}(U)$ and $\varphi_i : X|_i \xrightarrow{\cong} X_i$ such that $\varphi_{ji} \circ \varphi_i|_{ij} = \varphi_j|_{ij}$.

At first sight this might seem very complicated, but if we check in a particular example we will see that it is a very natural definition:

Example 2.12 (*Stable curves*). It is easy to check that the presheaf $\overline{\mathcal{M}}_g$ defined in 2.10 is a stack (all properties hold because of descent theory). We take the étale topology on (Sch/S) (we will see that the reason for this is that the automorphism group of a stable curve is finite). Let $\{U_i \to U\}_{i\in I}$ be a cover of U. Item 1 says that if we have two curves X and Y over U, and we have isomorphisms $\varphi_i : X|_i \to Y|_i$ on the restriction for each U_i, then these isomorphisms glue to give an isomorphism $\eta : X \to Y$ over U if the restrictions to the intersections $\varphi_i|_{ij}$ and $\varphi_j|_{ij}$ coincide.

Item 2 says that two morphisms of curves over U coincide if the restrictions to all U_i coincide.

Finally, item 3 says that if we have curves X_i over U_i and we are given isomorphisms φ_{ij} over the intersections U_{ij}, then we can glue the curves to get a curve over U if the isomorphisms satisfy the cocycle condition.

Example 2.13 (*Vector bundles*). It is also easy to check that the presheaf of vector bundles $\mathcal{M}_X$ is a sheaf. In this case we take the fppf topology on (Sch/S) (we will see that the reason for this choice is that the automorphism group of a vector bundle is not finite, because it includes multiplication by scalars).

Let us stop for a moment and look at how we have enlarged the category of schemes by defining the category of stacks. We can draw the following diagram

$$\begin{tikzcd}
 & \text{Algebraic Stacks} \arrow[r] & \text{Stacks} \arrow[r] & \text{Presheaves of groupoids} \\
\mathrm{Sch}/S \arrow[r] \arrow[ur] & \text{Algebraic Spaces} \arrow[r] \arrow[u] & \text{Spaces} \arrow[r] \arrow[u] & \text{Presheaves of sets} \arrow[u]
\end{tikzcd}$$

where $A \to B$ means that the category A is a subcategory B. Recall that a presheaf of sets is just a functor from (Sch/S) to the category (Sets), a presheaf of groupoids is just a 2-functor to the 2-category (groupoids). A sheaf (for example a space or a stack) is a presheaf that satisfies the sheaf axioms (these axioms are slightly different in the context of categories or 2-categories), and if this sheaf satisfies some geometric conditions (that we have not yet specified), we will have an algebraic stack or algebraic space.

2.2 *Stacks as categories: Groupoids*

There is an alternative way of defining a stack. From this point of view a stack will be a category, instead of a functor.

DEFINITION 2.14

A category over (Sch/S) is a category $\mathcal{F}$ and a covariant functor $p_{\mathcal{F}} : \mathcal{F} \longrightarrow (\mathrm{Sch}/S)$ (called the structure functor). If X is an object (resp. ϕ is a morphism) of $\mathcal{F}$, and $p_{\mathcal{F}}(X) = B$ (resp. $p_{\mathcal{F}}(\phi) = f$), then we say that X lies over B (resp. ϕ lies over f).

DEFINITION 2.15 (Groupoid)

A category $\mathcal{F}$ over (Sch/S) is called a category fibered on groupoids (or just groupoid) if

1. For every $f : B' \longrightarrow B$ in (Sch/S) and every object X with $p_{\mathcal{F}}(X) = B$, there exists at least one object X' and a morphism $\phi : X' \longrightarrow X$ such that $p_{\mathcal{F}}(X') = B'$ and $p_{\mathcal{F}}(\phi) = f$.

$$\begin{tikzcd}
X' \arrow[r, dashed, "\phi"] \arrow[d, mapsto, dashed] & X \arrow[d, mapsto] \\
B' \arrow[r, "f"] & B
\end{tikzcd}$$

2. For every diagram

$$\begin{tikzcd}
X_3 \arrow[rr, "\psi"] \arrow[dd, mapsto] & & X_1 \arrow[dd, mapsto] \\
& X_2 \arrow[ur, "\phi"] \arrow[dd, mapsto] & \\
B_3 \arrow[rr, "f\circ f'"] \arrow[dr, "f'"'] & & B_1 \\
& B_2 \arrow[ur, "f"'] &
\end{tikzcd}$$

(where $p_{\mathcal{F}}(X_i) = B_i$, $p_{\mathcal{F}}(\phi) = f$, $p_{\mathcal{F}}(\psi) = f \circ f'$), there exists a unique $\varphi : X_3 \longrightarrow X_2$ with $\psi = \phi \circ \varphi$ and $p_{\mathcal{F}}(\varphi) = f'$.

Condition 2 implies that the object X' whose existence is asserted in condition 1 is unique up to canonical isomorphism. For each X and f we choose once and for all such an X' and call it f^*X. Another consequence of condition 2 is that ϕ is an isomorphism if and only if $p_{\mathcal{F}}(\phi) = f$ is an isomorphism.

Let B be an object of (Sch/S). We define $\mathcal{F}(B)$, the fiber of $\mathcal{F}$ over B, to be the subcategory of $\mathcal{F}$ whose objects lie over B and whose morphisms lie over id_B. It is a groupoid.

The association $B \longrightarrow \mathcal{F}(B)$ in fact defines a presheaf of groupoids (note that the 2-isomorphisms $\epsilon_{f,g}$ required in the definition of presheaf of groupoids are well defined thanks to condition 2). Conversely, given a presheaf of groupoids $\mathcal{G}$ on (Sch/S), we can define the category $\mathcal{F}$ whose objects are pairs (B, X) where B is an object of (Sch/S) and

X is an object of $\mathcal{G}(B)$, and whose morphisms $(B',X') \to (B,X)$ are pairs (f,α) where $f: B' \to B$ is a morphism in (Sch/S) and $\alpha: f^*X \to X'$ is an isomorphism, where $f^* = \mathcal{G}(f)$. This gives the relationship between both points of view. Since we have a canonical one-to-one relationship between presheaves of groupoids and groupoids over S, by a slight abuse of language, we denote both by the same letter.

Example 2.16 (*Vector bundles*). The groupoid of vector bundles $\mathcal{M}_X$ on a scheme X is the category whose objects are vector bundles over $B \times X$ (for B a scheme), and whose morphisms are isomorphisms

$$\varphi : V' \xrightarrow{\cong} (f \times \mathrm{id})^* V,$$

where V (resp. V') is a vector bundle over $B \times X$ (resp. $B' \times X$) and $f: B' \to B$ is a morphism of schemes. The structure functor sends a vector bundle over $B \times X$ to the scheme B, and a morphism φ to the corresponding morphism of schemes f.

Example 2.17 (*Stable curves*) ([DM], Definition 1.1). We define $\overline{\mathcal{M}}_g$, the groupoid over S whose objects are stable curves over B of genus g (see Definition 2.10), and whose morphisms are Cartesian diagrams

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & X \arrow[d] \\ B' \arrow[r, "f"] & B \end{tikzcd} \tag{3}$$

The structure functor sends a curve over B to the scheme B, and a morphism as in (3) to f.

Example 2.18 (*Quotient by group action*) ([La], 1.3.2), ([DM], Example 4.8), ([E], Example 2.2). Let X be an S-scheme (assume all schemes are Noetherian), and G an affine flat group S-scheme acting on the right on X. We define the groupoid $[X/G]$ whose objects are principal G-bundles $\pi: E \to B$ together with a G-equivariant morphism $f: E \to X$. A morphism is Cartesian diagram

$$\begin{tikzcd} E' \arrow[r, "p"] \arrow[d, "\pi'"'] & E \arrow[d, "\pi"] \\ B' \arrow[r, "g"] & B \end{tikzcd} \tag{4}$$

such that $f \circ p = f'$.

The structure functor sends an object $(\pi: E \to B, f: E \to X)$ to the scheme B, and a morphism as in (4) to g.

DEFINITION 2.19 (Stack)

A stack is a groupoid that satisfies

1. (*Prestack*). For all scheme B and pair of objects X, Y of $\mathcal{F}$ over B, the contravariant functor

$$\begin{array}{rccc} \mathrm{Iso}_B(X,Y): & (\mathrm{Sch}/B) & \longrightarrow & (\mathrm{Sets}) \\ & (f: B' \to B) & \longmapsto & \mathrm{Hom}(f^*X, f^*Y) \end{array}$$

   is a sheaf on the site (Sch/B).
2. Descent data is effective (this is just condition 3 in the Definition 2.11 of sheaf).

Example 2.20. If G is smooth and affine, the groupoid $[X/G]$ is a stack ([La], 2.4.2), ([Vi], Example 7.17), ([E], Proposition 2.2). Then also $\overline{\mathcal{M}}_g$ (cf. Example 2.17) is a stack, because it is isomorphic to a quotient stack of a subscheme of a Hilbert scheme by $PGL(N)$ ([E], Theorem 3.2), [DM]. The groupoid $\mathcal{M}_X$ defined in Example 2.16 is also a stack ([La], 2.4.4).

From now on we will mainly use this approach. Now we will give some definitions for stacks.

Morphisms of stacks. A morphism of stacks $f : \mathcal{F} \to \mathcal{G}$ is a functor between the categories, such that $p_{\mathcal{G}} \circ f = p_{\mathcal{F}}$. A commutative diagram of stacks is a diagram

$$\begin{tikzcd}
& \mathcal{G} \arrow[dr, "g"] \arrow[d, Rightarrow, "\alpha"] & \\
\mathcal{F} \arrow[ur, "f"] \arrow[rr, "h"'] & {} & \mathcal{H}
\end{tikzcd}$$

such that $\alpha : g \circ f \to h$ is an isomorphism of functors. If f is an equivalence of categories, then we say that the stacks $\mathcal{F}$ and $\mathcal{G}$ are isomorphic. We denote by $\mathrm{Hom}_S(\mathcal{F}, \mathcal{G})$ the category whose objects are morphisms of stacks and whose morphisms are natural transformations.

Stack associated to a scheme. Given a scheme U over S, consider the category (Sch/U). Define the functor $p_U : (\mathrm{Sch}/U) \to (\mathrm{Sch}/S)$ which sends the U-scheme $f : B \to U$ to the composition $B \xrightarrow{f} U \to S$, and sends the U-morphism $(B' \to U) \to (B \to U)$ to the S-morphism $(B' \to S) \to (B \to S)$. Then (Sch/U) becomes a stack. Usually we denote this stack also by U. From the point of view of 2-functors, the stack associated to U is the 2-functor that for each scheme B gives the category whose objects are the elements of the set $\mathrm{Hom}_S(B, U)$, and whose only morphisms are identities.

We say that a stack is represented by a scheme U when it is isomorphic to the stack associated to U. We have the following very useful lemmas:

Lemma 2.21. *If a stack has an object with an automorphism other that the identity, then the stack cannot be represented by a scheme.*

Proof. In the definition of stack associated with a scheme we see that the only automorphisms are identities. ☐

Lemma 2.22 ([Vi], 7.10). *Let $\mathcal{F}$ be a stack and U a scheme. The functor*

$$u : \mathrm{Hom}_S(U, \mathcal{F}) \to \mathcal{F}(U)$$

that sends a morphism of stacks $f : U \to \mathcal{F}$ to $f(id_U)$ is an equivalence of categories.

Proof. Follows from Yoneda lemma. ☐

This useful observation that we will use very often means that an object of $\mathcal{F}$ that lies over U is equivalent to a morphism (of stacks) from U to $\mathcal{F}$.

Fiber product. Given two morphisms $f_1 : \mathcal{F}_1 \to \mathcal{G}$, $f_2 : \mathcal{F}_2 \to \mathcal{G}$, we define a new stack $\mathcal{F}_1 \times_{\mathcal{G}} \mathcal{F}_2$ (with projections to $\mathcal{F}_1$ and $\mathcal{F}_2$) as follows. The objects are triples (X_1, X_2, α) where X_1 and X_2 are objects of $\mathcal{F}_1$ and $\mathcal{F}_2$ that lie over the same scheme U, and $\alpha : f_1(X_1) \to f_2(X_2)$ is an isomorphism in $\mathcal{G}$ (equivalently, $p_{\mathcal{G}}(\alpha) = \mathrm{id}_U$). A morphism from (X_1, X_2, α) to (Y_1, Y_2, β) is a pair (ϕ_1, ϕ_2) of morphisms $\phi_i : X_i \longrightarrow Y_i$ that lie over the same morphism of schemes $f : U \longrightarrow V$, and such that $\beta \circ f_1(\phi_1) = f_2(\phi_2) \circ \alpha$. The fiber product satisfies the usual universal property.

Representability. A stack $\mathcal{X}$ is said to be representable by an algebraic space (resp. scheme) if there is an algebraic space (resp. scheme) X such that the stack associated to X is isomorphic to $\mathcal{X}$. If 'P' is a property of algebraic spaces (resp. schemes) and $\mathcal{X}$ is a representable stack, we will say that $\mathcal{X}$ has 'P' iff X has 'P'.

A morphism of stacks $f : \mathcal{F} \longrightarrow \mathcal{G}$ is said to be representable if for all objects U in (Sch/S) and morphisms $U \longrightarrow \mathcal{G}$, the fiber product stack $U \times_{\mathcal{G}} \mathcal{F}$ is representable by an algebraic space. Let P be a property of morphisms of schemes that is local in nature on the target for the topology chosen on (Sch/S) (étale or fppf), and it is stable under arbitrary base change. For instance: separated, quasi-compact, unramified, flat, smooth, étale, surjective, finite type, locally of finite type,.... Then, for a representable morphism f, we say that f has P if for every $U \longrightarrow \mathcal{G}$, the pullback $U \times_{\mathcal{G}} \mathcal{F} \longrightarrow U$ has P ([La], p.17, [DM], p. 98).

Diagonal. Let $\Delta_{\mathcal{F}} : \mathcal{F} \longrightarrow \mathcal{F} \times_S \mathcal{F}$ be the obvious diagonal morphism. A morphism from a scheme U to $\mathcal{F} \times_S \mathcal{F}$ is equivalent to two objects X_1, X_2 of $\mathcal{F}(U)$. Taking the fiber product of these we have

$$\begin{tikzcd}
\mathrm{Iso}_U(X_1, X_2) \arrow[r] \arrow[d] & \mathcal{F} \arrow[d, "\Delta_{\mathcal{F}}"] \\
U \arrow[r, "{(X_1,X_2)}"] & \mathcal{F} \times_S \mathcal{F}
\end{tikzcd}$$

hence the group of automorphisms of an object is encoded in the diagonal morphism.

PROPOSITION 2.23 ([La], Corollary 2.12), ([Vi], Proposition 7.13)

The following are equivalent

1. *The morphism* $\Delta_{\mathcal{F}}$ *is representable.*
2. *The stack* $\mathrm{Iso}_U(X_1, X_2)$ *is representable for all* U, X_1 *and* X_2.
3. *For all scheme* U, *every morphism* $U \longrightarrow \mathcal{F}$ *is representable.*
4. *For all schemes* U, V *and morphisms* $U \longrightarrow \mathcal{F}$ *and* $V \longrightarrow \mathcal{F}$, *the fiber product* $U \times_{\mathcal{F}} V$ *is representable.*

Proof. The implications $1 \Leftrightarrow 2$ and $3 \Leftrightarrow 4$ follow easily from the definitions.
$(1 \Rightarrow 4)$ Assume that $\Delta_{\mathcal{F}}$ is representable. We have to show that $U \times_{\mathcal{F}} V$ is representable for any $f : U \longrightarrow \mathcal{F}$ and $g : V \longrightarrow \mathcal{F}$. Check that the following diagram is Cartesian

$$\begin{tikzcd}
U \times_{\mathcal{F}} V \arrow[r] \arrow[d] & \mathcal{F} \arrow[d, "\Delta_{\mathcal{F}}"] \\
U \times_S V \arrow[r, "f \times g"] & \mathcal{F} \times_S \mathcal{F}
\end{tikzcd} .$$

Then $U \times_{\mathcal{F}} V$ is representable.
$(1 \Leftarrow 4)$ First note that the Cartesian diagram defined by $h : U \longrightarrow \mathcal{F} \times_S \mathcal{F}$ and $\Delta_{\mathcal{F}}$ factors as follows:

$$\begin{tikzcd}
U \times_{\mathcal{F} \times_S \mathcal{F}} \mathcal{F} \arrow[r] \arrow[d] & U \times_{\mathcal{F}} U \arrow[r] \arrow[d] & \mathcal{F} \arrow[d] \\
U \arrow[r, "\Delta_U"] & U \times_S U \arrow[r] & \mathcal{F} \times_S \mathcal{F}
\end{tikzcd} .$$

The outer (big) rectangle and the right square are Cartesian, so the left square is also Cartesian. By hypothesis $U \times_{\mathcal{F}} U$ is representable, then $U \times_{\mathcal{F} \times_S \mathcal{F}} \mathcal{F}$ is also representable. □

2.3 *Algebraic stacks*

Now we will define the notion of algebraic stack. As we have said, first we have to choose a topology on (Sch/S). Depending of whether we choose the étale or fppf topology, we get different notions.

DEFINITION 2.24 (Deligne-Mumford stack)

Let (Sch/S) be the category of S-schemes with the étale topology. Let $\mathcal{F}$ be a stack. Assume

1. *Quasi-separatedness*. The diagonal $\Delta_{\mathcal{F}}$ is representable, quasi-compact and separated.
2. There exists a scheme U (called atlas) and an étale surjective morphism $u : U \longrightarrow \mathcal{F}$.

Then we say that $\mathcal{F}$ is a Deligne-Mumford stack.

The morphism of stacks u is representable because of Proposition 2.23 and the fact that the diagonal $\Delta_{\mathcal{F}}$ is representable. Then the notion of étale is well defined for u. In [DM] this was called an algebraic stack. In the literature, algebraic stack usually refers to Artin stack (that we will define later). To avoid confusion, we will use 'algebraic stack' only when we refer in general to both notions, and we will use 'Deligne-Mumford' or 'Artin' stack when we want to be specific.

Note that the definition of Deligne-Mumford stack is the same as the definition of algebraic space, but in the context of stacks instead of spaces. Following the terminology used in scheme theory, a stack such that the diagonal $\Delta_{\mathcal{F}}$ is quasi-compact and separated is called quasi-separated. We always assume this technical condition, as it is usually done both with schemes and algebraic spaces.

Sometimes it is difficult to find explicitly an étale atlas, and the following proposition is useful.

PROPOSITION 2.25 ([DM], Theorem 4.21), [E]

Let $\mathcal{F}$ be a stack over the étale site (Sch/S). *Assume*

1. *The diagonal $\Delta_{\mathcal{F}}$ is representable, quasi-compact, separated and unramified.*
2. *There exists a scheme U of finite type over S and a smooth surjective morphism $u : U \rightarrow \mathcal{F}$.*

Then $\mathcal{F}$ is a Deligne-Mumford stack.

Now we define the analog for the fppf topology [Ar2].

DEFINITION 2.26 (Artin stack)

Let (Sch/S) be the category of S-schemes with the fppf topology. Let $\mathcal{F}$ be a stack. Assume

1. *Quasi-separatedness*. The diagonal $\Delta_{\mathcal{F}}$ is representable, quasi-compact and separated.
2. There exists a scheme U (called atlas) and a smooth (hence locally of finite type) and surjective morphism $u : U \longrightarrow \mathcal{F}$.

Then we say that $\mathcal{F}$ is an Artin stack.

For propositions analogous to proposition 2.25, see [La, 4].

PROPOSITION 2.27 ([Vi], Proposition 7.15), ([La], Lemma 3.3)

If $\mathcal{F}$ is a Deligne-Mumford (resp. Artin) stack, then the diagonal $\Delta_{\mathcal{F}}$ is unramified (resp. finite type).

Recall that $\Delta_{\mathcal{F}}$ is unramified (resp. finite type) if for every scheme B and objects X, Y of $\mathcal{F}(B)$, the morphism $\mathrm{Iso}_B(X,Y) \longrightarrow B$ is unramified (resp. finite type). If $B = \operatorname{Spec} S$ and $X = Y$, then this means that the automorphism group of X is discrete and reduced for a Deligne-Mumford stack, and it is of finite type for an Artin stack.

Example 2.28 (*Vector bundles*). The stack $\mathcal{M}_X$ is an Artin stack, locally of finite type ([La], 4.14.2.1). The atlas is constructed as follows: Let P^H_{r,c_i} be the Hilbert polynomial corresponding to locally free sheaves on X with rank r and Chern classes c_i. Let Quot $(\mathcal{O}(-m)^{\oplus N}, P^H_{r,c_i})$ be the Quot scheme parametrizing quotients of sheaves on X,

$$\mathcal{O}(-m)^{\oplus N} \longrightarrow V, \tag{5}$$

where V is a coherent sheaf on X with Hilbert polynomial P^H_{r,c_i}. Let $R_{N,m}$ be the subscheme corresponding to quotients (5) such that V is a vector bundle with $H^p(V(m)) = 0$ for $p > 0$ and the morphism (5) induces an isomorphism on global sections

$$H^0(\mathcal{O})^{\oplus N} \xrightarrow{\cong} H^0(V(m)).$$

The scheme $R_{N,m}$ has a universal vector bundle, induced from the universal bundle of the Quot scheme, and then there is a morphism $u_{N,m} : R_{N,m} \longrightarrow \mathcal{M}_X$. Since H is ample, for every vector bundle V, there exist integers N and m such that $R_{N,m}$ has a point whose corresponding quotient is V, and then if we take the infinite disjoint union of these morphisms we get a surjective morphism

$$u : \left(\coprod_{N,m>0} R_{N,m} \right) \longrightarrow \mathcal{M}_X.$$

It can be shown that this morphism is smooth, and then it gives an atlas. Each scheme $R_{N,m}$ is of finite type, so the union is locally of finite type, which in turn implies that the stack $\mathcal{M}_X$ is locally of finite type.

Example 2.29 (*Quotient by group action*). The stack $[X/G]$ is an Artin stack ([La], 4.14.1.1). If G is smooth, an atlas is defined as follows (for more general G, see ([La], 4.14.1.1)): Take the trivial principal G-bundle $X \times G$ over X, and let the map $f : X \times G \longrightarrow X$ be the action of the group. This defines an object of $[X/G](X)$, and by Lemma 2.22, it defines a morphism $u : X \longrightarrow [X/G]$. It is representable, because if B is a scheme and $g : B \longrightarrow [X/G]$ is the morphism corresponding to a principal G-bundle E over B with an equivariant morphism $f : E \longrightarrow X$, then $B \times_{[X/G]} X$ is isomorphic to the scheme E, and in fact we have a Cartesian diagram

$$\begin{tikzcd}
E \arrow[r, "f"] \arrow[d, "\pi"'] & X \arrow[d, "u"] \\
B \arrow[r, "g"] & {[X/G]}
\end{tikzcd} .$$

The morphism u is surjective and smooth because π is surjective and smooth for every g (if G is not smooth, but only separated, flat and of finite presentation, then u is not an atlas, but if we apply Artin's theorem ([Ar2], Theorem 6.1), ([La], Theorem 4.1), we conclude that there is a smooth atlas).

If either G is étale over S ([DM], Example 4.8) or the stabilizers of the geometric points of X are finite and reduced ([Vi], Example 7.17), then $[X/G]$ is a Deligne-Mumford stack.

Note that if the action is not free, then $[X/G]$ is not representable by Lemma 2.21. On the other hand, if there is a scheme Y such that $X \longrightarrow Y$ is a principal G-bundle, then $[X/G]$ is represented by Y.

Let G be a reductive group acting on X. Let H be an ample line bundle on X, and assume that the action is polarized. Let X^s and X^{ss} be the subschemes of stable and semistable points. Let $Y = X//G$ be the GIT quotient. Recall that there is a good quotient $X^{ss} \longrightarrow Y$, and that the restriction to the stable part $X^s \longrightarrow Y$ is a principal bundle. There is a natural morphism $[X^{ss}/G] \longrightarrow X^{ss}//G$. By the previous remark, the restriction $[X^s/G] \longrightarrow Y^s$ is an isomorphism of stacks.

If $X = S$ (with trivial action of G on S), then $[S/G]$ is denoted BG, the classifying groupoid of principal G-bundles.

Example 2.30 (*Stable curves*). The stack $\overline{\mathcal{M}}_g$ is a Deligne-Mumford stack ([DM], Proposition 5.1), [E]. The idea of the proof is to show that $\overline{\mathcal{M}}_g$ is the quotient stack $[\overline{H}_g/PGL(N)]$ of a scheme $\overline{H}_g$ by a smooth group $PGL(N)$. This gives a smooth atlas. Then one shows that the diagonal is unramified, and finally we apply Proposition 2.25.

2.4 *Algebraic stacks as groupoid spaces*

We will introduce a third equivalent definition of stack. First consider a category C. Let U be the set of objects and R the set of morphisms. The axioms of a category give us four maps of sets

$$R \underset{t}{\overset{s}{\rightrightarrows}} U \xrightarrow{e} R \qquad R \times_{s,U,t} R \xrightarrow{m} R,$$

where s and t give the source and target for each morphism, e gives the identity morphism, and m is composition of morphisms. If the category is a groupoid then we have a fifth morphism

$$R \xrightarrow{i} R$$

that gives the inverse. These maps satisfy

1. $s \circ e = t \circ e = \mathrm{id}_U$, $s \circ i = t$, $t \circ i = s$, $s \circ m = s \circ p_2$, $t \circ m = t \circ p_1$.
2. *Associativity.* $m \circ (m \times \mathrm{id}_R) = m \circ (\mathrm{id}_R \times m)$.
3. *Identity.* Both compositions

$$R = R \times_{s,U} U = U \times_{U,t} R \underset{e \times \mathrm{id}_R}{\overset{\mathrm{id}_R \times e}{\rightrightarrows}} R \times_{s,U,t} R \xrightarrow{m} R$$

are equal to the identity map on R.

4. *Inverse.* $m \circ (i \times \mathrm{id}_R) = e \circ s$, $m \circ (\mathrm{id}_R \times i) = e \circ t$.

DEFINITION 2.31 (Groupoid space) ([La], 1.3.3), ([DM], pp. 668–669)

A groupoid space is a pair of spaces (sheaves of sets) U, R, with five morphisms s, t, e, m, i with the same properties as above.

DEFINITION 2.32 ([La], 1.3.3).

Given a groupoid space, define the groupoid over (Sch/S) as the category $[R,U]'$ over (Sch/S) whose objects over the scheme B are elements of the set $U(B)$ and whose morphisms over B are elements of the set $R(B)$. Given $f : B' \to B$ we define a functor $f^* : [R,U]'(B) \to [R,U]'(B')$ using the maps $U(B) \to U(B')$ and $R(B) \to R(B')$.

The groupoid $[R,U]'$ is in general only a prestack. We denote by $[R,U]$ the associated stack. The stack $[R,U]$ can be thought of as the sheaf associated to the presheaf of groupoids $B \mapsto [R,U]'(B)$ ([La], 2.4.3).

Example 2.33 (*Quotient by group action*). Let X be a scheme and G an affine group scheme. We denote by the same letters the associated spaces (functors of points). We take $U = X$ and $R = X \times G$. Using the group action we can define the five morphisms (t is the action of the group, $s = p_1$, m is the product in the group, e is defined with the identity of G, and i with the inverse).

The objects of $[X \times G, X]'(B)$ are morphisms $f : B \to X$. Equivalently, they are trivial principal G-bundles $B \times G$ over B and a map $B \times G \to X$ defined as the composition of the action of G and f. The stack $[X \times G, X]$ is isomorphic to $[X/G]$.

Example 2.34 (*Algebraic stacks*). Let R, U be a groupoid space such that R and U are algebraic spaces, locally of finite presentation (equivalently locally of finite type if S is noetherian). Assume that the morphisms s, t are flat, and that $\delta = (s,t) : R \to U \times_S U$ is separated and quasi-compact. Then $[R,U]$ is an Artin stack, locally of finite type ([La], Corollary 4.7).

In fact, any Artin stack $\mathcal{F}$ can be defined in this fashion. The algebraic space U will be the atlas of $\mathcal{F}$, and we set $R = U \times_{\mathcal{F}} U$. The morphisms s and t are the two projections, i exchanges the factors, e is the diagonal, and m is defined by projection to the first and third factor.

Let $\delta : R \to U \times_S U$ be an equivalence relation in the category of spaces. One can define a groupoid space, and $[R,U]$ is to be thought of as the stack-theoretic quotient of this equivalence relation, as opposed to the quotient space, used for instance to define algebraic spaces (for more details and the definition of equivalence relation see appendix A).

2.5 *Properties of algebraic stacks*

So far we have only defined scheme-theoretic properties for representable stacks and morphisms. We can define some properties for arbitrary algebraic stacks (and morphisms among them) using the atlas.

Let P be a property of schemes, local in nature for the smooth (resp. étale) topology. For example: regular, normal, reduced, of characteristic p,... Then we say that an Artin (resp. Deligne-Mumford) stack has P iff the atlas has P ([La], p. 25), ([DM], p. 100).

Let P be a property of morphisms of schemes, local on source and target for the smooth (resp. étale) topology, i.e. for any commutative diagram

$$\begin{tikzcd}
X' \arrow[r, "p"] \arrow[dr, "f''"'] & Y' \times_Y X \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\
& Y' \arrow[r, "g"] & Y
\end{tikzcd}$$

with p and g smooth (resp. étale) and surjective, f has P iff f'' has P. For example: flat, smooth, locally of finite type,.... For the étale topology we also have: étale, unramified,.... Then if $f : \mathcal{X} \to \mathcal{Y}$ is a morphism of Artin (resp. Deligne-Mumford) stacks, we say that f has P iff for one (and then for all) commutative diagram of stacks

$$\begin{tikzcd}
X' \arrow[r, "p"] \arrow[dr, "f''"'] & Y' \times_Y \mathcal{X} \arrow[r, "g'"] \arrow[d, "f'"'] & \mathcal{X} \arrow[d, "f"] \\
& Y' \arrow[r, "g"] & \mathcal{Y}
\end{tikzcd},$$

where X', Y' are schemes and p, g are smooth (resp. étale) and surjective, f'' has P ([La], pp. 27–29).

For Deligne-Mumford stacks it is enough to find a commutative diagram

$$\begin{tikzcd}
X' \arrow[r, "p"] \arrow[d, "f''"'] & \mathcal{X} \arrow[d, "f"] \\
Y' \arrow[r, "g"] & \mathcal{Y}
\end{tikzcd},$$

where p and g are étale and surjective and f'' has P. Then it follows that f has P ([DM], p. 100).

Other notions are defined as follows.

DEFINITION 2.35 (Substack) ([La], Definition 2.5), ([DM], p. 102).

A stack $\mathcal{E}$ is a substack of $\mathcal{F}$ if it is a full subcategory of $\mathcal{F}$ and

1. If an object X of $\mathcal{F}$ is in $\mathcal{E}$, then all isomorphic objects are also in $\mathcal{E}$.
2. For all morphisms of schemes $f : U \to V$, if X is in $\mathcal{E}(V)$, then f^*X is in $\mathcal{E}(U)$.
3. Let $\{U_i \to U\}$ be a cover of U in the site (Sch/S). Then X is in $\mathcal{E}$ iff $X|_i$ is in $\mathcal{E}$ for all i.

DEFINITION 2.36 ([La], Definition 2.13)

A substack $\mathcal{E}$ of $\mathcal{F}$ is called open (resp. closed, resp. locally closed) if the inclusion morphism $\mathcal{E} \to \mathcal{F}$ is *representable* and it is an open immersion (resp. closed immersion, resp. locally closed immersion).

DEFINITION 2.37 (Irreducibility) ([La], Definition 3.10), ([DM], p. 102)

An algebraic stack $\mathcal{F}$ is irreducible if it is not the union of two distinct and nonempty proper closed substacks.

DEFINITION 2.38 (Separatedness) ([La], Definition 3.17), ([DM], Definition 4.7)

An algebraic stack $\mathcal{F}$ is separated, if the (representable) diagonal morphism $\Delta_{\mathcal{F}}$ is universally closed (and hence proper, because it is automatically separated and of finite type).

A morphism $f : \mathcal{F} \to \mathcal{G}$ of algebraic stacks is separated if for all $U \to \mathcal{G}$ with U affine, $U \times_{\mathcal{G}} \mathcal{F}$ is a separated (algebraic) stack.

For Deligne-Mumford stacks, $\Delta_{\mathcal{F}}$ is universally closed iff it is finite. There is a valuative criterion of separatedness, similar to the criterion for schemes. Recall that by Yoneda lemma (Lemma 2.22), a morphism $f : U \to \mathcal{F}$ between a scheme and a stack is equivalent to an object in $\mathcal{F}(U)$. Then we will say that α is an isomorphism between two morphisms $f_1,f_2 : U \to \mathcal{F}$ when α is an isomorphism between the corresponding objects of $\mathcal{F}(U)$.

PROPOSITION 2.39 (Valuative criterion of separatedness (stacks)) ([La], Proposition 3.19), ([DM], Theorem 4.18)

An algebraic stack $\mathcal{F}$ is separated (over S) if and only if the following holds. Let A be a valuation ring with fraction field K. Let $g_1 : \operatorname{Spec} A \to \mathcal{F}$ and $g_2 : \operatorname{Spec} A \to \mathcal{F}$ be two morphisms such that:

1. $f_{p_{\mathcal{F}}} \circ g_1 = f_{p_{\mathcal{F}}} \circ g_2$.
2. *There exists an isomorphism* $\alpha : g_1|_{\operatorname{Spec} K} \to g_2|_{\operatorname{Spec} K}$.

$$\begin{tikzcd}
 & & \mathcal{F} \arrow[d, "p_{\mathcal{F}}"] \\
\operatorname{Spec} K \arrow[r, "i"] \arrow[urr, bend left] & \operatorname{Spec} A \arrow[r] \arrow[ur, shift left, "g_1"] \arrow[ur, shift right, "g_2"'] & S
\end{tikzcd}$$

then there exists an isomorphism (in fact unique) $\tilde{\alpha} : g_1 \to g_2$ that extends α, i.e. $\tilde{\alpha}|_{\operatorname{Spec} K} = \alpha$.

Remark 2.40. It is enough to consider complete valuation rings A with algebraically closed residue field ([La], 3.20.1). If furthermore S is locally Noetherian and $\mathcal{F}$ is locally of finite type, it is enough to consider discrete valuation rings A ([La], 3.20.2).

Example 2.41. The stack BG will not be separated if G is not proper over S ([La], 3.20.3), and since we assumed G to be affine, this will not happen if it is not finite.

In general the moduli stack of vector bundles $\mathcal{M}_X$ is not separated. It is easy to find families of vector bundles that contradict the criterion.

The stack of stable curves $\overline{\mathcal{M}}_g$ is separated ([DM], Proposition 5.1).

The criterion for morphisms is more involved because we are working with stacks and we have to keep track of the isomorphisms.

PROPOSITION 2.42 (Valuative criterion of separatedness (morphisms)) ([La], Proposition 3.19)

A morphism of algebraic stacks $f : \mathcal{F} \to \mathcal{G}$ is separated if and only if the following holds. Let A be a valuation ring with fraction field K. Let $g_1 : \operatorname{Spec} A \to \mathcal{F}$ and $g_2 : \operatorname{Spec} A \to \mathcal{F}$ be two morphisms such that:

1. *There exists an isomorphism* $\beta : f \circ g_1 \to f \circ g_2$.
2. *There exists an isomorphism* $\alpha : g_1|_{\operatorname{Spec} K} \to g_2|_{\operatorname{Spec} K}$.
3. $f(\alpha) = \beta|_{\operatorname{Spec} K}$.

Then there exists an isomorphism (in fact unique) $\tilde{\alpha} : g_1 \to g_2$ that extends α, i.e. $\tilde{\alpha}|_{\operatorname{Spec} K} = \alpha$ and $f(\tilde{\alpha}) = \beta$.

Remark 2.40 is also true in this case.

DEFINITION 2.43 ([La], Definition 3.21), ([DM], Definition 4.11)

An algebraic stack $\mathcal{F}$ is proper (over S) if it is separated and of finite type, and if there is a scheme X proper over S and a (representable) surjective morphism $X \to \mathcal{F}$.

A morphism $\mathcal{F} \to \mathcal{G}$ is proper if for any affine scheme U and morphism $U \to \mathcal{G}$, the fiber product $U \times_{\mathcal{G}} \mathcal{F}$ is proper over U.

For properness we only have a satisfactory criterion for stacks (see ([La], Proposition 3.23 and Conjecture 3.25) for a generalization for morphisms).

PROPOSITION 2.44 (Valuative criterion of properness) ([La], Proposition 3.23), ([DM], Theorem 4.19)

Let $\mathcal{F}$ be a separated algebraic stack (over S). It is proper (over S) if and only if the following condition holds. Let A be a valuation ring with fraction field K. For any commutative diagram

$$\begin{tikzcd}
& & \mathcal{F} \arrow[d, "p_{\mathcal{F}}"] \\
\operatorname{Spec} K \arrow[r, "i"] \arrow[urr, "g"] & \operatorname{Spec} A \arrow[r] & S
\end{tikzcd}$$

there exists a finite field extension K' of K such that g extends to $\operatorname{Spec}(A')$, *where A' is the integral closure of A in K'.*

$$\begin{tikzcd}
& & \mathcal{F} \arrow[dd, "p_{\mathcal{F}}."] \\
\operatorname{Spec} K' \arrow[r] \arrow[urr, "g\circ u"] \arrow[d, "u"'] & \operatorname{Spec} A' \arrow[ur, dashed] \arrow[d] & \\
\operatorname{Spec} K \arrow[r, "i"] & \operatorname{Spec} A \arrow[r] & S
\end{tikzcd}$$

Example 2.45 (*Stable curves*). The Deligne-Mumford stack of stable curves $\overline{\mathcal{M}}_g$ is proper ([DM], Theorem 5.2).

2.6 *Points and dimension*

We will introduce the concept of point of an algebraic stack and dimension of a stack at a point. The reference for this is ([La], Chapter 5).

DEFINITION 2.46

Let $\mathcal{F}$ be an algebraic stack over S. The set of points of $\mathcal{F}$ is the set of equivalence classes of pairs (K,x), with K a field over S (i.e. a field with a morphism of schemes $\operatorname{Spec}K \longrightarrow S$) and $x : \operatorname{Spec}K \longrightarrow \mathcal{F}$ a morphism of stacks over S. Two pairs (K',x') and (K'',x'') are equivalent if there is a field K extension of K' and K'' and a commutative diagram

$$\begin{tikzcd}
\operatorname{Spec} K \arrow[r] \arrow[d] & \operatorname{Spec} K' \arrow[d, "x'"] \\
\operatorname{Spec} K'' \arrow[r, "x''"] & \mathcal{F}
\end{tikzcd} .$$

Given a morphism $\mathcal{F} \longrightarrow \mathcal{G}$ of algebraic stacks and a point of $\mathcal{F}$, we define the image of that point in $\mathcal{G}$ by composition.

Every point of an algebraic stack is the image of a point of an atlas. To see this, given a point represented by $\operatorname{Spec}K \longrightarrow \mathcal{F}$ and an atlas $X \longrightarrow \mathcal{F}$, take any point $\operatorname{Spec}K' \longrightarrow X \times_{\mathcal{F}} \operatorname{Spec}K$. The image of this point in X maps to the given point.

To define the concept of dimension, recall that if X and Y are locally Noetherian schemes and $f : X \longrightarrow Y$ is flat, then for any point $x \in X$ we have

$$\dim_x(X) = \dim_x(f) + \dim_{f(x)}(Y),$$

with $\dim_x(f) = \dim_x(X_{f(x)})$, where X_y is the fiber of f over y.

DEFINITION 2.47

Let $f : \mathcal{F} \to \mathcal{G}$ be a representable morphism, locally of finite type, between two algebraic spaces. Let ξ be a point of $\mathcal{F}$. Let Y be an atlas of $\mathcal{G}$. Take a point x in the algebraic space $Y \times_{\mathcal{G}} \mathcal{F}$ that maps to ξ,

$$\begin{tikzcd}
Y \times_{\mathcal{G}} \mathcal{F} \arrow[r] \arrow[d, "\tilde{f}"'] & \mathcal{F} \arrow[d, "f"] \\
Y \arrow[r] & \mathcal{G}
\end{tikzcd}$$

and define the dimension of the morphism f at the point ξ as

$$\dim_\xi(f) = \dim_x(\tilde{f}).$$

It can be shown that this definition is independent of the choices made.

DEFINITION 2.48

Let $\mathcal{F}$ be a locally Noetherian algebraic stack and ξ a point of $\mathcal{F}$. Let $u : X \to \mathcal{F}$ be an atlas, and x a point of X mapping to ξ. We define the dimension of $\mathcal{F}$ at the point ξ as

$$\dim_\xi(\mathcal{F}) = \dim_x(X) - \dim_x(u).$$

The dimension of $\mathcal{F}$ is defined as

$$\dim(\mathcal{F}) = \mathrm{Sup}_\xi(\dim_\xi(\mathcal{F})).$$

Again, this is independent of the choices made.

Example 2.49 (*Quotient by group action*). Let X be a smooth scheme of dimension $\dim(X)$ and G a smooth group of dimension $\dim(G)$ acting on X. Let $[X/G]$ be the quotient stack defined in Example 2.18. Using the atlas defined in Example 2.29, we see that

$$\dim[X/G] = \dim(X) - \dim(G).$$

Note that we have not made any assumption on the action. In particular, the action could be trivial. The dimension of an algebraic stack can then be negative. For instance, the dimension of the classifying stack BG defined in Example 2.18 has dimension $\dim(BG) = -\dim(G)$.

2.7 *Quasi-coherent sheaves on stacks*

DEFINITION 2.50 ([Vi], Definition 7.18), ([La], Definition 6.11, Proposition 6.16). A quasi-coherent sheaf $\mathcal{S}$ on an algebraic stack $\mathcal{F}$ is the following set of data:

1. For each morphism $X \to \mathcal{F}$ where X is a scheme, a quasi-coherent sheaf $\mathcal{S}_X$ on X.
2. For each commutative diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[dr] & Y \arrow[d] \\
 & \mathcal{F}
\end{tikzcd}$$

an isomorphism $\varphi_f : \mathcal{S}_X \xrightarrow{\cong} f^*\mathcal{S}_Y$, satisfying the cocycle condition, i.e. for any commutative diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[dr] & Y \arrow[r, "g"] \arrow[d] & Z \arrow[dl] \\
& \mathcal{F} &
\end{tikzcd} \tag{6}$$

we have $\varphi_{g\circ f} = \varphi_f \circ f^*\varphi_g$.

We say that $\mathcal{S}$ is coherent (resp. finite type, finite presentation, locally free) if $\mathcal{S}_X$ is coherent (resp. finite type, finite presentation, locally free) for all X.

A morphism of quasi-coherent sheaves $h : \mathcal{S} \to \mathcal{S}'$ is a collection of morphisms of sheaves $h_X : \mathcal{S}_X \to \mathcal{S}'_X$ compatible with the isomorphisms φ

Remark 2.51. Since a sheaf on a scheme can be obtained by glueing the restriction to an affine cover, it is enough to consider affine schemes.

Example 2.52 (*Structure sheaf*). Let $\mathcal{F}$ be an algebraic stack. The structure sheaf $\mathcal{O}_{\mathcal{F}}$ is defined by taking $(\mathcal{O}_{\mathcal{F}})_X = \mathcal{O}_X$.

Example 2.53 (*Sheaf of differentials*). Let $\mathcal{F}$ be a Deligne-Mumford stack. To define the sheaf of differentials $\Omega_{\mathcal{F}}$, if $U \to \mathcal{F}$ is an étale morphism we set $(\Omega_{\mathcal{F}})_U = \Omega_U$, the sheaf of differentials of the scheme U. If $V \to \mathcal{F}$ is another étale morphism and we have a commutative diagram

$$\begin{tikzcd}
U \arrow[r, "f"] \arrow[dr] & V \arrow[d] \\
& \mathcal{F}
\end{tikzcd}$$

then f has to be étale, there is a canonical isomorphism $\varphi_f : \Omega_{U/S} \to f^*\Omega_{V/S}$, and these canonical isomorphisms satisfy the cocycle condition.

Once we have defined $(\Omega_{\mathcal{F}})_U$ for étale morphisms $U \to \mathcal{F}$, we can extend the definition for any morphism $X \to \mathcal{F}$ with X an arbitrary scheme as follows: take an (étale) atlas $U = \coprod U_i \to \mathcal{F}$. Consider the composition morphism

$$X \times_{\mathcal{F}} U \xrightarrow{p_2} U \to \mathcal{F},$$

and define $(\Omega_{\mathcal{F}})_{X\times_{\mathcal{F}}U} = p_2^*\Omega_U$. The cocycle condition for Ω_{U_i} and étale descent implies that $(\Omega_{\mathcal{F}})_{X\times_{\mathcal{F}}U}$ descends to give a sheaf $(\Omega_{\mathcal{F}})_X$ on X. It is easy to check that this doesn't depend on the atlas U used, and that given a commutative diagram like (6), there are canonical isomorphisms φ satisfying the cocycle condition.

Example 2.54 (*Universal vector bundle*). Let $\mathcal{M}_X$ be the moduli stack of vector bundles on a scheme X defined in 2.9. The universal vector bundle V on $\mathcal{M}_X \times X$ is defined as follows:

Let U be a scheme and $f = (f_1, f_2) : U \to \mathcal{M}_X \times X$ a morphism. By Lemma 2.22, the morphism $f_1 : U \to \mathcal{M}_X$ is equivalent to a vector bundle W on $U \times X$. We define V_U as $\tilde{f}^* W$, where $\tilde{f} = (\mathrm{id}_U, f_2) : U \to U \times X$. Let

$$\begin{tikzcd}
U' \arrow[r, "g"] \arrow[dr, "f'"'] & U \arrow[d, "f"] \\
& \mathcal{M}_X \times X
\end{tikzcd}$$

be a commutative diagram. Recall that this means that there is an isomorphism $\alpha : f \circ g \to f'$, and looking at the projectiôn to $\mathcal{M}_X$ we have an isomorphism $\alpha_1 : f_1 \circ g \to f_1'$. Using Lemma 2.22, $f_1 \circ g$ and f_1' correspond respectively to the vector bundles $(g \times \mathrm{id}_X)^* W$ and W' on $U' \times X$, and (again by Lemma 2.22) α_1 gives an isomorphism between them. It is easy to check that these isomorphisms satisfy the cocycle condition for diagrams of the form (6).

3. Vector bundles: Moduli stack vs. moduli scheme

In this section we will compare, in the context of vector bundles, the new approach of stacks versus the standard approach of moduli schemes via geometric invariant theory (GIT) (for background on moduli schemes of vector bundles, see [HL]).

Fix a scheme X over , a positive integer r and classes $c_i \in H^{2i}(X)$. All vector bundles over X in this section will have rank r and Chern classes c_i. We will also consider vector bundles on products $B \times X$ where B is a scheme. We will always assume that these vector bundles are flat over B, and that the restriction to the slices $\{p\} \times X$ are vector bundles with rank r and Chern classes c_i. Fix also a polarization on X. All references to stability or semistability of vector bundles will mean Gieseker stability with respect to this fixed polarization.

Recall that the functor $\underline{M}^s_X$ (resp. $\underline{M}^{ss}_X$) is the functor from (Sch/S) to (Sets) that for each scheme B gives the set of *equivalence* classes of vector bundles over $B \times X$, flat over B and such that the restrictions $V|_b$ to the slices $p \times X$ are stable (resp. semistable) vector bundles with fixed rank and Chern classes, where two vector bundles V and V' on $B \times X$ are considered *equivalent* if there is a line bundle L on B such that V is isomorphic to $V' \otimes p_B^* L$.

Theorem 3.1. *There are schemes M^s_X and M^{ss}_X, called moduli schemes, corepresenting the functors $\underline{M}^s_X$ and $\underline{M}^{ss}_X$.*

The moduli scheme M^{ss}_X is constructed using the Quot schemes introduced in Example 2.28 (for a detailed exposition of the construction, see [HL]). Since the set of *semistable* vector bundles is bounded, we can choose once and for all N and m (depending only on the Chern classes and rank) with the property that for any semistable vector bundle V there is a point in $R = R_{N,m}$ whose corresponding quotient is isomorphic to V.

The scheme R parametrizes vector bundles V on X together with a basis of $H^0(V(m))$ (up to multiplication by scalar). Recall that $N = h^0(V(m))$. There is an action of $GL(N)$ on R, corresponding to change of basis but since two basis that only differ by a scalar give the same point on R, this $GL(N)$ action factors through $PGL(N)$. Then the moduli scheme M^{ss}_X is defined as the GIT quotient $R//PGL(N)$.

The closed points of M^{ss}_X correspond to S-equivalence classes of vector bundles, so if there is a strictly semistable vector bundle, the functor $\underline{M}^{ss}_X$ is not representable.

Now we will compare this scheme with the moduli stack $\mathcal{M}_X$ defined on Example 2.9. We will also consider the moduli stack $\mathcal{M}^s_X$ defined in the same way, but with the extra requirement that the vector bundles should be stable. The moduli stack $\mathcal{M}^s_X$ is a substack (Definition 2.35) of $\mathcal{M}_X$. The following are some of the differences between the moduli scheme and the moduli stack:

1. The stack $\mathcal{M}_X$ parametrizes all vector bundles, but the scheme M^{ss}_X only parametrizes semistable vector bundles.

2. From the point of view of the scheme M^{ss}_X, we identify two vector bundles on X (i.e. they give the same closed point on M^{ss}_X) if they are S-equivalent. On the other hand, from the point of view of the moduli stack, two vector bundles are identified (i.e. give isomorphic objects on $\mathcal{M}_X(\operatorname{Spec} k)$) only if they are isomorphic as vector bundles.
3. Let V and V' be two families of vector bundles parametrized by a scheme B, i.e. two vector bundles (flat over B) on $B \times X$. If there is a line bundle L on B such that V is isomorphic to $V' \otimes p_B^* L$, then from the point of view of the moduli scheme, V and V' are identified as being the same family. On the other hand, from the point of view of the moduli stack, V and V' are identified only if they are isomorphic as vector bundles on $B \times X$.
4. The subscheme M^s_X corresponding to stable vector bundles is sometimes representable by a scheme, but the moduli stack $\mathcal{M}^s_X$ is never representable by a scheme. To see this, note that any vector bundle has automorphisms different from the identity (multiplication by scalars) and apply Lemma 2.21.

Now we will restrict our attention to stable bundles, i.e. to the scheme M^s_X and the stack $\mathcal{M}^s_X$. For stable bundles the notions of S-equivalence and isomorphism coincide, so the points of M^s_X correspond to isomorphism classes of vector bundles. Consider $R^s \subset R$, the subscheme corresponding to stable bundles. There is a map $\pi : R^s \longrightarrow M^s_X = R^s/PGL(N)$, and π is in fact a principal $PGL(N)$-bundle (this is a consequence of Luna's étale slice theorem).

Remark 3.2 (*Universal bundle on moduli scheme*). The scheme M^s_X represents the functor $\underline{M}^s_X$ if there is a universal family. Recall that a universal family for this functor is a vector bundle E on $M^s_X \times X$ such that the isomorphism class of $E|_{p\times X}$ is the isomorphism class corresponding to the point $p \in M^s_X$, and for any family of vector bundles V on $B \times X$ there is a morphism $f : B \longrightarrow M^s_X$ and a line bundle L on B such that $V \otimes p_B^* L$ is isomorphic to $(f \times \mathrm{id}_X)^* E$. Note that if E is a universal family, then $E \otimes p_{M^s_X}^* L$ will also be a universal family for any line bundle L on M^s_X.

The universal bundle for the Quot scheme gives a universal family $\tilde{V}$ on $R^s \times X$, but this family does not always descend to give a universal family on the quotient M^s_X.

Let $X \xrightarrow{G} Y$ be a principal G-bundle. A vector bundle V on X descends to Y if the action of G on X can be lifted to V. In our case, if certain numerical criterion involving r and c_i is satisfied (if X is a smooth curve this criterion is $\gcd(r, c_1) = 1$), then we can find a line bundle L on R^s such that the $PGL(N)$ action on R^s can be lifted to $\tilde{V} \otimes p_{R^s}^* L$, and then this vector bundle descends to give a universal family on $M^s_X \times X$. But in general the best that we can get is a universal family on an étale cover of M^s_X.

Recall from Example 2.29 that there is a morphism $[R^{ss}/PGL(N)] \longrightarrow M^{ss}_X$, and that the morphism $[R^s/PGL(N)] \longrightarrow M^s_X$ is an isomorphism of stacks.

PROPOSITION 3.3

There is a commutative diagram of stacks

$$\begin{tikzcd}
{[R^s/GL(N)]} \arrow[r, "q"] \arrow[d, "g"', "\simeq"] & {[R^s/PGL(N)]} \arrow[d, "h", "\simeq"'] \\
\mathcal{M}^s_X \arrow[r, "\varphi"'] & M^s_X,
\end{tikzcd}$$

where g and h are isomorphisms of stacks, but q and φ are not. If we change 'stable' by 'semistable' we still have a commutative diagram, but the corresponding morphism h^{ss} is not an isomorphism of stacks.

Proof. The morphism φ is the composition of the natural morphism $\mathcal{M}^s_X \to \underline{M}^s_X$ (sending each category to the set of isomorphism classes of objects) and the morphism $\underline{M}^s_X \to M^s_X$ given by the fact that the scheme $M^s_X = R^s//PGL(N)$ corepresents the functor $\underline{M}^s_X$.

The morphism h was constructed in Example 2.18.

The key ingredient needed to define g is the fact that the $GL(N)$ action on the Quot scheme lifts to the universal bundle, i.e. the universal bundle on the Quot scheme has a $GL(N)$-linearization. Let

$$\begin{tikzcd}
\widetilde{B} \arrow[r, "f"] \arrow[d] & R^{ss} \\
B &
\end{tikzcd}$$

be an object of $[R^{ss}/GL(N)]$. Since R^{ss} is a subscheme of a Quot scheme, by restriction we have a universal bundle on $R^{ss} \times X$, and this universal bundle has a $GL(N)$-linearization. Let $\tilde{E}$ be the vector bundle on $\tilde{B} \times X$ defined by the pullback of this universal bundle. Since f is $GL(N)$-equivariant, $\tilde{E}$ is also $GL(N)$-linearized. Since $\tilde{B} \times X \to B \times X$ is a principal bundle, the vector bundle $\tilde{E}$ descends to give a vector bundle E on $B \times X$, i.e. an object of $\mathcal{M}^{ss}_X$. Let

$$\begin{tikzcd}
\widetilde{B} \arrow[r, "\phi"'] \arrow[d] \arrow[rr, bend left, "f"] & \widetilde{B}' \arrow[r, "f'"'] \arrow[d] & R^{ss} \\
B \arrow[r, equal] & B &
\end{tikzcd}$$

be a morphism in $[R^{ss}/GL(N)]$. Consider the vector bundles $\tilde{E}$ and $\tilde{E}'$ defined as before. Since $f' \circ \phi = f$, we get an isomorphism of $\tilde{E}$ with $(\phi \times \mathrm{id})^* \tilde{E}'$. Furthermore this isomorphism is $GL(N)$-equivariant, and then it descends to give an isomorphism of the vector bundles E and E' on $B \times X$, and we get a morphism in $\mathcal{M}^{ss}_X$.

To prove that this gives an equivalence of categories, we construct a functor $\overline{g}$ from $\mathcal{M}^{ss}_X$ to $[R^{ss}/GL(N)]$. Given a vector bundle E on $B \times X$, let $q : \tilde{B} \to B$ be the $GL(N)$-principal bundle associated with the vector bundle $p_{B*}E$ on B. Let $\tilde{E} = (q \times \mathrm{id})^* E$ be the pullback of E to $\tilde{B} \times X$. It has a canonical $GL(N)$-linearization because it is defined as a pullback by a principal $GL(N)$-bundle. The vector bundle $p_{\tilde{B}*}\tilde{E}$ is canonically isomorphic to the trivial bundle $\mathcal{O}^N_{\tilde{B}}$, and this isomorphism is $GL(N)$-equivariant, so we get an *equivariant* morphism $\tilde{B} \to R^{ss}$, and hence an object of $[R^{ss}/GL(N)]$.

If we have an isomorphism between two vector bundles E and E' on $B \times X$, it is easy to check that it induces an isomorphism between the associated objects of $[R^{ss}/GL(N)]$.

It is easy to check that there are natural isomorphisms of functors $g \circ \tilde{g} \cong \mathrm{id}$ and $\tilde{g} \circ g \cong \mathrm{id}$, and then g is an equivalence of categories.

The morphism q is defined using the following lemma, with $G = GL(N)$, H the subgroup consisting of scalar multiples of the identity, $\overline{G} = PGL(N)$ and Y=R^{ss}. ☐

Lemma 3.4. *Let Y be an S-scheme and G an affine flat group S-scheme, acting on Y on the right. Let H be a normal closed subgroup of G. Assume that $\overline{G} = G/H$ is affine. If H* *acts trivially on Y, then there is a morphism of stacks*

$$[Y/G] \longrightarrow [Y/\overline{G}].$$

If H is nontrivial, then this morphism is not faithful, so it is not an isomorphism.

Proof. Let

$$\begin{tikzcd} E \arrow[r, "f"] \arrow[d, "\pi"] & Y \\ B & \end{tikzcd}$$

be an object of $[Y/G]$. There is a scheme E/H such that π factors

$$E \xrightarrow{q} E/H \xrightarrow{\pi'} B.$$

To construct E/H, note that there is a local étale cover U_i of B and isomorphisms $\phi_i : \pi^{-1}(U_i) \longrightarrow U_i \times G$, with transition functions $\psi_{ij} = \phi_i \circ \phi_j^{-1}$. Since these isomorphisms are G-equivariant, they descend to give isomorphisms $\overline{\psi}_{ij} : U_j \times G/H \longrightarrow U_i \times G/H$, and using these transition functions we get E/H. This construction shows that π' is a principal $\overline{G}$-bundle. Furthermore, q is also a principal H-bundle ([HL], Example 4.2.4), and in particular it is a categorical quotient.

Since f is H-invariant, there is a morphism $\overline{f} : E/H \longrightarrow Y$, and this gives an object of $[Y/\overline{G}]$.

If we have a morphism in $[Y/G]$, given by a morphism $g : E \longrightarrow E'$ of principal G-bundles over B, it is easy to see that it descends (since g is equivariant) to a morphism $\overline{g} : E/H \longrightarrow E'/H$, giving a morphism in $[Y/\overline{G}]$.

This morphism is not faithful, since the automorphism $E \xrightarrow{\cdot z} E$ given by multiplication on the right by a nontrivial element $z \in H$ is sent to the identity automorphism $E/H \longrightarrow E/H$, and then $\mathrm{Hom}(E, E) \longrightarrow \mathrm{Hom}(E/H, E/H)$ is not injective. □

If X is a smooth curve, then it can be shown that $\mathcal{M}_X$ is a smooth stack of dimension $r^2(g-1)$, where r is the rank and g is the genus of X. In particular, the open substack $\mathcal{M}_X^{ss}$ is also smooth of dimension $r^2(g-1)$, but the moduli scheme M_X^{ss} is of dimension $r^2(g-1)+1$ and might not be smooth. Proposition 3.3 explains the difference in the dimensions (at least on the smooth part): we obtain the moduli stack by taking the quotient by the group $GL(N)$, of dimension N^2, but the moduli scheme is obtained by a quotient by the group $PGL(N)$, of dimension N^2-1. The moduli scheme M_X^{ss} is not smooth in general because in the strictly semistable part of R^{ss} the action of $PGL(N)$ is not free. On the other hand, the smoothness of a stack quotient doesn't depend on the freeness of the action of the group.

Appendix A: Grothendieck topologies, sheaves and algebraic spaces

The standard reference for Grothendieck topologies is SGA (*Séminaire de Géométrie Algébrique*). For an introduction see [T] or [MM]. For algebraic spaces, see [K] or [Ar1].

An open cover in a topological space U can be seen as family of morphisms in the category of topological spaces $f_i : U_i \longrightarrow U$, with the property that f_i is an open inclusion and the union of their images is U, i.e we are choosing a class of morphisms (open inclusions) in the category of topological spaces. A Grothendieck topology on an arbitrary category is basically a choice of a class of morphisms, that play the role of 'open sets'. A morphism $f : V \to U$ in this class is to be thought of as an 'open set' in the object U. The concept of intersection of open sets is replaced by the fiber product: the 'intersection' of $f_1 : U_1 \to U$ and $f_2 : U_2 \to U$ is $f_{12} : U_1 \times_U U_2 \to U$.

A category with a Grothendieck topology is called a site. We will consider two topologies on (Sch/S).

fppf topology. Let U be a scheme. Then a cover of U is a finite collection of morphisms $\{f_i : U_i \to U\}_{i\in I}$ such that each f_i is a finitely presented flat morphism (for Noetherian schemes, this is equivalent to flat and finite type), and U is the (set theoretic) union of the images of f_i. In other words, $\coprod U_i \to U$ is *'fidèlement plat de présentation finie'*.

Étale topology. Same definition, but substituting flat by étale.

DEFINITION 4.1 (Presheaf of sets)

A presheaf of sets on (Sch/S) is a contravariant functor F from (Sch/S) to (Sets).

As usual, we will use the following notation: if $X \in F(U)$ and $f_i : U_i \to U$ is a morphism, then $X|_i$ is the element of $F(U_i)$ given by $F(f_i)(X)$, and we will call $X|_i$ the 'restriction of X to U_i', even if f_i is not an inclusion. If $X_i \in F(U_i)$, then $X_i|_{ij}$ is the element of $F(U_{ij})$ given by $F(f_{ij,i})(X_i)$ where $f_{ij,i} : U_i \times_U U_j \to U_i$ is the pullback of f_j.

DEFINITION 4.2 (Sheaf of sets)

Choose a topology on (Sch/S). We say that F is a sheaf (or an S-space) with respect to that topology if for every cover $\{f_i : U_i \to U\}_{i\in I}$ in the topology the following two axioms are satisfied:

1. *Mono.* Let X and Y be two elements of $F(U)$. If $X|_i = Y|_i$ for all i, then $X = Y$.
2. *Glueing.* Let X_i be an object of $F(U_i)$ for each i such that $X_i|_{ij} = X_j|_{ij}$, then there exists $X \in F(U)$ such that $X|_i = X_i$ for each i.

We define morphisms of S-spaces as morphisms of sheaves (i.e. natural transformations of functors). Note that a scheme M can be viewed as an S-space via its functor of points $\mathrm{Hom}_S(-, M)$, and a morphism between two such S-spaces is equivalent to a scheme morphism between the schemes (by the Yoneda embedding lemma), then the category of S-schemes is a full subcategory of the category of S-spaces.

Equivalence relation and quotient space. An equivalence relation in the category of S-spaces consists of two S-spaces R and U and a monomorphism of S-spaces

$$\delta : R \to U \times_S U$$

such that for all S-scheme B, the map $\delta(B) : R(B) \to U(B) \times U(B)$ is the graph of an equivalence relation between sets. A quotient S-space for such an equivalence relation is by definition the sheaf cokernel of the diagram

$$\begin{tikzcd} R \arrow[r, shift left, "p_2\circ\delta"] \arrow[r, shift right, "p_1\circ\delta"'] & U. \end{tikzcd}$$

DEFINITION 4.3 (Algebraic space) ([La], 0).

An S-space F is called an algebraic space if it is the quotient S-space for an equivalence relation such that R and U are S-schemes, $p_1 \circ \delta$, $p_2 \circ \delta$ are étale (morphisms of S-schemes), and δ is a quasi-compact morphism (of S-schemes).

Roughly speaking, an algebraic space is a quotient of a scheme by an étale equivalence relation. The following is an equivalent definition.

DEFINITION 4.4 ([K], Definition 1.1)

An S-space F is called an algebraic space if there exists a scheme U (atlas) and a morphism of S-spaces $u : U \longrightarrow F$ such that

1. *The morphism u is étale.* For any S-scheme V and morphism $V \longrightarrow F$, the (sheaf) fiber product $U \times_F V$ is representable by a scheme, and the map $U \times_F V \longrightarrow V$ is an étale morphism of schemes.
2. *Quasi-separatedness.* The morphism $U \times_F U \longrightarrow U \times_S U$ is quasi-compact.

We recover the first definition by taking $R = U \times_F U$. Then roughly speaking, we can also think of an algebraic space as 'something' that looks locally in the étale topology like an affine scheme, in the same sense that a scheme is something that looks locally in the Zariski topology like an affine scheme.

Algebraic spaces are used, for instance, to give algebraic structure to certain complex manifolds (for instance Moishezon manifolds) that are not schemes, but can be realized as algebraic spaces. All smooth algebraic spaces of dimension 1 and 2 are actually schemes. An example of a smooth algebraic space of dimension 3 that is not a scheme can be found in [H].

But étale topology is useful even if we are only interested in schemes. The idea is that the étale topology is finer than the Zariski topology, and in many situations it is 'fine enough' to do the analog of the manipulations that can be done with the analytic topology of complex manifolds. As an example, consider the affine complex line $\mathrm{Spec}(\mathbb{C}[x])$, and take a (closed) point x_0 different from 0. Assume that we want to define the function $\sqrt{x}$ in a neighborhood of x_0. In the analytic topology we only need to take a neighborhood small enough so that it does not contain a loop that goes around the origin, then we choose one of the branches (a sign) of the square root. In the Zariski topology this cannot be done, because all open sets are too large (have loops going around the origin, so the sign of the square root will change, and $\sqrt{x}$ will be multivaluated). But take the 2:1 étale map $V = \mathrm{Spec}(\mathbb{C}[y,x,x^{-1}]/(y-x^2)) \longrightarrow \mathrm{Spec}(\mathbb{C}[x])$. The function $\sqrt{x}$ can certainly be defined on V, it is just equal to the function y, so it is in this sense that we say that the étale topology is finer: V is a 'small enough open subset' because the square root can be defined on it.

Appendix B: 2-categories

In this section we recall the notions of 2-category and 2-functor. A 2-category C consists of the following data [Hak]:

(i) A class of objects ob C.
(ii) For each pair X, $Y \in \mathrm{ob}\, C$, a category $\mathrm{Hom}(X, Y)$.

(iii) *Horizontal composition of 1-morphisms and 2-morphisms.* For each triple X, Y, $Z \in \mathrm{ob}C$, a functor

$$\mu_{X,Y,Z} : \mathrm{Hom}(X,Y) \times \mathrm{Hom}(Y,Z) \longrightarrow \mathrm{Hom}(X,Z)$$

with the following conditions

(i') *Identity 1-morphism.* For each object $X \in \mathrm{ob}C$, there exists an object $\mathrm{id}_X \in \mathrm{Hom}(X,X)$ such that

$$\mu_{X,X,Y}(\mathrm{id}_X,) = \mu_{X,Y,Y}(, \mathrm{id}_Y) = \mathrm{id}_{\mathrm{Hom}(X,Y)},$$

where $\mathrm{id}_{\mathrm{Hom}(X,Y)}$ is the identity functor on the category $\mathrm{Hom}(X,Y)$.

(ii') *Associativity of horizontal compositions.* For each quadruple X, Y, Z, $T \in \mathrm{ob}C$,

$$\mu_{X,Z,T} \circ (\mu_{X,Y,Z} \times \mathrm{id}_{\mathrm{Hom}(Z,T)}) = \mu_{X,Y,T} \circ (\mathrm{id}_{\mathrm{Hom}(X,Y)} \times \mu_{Y,Z,T}).$$

The example to keep in mind is the 2-category *Cat* of categories. The objects of *Cat* are categories, and for each pair X, Y of categories, $\mathrm{Hom}(X,Y)$ is the category of functors between X and Y.

Note that the main difference between a 1-category (a usual category) and a 2-category is that $\mathrm{Hom}(X,Y)$, instead of being a set, is a category.

Given a 2-category, an object f of the category $\mathrm{Hom}(X,Y)$ is called a 1-morphism of C, and is represented with a diagram

$$\begin{tikzcd} \overset{X}{\bullet} \arrow[r, "f"] & \overset{Y}{\bullet} \end{tikzcd}$$

and a morphism α of the category $\mathrm{Hom}(X,Y)$ is called a 2-morphisms of C, and is represented as

$$\begin{tikzcd} \overset{X}{\bullet} \arrow[r, bend left=50, "f"{name=U}] \arrow[r, bend right=50, "f'"'{name=D}] & \overset{Y}{\bullet} \arrow[Rightarrow, from=U, to=D, "\alpha"] \end{tikzcd}$$

Now we will rewrite the axioms of a 2-category using diagrams.

1. *Composition of 1-morphisms.* Given a diagram

$$\begin{tikzcd} \overset{X}{\bullet} \arrow[r, "f"] & \overset{Y}{\bullet} \arrow[r, "g"] & \overset{Z}{\bullet} & \text{there exist} & \overset{X}{\bullet} \arrow[r, "g\circ f"] & \overset{Z}{\bullet} \end{tikzcd}$$

(this is (iii) applied to objects) and this composition is associative: $(h \circ g) \circ f = h \circ (g \circ f)$ (this is (ii') applied to objects).

2. *Identity for* 1*-morphisms.* For each object X there is a 1-morphism id_X such that $f \circ \mathrm{id}_Y = \mathrm{id}_X \circ f = f$ (this is (i')).

3. *Vertical composition of* 2*-morphisms.* Given a diagram

$$\begin{tikzcd} \overset{X}{\bullet} \arrow[r, bend left=60, "f"{name=U}] \arrow[r, "g"{name=M, description}] \arrow[r, bend right=60, "h"'{name=D}] & \overset{Y}{\bullet} \arrow[Rightarrow, from=U, to=M, "\alpha"] \arrow[Rightarrow, from=M, to=D, "\beta"] & \text{there exists} & \overset{X}{\bullet} \arrow[r, bend left=50, "f"{name=U2}] \arrow[r, bend right=50, "h"'{name=D2}] & \overset{Y}{\bullet} \arrow[Rightarrow, from=U2, to=D2, "\beta\circ\alpha"] \end{tikzcd}$$

and this composition is associative $(\gamma \circ \beta) \circ \alpha = \gamma \circ (\beta \circ \alpha)$.

4. *Horizontal composition of* 2-*morphisms.* Given a diagram

$$\begin{tikzcd}
X \arrow[r, bend left=50, "f"{name=U1}] \arrow[r, bend right=50, "f'"'{name=D1}] \arrow[Rightarrow, from=U1, to=D1, "\alpha"] & Y \arrow[r, bend left=50, "g"{name=U2}] \arrow[r, bend right=50, "g'"'{name=D2}] \arrow[Rightarrow, from=U2, to=D2, "\beta"] & Z
\end{tikzcd} \tag{7}$$

there exists

$$\begin{tikzcd}
X \arrow[r, bend left=50, "g\circ f"{name=U}] \arrow[r, bend right=50, "g'\circ f'"'{name=D}] \arrow[Rightarrow, from=U, to=D, "\beta*\alpha"] & Z
\end{tikzcd}$$

(this is (iii) applied to morphisms) and it is associative $(\gamma * \beta) * \alpha = \gamma * (\beta * \alpha)$ (this is (ii') applied to morphisms).

5. *Identity for* 2-*morphisms.* For every 1-morphism f there is a 2-morphism id_f such that $\alpha \circ \mathrm{id}_g = \mathrm{id}_f \circ \alpha = \alpha$ (this and item are (ii)). We have $\mathrm{id}_g * \mathrm{id}_f = \mathrm{id}_{g\circ f}$ (this means that $\mu_{X,Y,Z}$ respects the identity).

6. *Compatibility between horizontal and vertical composition of* 2-*morphisms.* Given a diagram

$$\begin{tikzcd}
X \arrow[r, bend left=60, "f"{name=U1}] \arrow[r, "f'"{name=M1, description}] \arrow[r, bend right=60, "f''"'{name=D1}] \arrow[Rightarrow, from=U1, to=M1, "\alpha"] \arrow[Rightarrow, from=M1, to=D1, "\alpha'"] & Y \arrow[r, bend left=60, "g"{name=U2}] \arrow[r, "g'"{name=M2, description}] \arrow[r, bend right=60, "g''"'{name=D2}] \arrow[Rightarrow, from=U2, to=M2, "\beta"] \arrow[Rightarrow, from=M2, to=D2, "\beta'"] & Z
\end{tikzcd} ,$$

then $(\beta' \circ \beta) * (\alpha' \circ \alpha) = (\beta' * \alpha') \circ (\beta * \alpha)$ (this is (iii) applied to morphisms).

Two objects X and Y of a 2-category are called equivalent if there exist two 1-morphisms $f : X \longrightarrow Y$, $g : Y \longrightarrow X$ and two 2-isomorphisms (invertible 2-morphism) $\alpha : g \circ f \longrightarrow \mathrm{id}_X$ and $\beta : f \circ g \longrightarrow \mathrm{id}_Y$.

A commutative diagram of 1-morphisms in a 2-category is a diagram

$$\begin{tikzcd}
 & Y \arrow[dr, "g"] \arrow[d, Rightarrow, "\alpha"] & \\
X \arrow[ur, "f"] \arrow[rr, "h"'] & {} & Z
\end{tikzcd}$$

such that $\alpha : g \circ f \longrightarrow h$ is a 2-isomorphisms.

Remark 5.1 Note that we do not require $g \circ f = h$ to say that the diagram is commutative, but just require that there is a 2-isomorphisms between them. This is the reason why 2-categories are used to describe stacks.

On the other hand, a diagram of 2-morphisms will be called commutative only if the compositions are actually equal. Now we will define the concept of covariant 2-functor (a contravariant 2-functor is defined in a similar way).

A covariant 2-functor F between two 2-categories C and C' is a law that for each object X in C gives an object $F(X)$ in C'. For each 1-morphism $f : X \longrightarrow Y$ in C gives a 1-morphism $F(f) : F(X) \longrightarrow F(Y)$ in C', and for each 2-morphism $\alpha : f \Rightarrow g$ in C gives a

2-morphism $F(\alpha) : F(f) \Rightarrow F(g)$ in C', such that

1. *Respects identity 1-morphism.* $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$.
2. *Respects identity 2-morphism.* $F(\mathrm{id}_f) = \mathrm{id}_{F(f)}$.
3. *Respects composition of 1-morphism up to a 2-isomorphism.* For every diagram

$$\begin{tikzcd} \overset{X}{\bullet} \arrow[r, "f"] & \overset{Y}{\bullet} \arrow[r, "g"] & \overset{Z}{\bullet} \end{tikzcd}$$

there exists a 2-isomorphism $\epsilon_{g,f} : F(g) \circ F(f) \to F(g \circ f)$

$$\begin{tikzcd} & \overset{F(Y)}{\bullet} \arrow[dr, "F(g)"] \arrow[d, Rightarrow, "\epsilon_{g,f}"] & \\ F(X)\bullet \arrow[ur, "F(f)"] \arrow[rr, "F(g\circ f)"'] & {} & \bullet F(Z) \end{tikzcd}$$

   (a) $\epsilon_{f,\mathrm{id}_X} = \epsilon_{\mathrm{id}_Y,f} = \mathrm{id}_{F(f)}$.
   (b) *ϵ is associative.* The following diagram is commutative

$$\begin{tikzcd} F(h) \circ F(g) \circ F(f) \arrow[r, Rightarrow, "\epsilon_{h,g} \times \mathrm{id}"] \arrow[d, Rightarrow, "\mathrm{id} \times \epsilon_{g,f}"'] & F(h \circ g) \circ F(f) \arrow[d, Rightarrow, "\epsilon_{h\circ g,f}"] \\ F(h) \circ F(g \circ f) \arrow[r, Rightarrow, "\epsilon_{h,g\circ f}"] & F(h \circ g \circ f) \end{tikzcd} .$$

4. *Respects vertical composition of 2-morphisms.* For every pair of 2-morphisms $\alpha : f \to g$, $\beta : g \to h$, we have $F(\beta \circ \alpha) = F(\beta) \circ F(\alpha)$.
5. *Respects horizontal composition of 2-morphisms.* For every pair of 2-morphisms $\alpha : f \to f'$, $\beta : g \to g'$ as in (7) the following diagram commutes

$$\begin{tikzcd} F(g) \circ F(f) \arrow[r, Rightarrow, "F(\beta) * F(\alpha)"] \arrow[d, Rightarrow, "\epsilon_{g,f}"'] & F(g') \circ F(f') \arrow[d, Rightarrow, "\epsilon_{g',f'}"] \\ F(g \circ f) \arrow[r, Rightarrow, "F(\beta * \alpha)"] & F(g' \circ f') \end{tikzcd} .$$

By a slight abuse of language, condition 5 is usually written as $F(\beta) * F(\alpha) = F(\beta * \alpha)$. Note that strictly speaking this equality doesn't make sense, because the sources (and the targets) do not coincide, but if we chose once and for all the 2-isomorphisms ϵ of condition 3, then there is a unique way of making sense of this equality.

Remark 5.2. Since 2-functors only respect composition of 1-functors up to a 2-isomorphism (condition 3), sometimes they are called pseudofunctors or lax functors.

Remark 5.3. In the applications to stacks, the isomorphism $\epsilon_{g,f}$ of item 3 is canonically defined, and by abuse of language we will say that $F(g) \circ F(f) = F(g \circ f)$, instead of saying that they are isomorphic.

Given a 1-category C (a usual category), we can define a 2-category: we just have to make the set $\mathrm{Hom}(X, Y)$ into a category, and we do this just by defining the unit morphisms for each element.

On the other hand, given a 2-category C there are two ways of defining a 1-category. We have to make each category $\mathrm{Hom}(X, Y)$ into a set. The naive way is just to take the set of objects of $\mathrm{Hom}(X, Y)$, and then we obtain what is called the underlying category of C

(see [Hak]). This has the problem that a 2-functor $F : C \to C'$ is not in general a functor of the underlying categories (because in item 3 we only require the composition of 1-morphisms to be respected up to 2-isomorphism).

The best way of constructing a 1-category from a 2-category is to define the set of morphisms between the objects X and Y as the set of isomorphism classes of objects of $\mathrm{Hom}(X, Y)$: two objects f and g of $\mathrm{Hom}(X, Y)$ are isomorphic if there exists a 2-isomorphism $\alpha : f \Rightarrow g$ between them. We call the category obtained in this way the 1-category associated to C. Note that a 2-functor between 2-categories then becomes a functor between the associated 1-categories.

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Variational formulae for Fuchsian groups over families of algebraic curves

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Abstract. We study the problem of understanding the uniformizing Fuchsian groups for a family of plane algebraic curves by determining explicit first variational formulae for the generators.

Keywords. Riemann surfaces; Fuchsian groups; Ahlfors-Bers variational formulae.

1. Introduction

In this paper we make a contribution to the problem of understanding the uniformizing Fuchsian groups for a family of plane algebraic curves by determining explicit first variational formulae for the generators of the Fuchsian groups, say G_t, associated to a t-parameter family of compact Riemann surfaces X_t, where the X_t are the Riemann surfaces for the complex algebraic curves arising from a t-parameter family of irreducible polynomials. The main idea of our work is to utilize explicit quasiconformal mappings between algebraic curves, calculate the Beltrami coefficients, and hence utilize the Ahlfors-Bers variational formulae when applied to quasiconformal conjugates of Fuchsian groups.

We start with a compact Riemann surface X_0, corresponding to the plane algebraic curve $P(x,y) = \sum\sum a_{ij}x^iy^j = 0$, having genus say $g > 1$. Let us assume also that $X_0 = U/G_0$ where G_0 (i.e. the holomorphic deck-transformation group) is known. Then we consider the parametrized family of compact Riemann surfaces X_t corresponding to the polynomial equation $P_t(x,y) = 0$ where $P_t(x,y) = \sum\sum a_{ij}(t)x^iy^j$ such that $a_{ij}(t)$ are holomorphic functions of t (t in a small disk around the origin) with additional restriction that $a_{ij}(0) = a_{ij}$. For such X_t we determine first variational formula for $\gamma_t \in G_t$ where $X_t \equiv U/G_t$ (G_t is the uniformizing Fuchsian group corresponding to X_t)

$$\gamma_t = \gamma + t\dot{\gamma} + \bar{t}\dot{\gamma}^* + o(t), \tag{1}$$

where γ is an element of G_0 (and $\dot{\gamma}$, $\dot{\gamma}^*$ are as in eq. (16)).

Remark. Although we have dealt with compact Riemann surfaces and the torsion-free parabolic-free Fuchsian uniformizing group in the introduction above, the theory of Teichmüller spaces works exactly the same for Riemann surfaces of finite conformal type

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– namely we can allow distinguished points or punctures on the compact Riemann surfaces and correspondingly allow elliptic or parabolic elements in the Fuchsian groups under scrutiny. Those results are exactly parallel and nothing new needs to be said.

2. Invariance of sheet monodromy over families of curves

Monodromy Invariance Lemma. To solve our problems we have to find a correspondence between the ramification (branch) points of $P_t(x,y)=0$ lying on the x-sphere for different values of t. Also we will need to make a correspondence between the algebraic functions $y_t(x)=y(x,t)$ satisfying $P_t(x,y(x,t))=0$ for different values of t, so that the monodromy remains invariant at the corresponding branch points. That will guarantee that the topological structure of the branched covering is kept invariant as t changes.

In order to do this we assume certain restrictions on $P_t(x,y)$:

Assume $\deg P(x,y,t)=D$ for all t. Assume also that there exists r, s such that $r+s=D$ where $0\le r\le m$, $0\le s\le N$ and $a_{rs}(0)\ne 0$ i.e degree $P_0(x,y)=D$.

Assume

(1) $P_0(x,y)$ is irreducible in the polynomial ring $\mathbb{C}[x,y]$.

(2) If degree $P_t(x,y)=D$, then degree $P_0(x,y)=D$; that is if we substitute $t=0$ in $P_t(x,y)$ degree of the polynomial remains the same.

(3) Suppose P_t is of degree N in the y variable for all small t:

$$P_t(x,y)=P_N(x,t)y^N+P_{N-1}(x,t)y^{N-1}+\cdots+P_0(x,t),$$

where

$$P_N(x,t)=a_k(t)x^k+\cdots+a_0(t).$$

Let $D(t)$ denote the discriminant of $P_N(x,t)$. Then assume that $D(0)\ne 0$ and $a_k(0)\ne 0$.

(4) Let $D(x,t)$ be the discriminant of $P_t(x,y)=0$. Then $D(x,t)=P_N(x,t)Q(x,t)$ where

$$Q(x,t)=Q_0(t)x^r+\ldots+Q_r(t).$$

We assume that $Q_0(0)\ne 0$ and $\tilde{D}(0)\ne 0$, where $\tilde{D}(t)=$ discriminant of $Q(x,t)$.

(5) The resultant of $Q(x,t)$ and $P_N(x,t)$ does not vanish at $t=0$.

Assume

$$P(x,y,0)=P_0(x,y)$$

is an irreducible polynomial such that $x=0$ and $x=\infty$ are ordinary points, and the set of ramification points on the x-plane are say located at:

$$\{\zeta_1^0,\ldots,\zeta_k^0\}.$$

Then it is not hard to demonstrate that:

(i) For all t sufficiently close to 0, the polynomial $P_t(x,y)$ is irreducible and 0, ∞ are ordinary points.

(ii) The ramification points on the x-sphere for $P_t(x,y)$ are holomorphically dependent on t and are given by k holomorphic functions: $\{\zeta_1(t),\ldots,\zeta_k(t)\}$ such that $\zeta_j(0)=\zeta_j^0$ for $0\le j\le k$ and $\zeta_i(t)\ne\zeta_j(t)$ for $i\ne j$ and all t small enough.

(iii) Assume N is the degree of P_t in the y variable (this follows from the stability conditions mentioned above.) Then there exists holomorphic function germs $\{y_1(x,t),\dots,y_N(x,t)\}$ around $(x,t)=(0,0)\in \mathbf{C}^2$ such that

$$P_t(x,y_j(x,t))=0$$

for all (x,t) sufficiently close to (0,0) and such that N roots of the y equation $P(x,y,t)=0$ are given by $y_j(x,t)$.

(iv) Analytic continuation of $y_1(x,t)$ for every fixed t, $\mid t\mid\leq\epsilon$ in the x-sphere along the same route (avoiding the branch points) produces the *same permutation* of $\{y_1(x,t),\dots,y_N(x,t)\}$ – i.e., the monodromy permutations are independent of t.

Idea of the proof for (iv): Follow the construction, as in Siegel [S], for each $\zeta_i(0)$ we consider a circle C_i with center at $\zeta_i(0)$ such that any two of them does not intersect and we join the origin to $\zeta_i(0)$ by a simple curve l_i so that if we cut $\mathbf{CP}^1$ along these curves it remains simply connected. Since ζ_i's are holomorphic function of t we can find a neighborhood of $t=0$ say, $N=\{t:\mid t\mid<\epsilon\}$ such that $\zeta_1(N),\dots,\zeta_k(N)$ lies inside $C_1,\dots,C_k$ respectively and each $\zeta_i(N)$ is an open connected subset lying in the interior of $C_i\, 1\leq i\leq n$. Now for each point x_0 on C_i, $1\leq i\leq n$ we can find mutually disjoint neighborhood $W_1(x_0),\dots,W_N(x_0)$ of $\phi_i(x_0,0)$, $1\leq i\leq N$ (where $P(x_0,\phi_i(x_0,0),0)=0$ and $\phi_i(x,0)$ is an analytic function of $x\, 1\leq i\leq N$) and an open disc $U(x_0)$ of x_0 and an open disc $V(x_0)$ of $t=0$ such that $\forall x\in U(x_0)$, $\forall t\in V(x_0)$, $\phi_i(x,t)\in W(x_0)$ and the function germs are analytic on $U(x_0)$ and $U(x_0)\cap\zeta_i(N)=\varphi$ for all i. Again since the points on C_i $1\leq i\leq n$ form a compact set $D=\cup_{i=1}^k C_i$, the open cover $\{U(x):x\in D\}$ has a finite subcover where $D\subset\cup_{i=1}^n U(x_i)$. Set $V=\cap_{i=1}^n V(x_i)\cap N$. Note that $\phi_i(x,0)=y_j(x_0,0)$ for some j, $1\leq j\leq N$. Let us consider the monodromy permutation around $\zeta_1(0)$. For simplicity let $y_1(x,0)\to y_2(x,0)\to y_3(x,0)\to y_1(x,0)$. We shall prove that for each $t\in V$ $y_1(x,t)\to y_2(x,t)\to y_3(x,t)\to y_1(x,t)$.

Let $U(x_0)$ is a neighborhood of x_0 such that $U(x_0)=U_1(x_0)\cup U_2(x_0)$. Then

$$\begin{aligned}
&\forall x\in U_1(x_0),\quad \forall t\in V,\quad y_1(x,t)\in W_1(x_0)\\
&\forall x\in U_2(x_0),\quad \forall t\in V,\quad y_3(x,t)\in W_1(x_0)\\
&\text{as}\quad y_3(x,0)\longrightarrow y_1(x,0)\ \ \text{in the neighborhood of } x=x_0,\\
&\forall x\in U_1(x_0),\quad \forall t\in V,\quad y_2(x,t)\in W_2(x_0)\\
&\forall x\in U_2(x_0),\quad \forall t\in V,\quad y_1(x,t)\in W_2(x_0)\\
&\text{as}\, y_1(x,0)\longrightarrow y_2(x,0),
\end{aligned}$$

and

$$\begin{aligned}
&\forall x\in U_1(x_0),\quad \forall t\in V,\quad y_3(x,t)\in W_3(x_0)\\
&\forall x\in U_2(x_0),\quad \forall t\in V,\quad y_2(x,t)\in W_3(x_0)\\
&\text{as}\, y_2(x,0)\longrightarrow y_3(x,0).
\end{aligned}$$

By construction we can find finite number of points $x_0,\dots,x_k$ on C_1 and their neighborhood $U(x_0),\dots,U(x_k)$ and disjoint open set $W_1(x_i),\dots,W_N(x_i)$ for each fixed i, $0\leq i\leq k$ around $y_j(x_i,0)$, $1\leq j\leq N$ such that $\forall x\in U(x_i)$, $t\in V$, $y_j(x,t)\in W_j(x_i)$ $1\leq j\leq N$. Since $y_1(x,0)$ analytically continues to $y_2(x,0)$, $W_1(x_k)$ (i.e the neighborhood of $y_1(x_k,0)$) intersects $W_2(x_0)$ (which is the neighborhood of $y_2(x_0,0)$).

$$\forall x\in U_2(x_0),\quad y_1(x,0)\in W_2(x_0).$$

Choose

$$\begin{aligned}
&\tilde{x} \in U(x_k) \cap U_2(x_0) \\
&\Longrightarrow y_1(\tilde{x}, 0) \in W_2(x_0) \\
&\Longrightarrow y_1(\tilde{x}, t) \in W_2(x_0) \text{ for } t \text{ small (by continuity of } y_1 \text{ in } t) \\
&\text{as only } \phi_2(\tilde{x}, t) \in W_2(x_0)\ \forall t \in V \\
&\Longrightarrow \phi_2(\tilde{x}, t) = y_1(\tilde{x}, t) \text{ for } t \text{ small} \\
&\Longrightarrow \phi_2(\tilde{x}, t) = y_1(\tilde{x}, t) \quad \forall t \in V \quad (\text{as } y_1 \text{ and } \phi_2 \text{ are analytic function of } t) \\
&\Longrightarrow y_1(\tilde{x}, t) \in W_2(x_0) \quad \forall t \in V,\ \forall \tilde{x} \in U_2(x_0) \cap U(x_k) \\
&\Longrightarrow y_1(x, t) \in W_2(x_0) \quad \forall t \in V,\ x \in U_2(x_0) \\
&(\text{as for } t \text{ fixed } y_1(\tilde{x}, t) = \phi_2(\tilde{x}, t) \quad \forall x \in U_2(x_0) \cap U(x_k) \\
&\Longrightarrow y_1(x, t) = \phi_2(x, t) \quad \forall x \in U_2(x_0) \quad \text{by analyticity in } x).
\end{aligned}$$

So if we continue $y_1(x,t)$ along l_1 we get $\phi_2(x,t)$. Again only $y_2(x,t) \in W_2(x_0)$ $\forall x \in U_1(x_0)$. Let us fix $t \in V$. If we continue $y_1(x,t)$ across l_1 the function we get say $\tilde{y}(x,t)$ which is a solution of $P(x,y,t) = 0$ (for fixed t) and hence belong to either $W_1(x_0)$ or $W_2(x_0)$ or $W_3(x_0)$.

Since

$$y_1(x,t) \in W_2(x_0) \quad \forall x \in U_2(x_0)$$

and

$$W_2(x_0) \cap W_1(x_0) = \varphi, \quad W_2(x_0) \cap W_3(x_0) = \varphi.$$

So

$$\begin{aligned}
&\tilde{y}(x,t) \in W_2(x_0) \quad \forall x \in U_1(x_0) \\
&\Longrightarrow \tilde{y}(x,t) = y_2(x,t) \quad \forall x \in U_1(x_0) \\
&\text{as only } y_2(x,t) \in W_2(x_0) \quad \forall x \in U_1(x_0) \quad \forall t \in V.
\end{aligned}$$

Since $t \in V$ is arbitrary $y_1(x,t)$ continues to $y_2(x,t)$ and thus monodromy remains invariant. □

3. Construction of quasiconformal marking maps

3.1 *Construction of a piecewise-affine mapping* ϕ_t: $\mathbf{CP}^1 \to \mathbf{CP}^1$ *which carries ramification points of* $P_0(x,y)$ *to the ramification points of* $P_t(x,y)$

Recall that the ramification points on the Riemann sphere for the covering surface X_t, (i.e., the critical value set for the branched covering map x_t on X_t), are assumed to be located at precisely K points (for each t):

$$(\zeta_1(t), \ldots, \zeta_K(t)).$$

Let g denote the genus of each of the Riemann surfaces X_t.

The aim now is to consider X_0 as the base point for the Teichmüller space $T(X_0) = T_g$, and consequently realise each X_t as a point of the Teichmüller space by constructing an explicit quasiconformal (q.c) marking homeomorphism from X_0 onto X_t:

$$\tilde{\phi}_t : X_0 \longrightarrow X_t.$$

We shall have ϕ_0 as the identity mapping. For these see Nag [N].

Thus the equivalence class of the triple $[X_0, \tilde{\phi}_t, X_t]$ is a point of the Teichmüller space $T(X_0)$. In fact we shall construct a holomorphic 'classifying map' (as the coefficients of P_t vary holomorphically with t):

$$\eta : t \mapsto [X_0, \tilde{\phi}_t, X_t]$$

mapping the t disc $\{|t| < \epsilon\}$ into T_g.

Using the Bers projection

$$\beta : Bel(X_0) \to T(X_0)$$

we will have a lifting of the 'classifying map' η to a map

$$\tilde{\eta} : \{|t| < \epsilon\} \longrightarrow Bel(X_0).$$

The marking homeomorphism between the compact Riemann surfaces X_0 and X_t will be obtained by lifting a mapping ϕ_t between the Riemann spheres that carries corresponding ramification points to ramification points. Construction of $\phi_t : \mathbf{P}^1 \to \mathbf{P}^1$ is detailed below.

Recall that ∞ was set up as an ordinary point for the meromorphic function x on each X_t. Hence all the ramification points, $\zeta_i(t)$ $1 \leq i \leq k$ lie in the finite x-plane. Restrict the parameter t in a relatively compact sub-disc around $t = 0$: $t \in \Delta_\epsilon = \{t : | t | \leq \epsilon\}$. (To save on notation we still call the radius of the sub-disc as ϵ.)

Since the functions ζ_i are analytic in t, we can find a rectangle R containing in its interior all of the points $S = \{\zeta_i(t) : 1 \leq i \leq K, t \in \Delta_\epsilon\}$. Outside R we will define ϕ_t to be the identity mapping.

To define ϕ_t inside R we take the first (domain) copy of $\mathbf{CP}^1$ and triangulate R as follows: we divide R into non-degenerate triangular regions such that each of the points $\zeta_i(0)$ are used as vertices. Thus the triangulation utilizes a set of vertices containing all the K points $\zeta_i(0)$, as well as some extra points ζ_s for some index set $s = K+1, \ldots, K+L$. (The four vertices of the rectangle R are certainly included amongst these last L vertices. Also note that each triangle utilized is, by requirement, non-degenerate – namely the vertices are always three non-collinear points.)

Now consider another copy of $\mathbf{CP}^1$ (which will serve as the range of the map ϕ_t) and divide the region inside the rectangle R in this second copy into triangular regions in the natural 'corresponding' fashion, as detailed next: namely the vertices of the triangles of this second copy of R consist of the new ramification points $\zeta_i(t)$'s in place of the $\zeta_i(0)$, $1 \leq i \leq K$, – together with the same extra set of points ζ_s (for index set $s = K+1, \ldots, K+L$) that were used before. Note: these last L vertices are left undisturbed. Of course, the edges of the two triangulations correspond exactly since the vertices have the above correspondence. That is, if $(\zeta_i(0), \zeta_j(0), \zeta_k(0))$ form vertices of a triangle in the first copy then $(\zeta_i(t), \zeta_j(t), \zeta_k(t))$ form vertices of the corresponding triangle in the second copy; similarly, if $(\zeta_i(0), \zeta_p, \zeta_q)$ are vertices of a triangle in the first copy then $(\zeta_i(t), \zeta_p, \zeta_q)$ will be the vertices of the corresponding in the second copy, etc.

Remark. Since the initial triangulation is non-degenerate, namely the vertices of any triangle that was utilized were non-collinear, then, by continuity of the functions $\zeta_j(t)$, that non-degeneracy of the corresponding triangulation (on the range copy) remains valid for all small values of t near $t = 0$.

Affine mapping of one triangle onto another: If (z_1, z_2, z_3) are any three non-collinear points in the plane, then recall that their *closed convex hull*, (smallest closed convex set in the plane containing these points), is precisely the triangle T (includes the interior and the edges) with the given points as vertices. From elementary linear geometry one knows that every point of T has a unique representation as a convex combination of the vertex vectors; namely, each point of T is representable as $\lambda z_1 + \mu z_2 + \nu z_3$, where λ, μ and ν are real numbers in the closed unit interval $[0,1]$ such that $\lambda + \mu + \nu = 1$.

Clearly then, given any other set of three non-collinear vertices (w_1, w_2, w_3) for a second triangle T', there is a natural *affine mapping* of the first triangle onto the second which simply sends the point $\lambda z_1 + \mu z_2 + \nu z_3$ of T to the point $\lambda w_1 + \mu w_2 + \nu w_3$ of T'.

DEFINITION OF ϕ_t

We therefore define the desired homeomorphism ϕ_t inside the rectangle R by taking the triangles of the first triangulation, by the above affine mappings, onto the corresponding triangles of the second triangulation. Notice that if two triangles share a common edge, then the affine mappings defined on the two abutting triangles will coincide in their definition along the common edge. That is crucial. Consequently we clearly get a well defined homeomorphism ϕ_t of the rectangle R on itself, and outside R we simply extend ϕ_t by the identity map to the whole Riemann sphere.

It is clear that ϕ_t is a C^∞-diffeomorphism when restricted to the interiors of the triangles used in triangulating R, and also, of course, on the exterior of R.

Lemma. *ϕ_t is quasiconformal for each t in the ϵ disc. The Beltrami coefficient of ϕ_t, is a complex constant (of modulus less than unity) when restricted to the interior of each triangle in the initial triangulation of the rectangle R. Of course, the Beltrami coefficient is identically zero in the exterior of R.*

3.2 *Lifting of* $\phi_t : \mathbf{CP}^1 \longrightarrow \mathbf{CP}^1$ *to* $\tilde{\phi}_t : X_0 \longrightarrow X_t$

Consider the following diagram of Riemann surfaces with the vertical arrows being, as we know, holomorphic branched coverings:

$$\begin{tikzcd}
X_0 \arrow[r, "\tilde{\phi}_t"] \arrow[d, "x"'] & X_t \arrow[d, "x_t"] \\
\mathbf{CP}^1 \arrow[r, "\phi_t"] & \mathbf{CP}^1
\end{tikzcd}$$

PROPOSITION

There exists a quasiconformal, orientation preserving homeomorphism:

$$\tilde{\phi}_t : X_0 \longrightarrow X_t$$

lifting the map $\phi_t : \mathbf{CP}^1 \to \mathbf{CP}^1$ and making the above diagram commute. (Note that $\tilde{\phi}_0$ is the identity.)

Proof. In fact, in order to deal with unbranched covering spaces, we define the following punctured Riemann surfaces:

$$X_0' = x^{-1}\{\mathbf{CP}^1 - \text{all critical values of } x\}$$

and

$$X_t' = x_t^{-1}\{\mathbf{CP}^1 - \text{all critical values of } x_t\}.$$

Restricted to X_0' and X_t', the vertical mappings are now smooth (=unbranched) covering projections. Observe that the ϕ_t was designed so as to map the critical values of x onto hose of x_t. Now we can apply the standard lifting criterion for maps from the theory of covering spaces to demonstrate that ϕ_t lifts. Consequently, at the level of fundamental groups we need to look at the image of the action on π_1 of $(\phi_t \circ x)$ as compared with that of x_t. (See, for instance, Theorem 5.1, p. 128, of Massey [M] for the statement of the usual lifting criterion.)

Since the monodromy permutation at any critical point say $\zeta_m(0)$ is the same as that around the perturbed critical point $\zeta_m(t)$, and since $\phi_t(\zeta_m(0)) = \zeta_m(t)$, we see that:

$$\pi_1(\phi_t \circ x)\pi_1(X_0', w_0) = \pi_1(x_t)\pi_1(X_t', \beta_0),$$

(where $w_0 \in X_0'$ and $x(w_0) = z_0$ and $\beta_0 \in X_t'$ such that $x_t(\beta_0) = \phi_t(z_0)$).

Clearly then the lifting criterion is satisfied, and hence the homeomorphism ϕ_t lifts to a homeomorphism $\tilde{\phi}_t$, as desired. Certainly the lift is quasiconformal since the vertical mappings are holomorphic. This completes the proof of the proposition. In this connection recall the following result.

Theorem. *If U and V are open subsets of compact surfaces X and Y respectively with finite complements, then any homeomorphism from of U onto V extends uniquely to one of X onto Y.* □

Finally then, for our applications to the variation of Fuchsian groups we may *lift all the way to the universal covering upper half-planes* and obtain the quasiconformal homeomorphism $\Phi_t(z) = \Phi(z,t)$ from U to U, obtained by lifting the mapping to $\tilde{\phi}_t : X_0 \longrightarrow X_t$.

Thus we have determined $\Phi_t(z)$ so that the following diagram commutes:

$$\begin{tikzcd}
z \in U \arrow[r, "\Phi_t(z) = \Phi(z,t)"] \arrow[d, "\pi"] & U \arrow[d, "\pi_t"] \\
U/G_0 \equiv X_0 \arrow[r, "\tilde{\phi}_t"] \arrow[d, "x_0"] & X_t \equiv U/G_t \arrow[d, "x_t"] \\
x \in \mathbf{CP}^1 \arrow[r, "\phi_t(z)", "= \phi(z,t)"'] & \mathbf{CP}^1
\end{tikzcd}$$

.

4. Variational formulae for the Fuchsian groups of varying curve

4.1 *The fundamental variational term*

Let $\mu_t(z)$ denote a one-parameter family of Beltrami coefficients on the upper half-plane depending real or complex analytically on the (real or complex) parameter t near $t = 0$. Suppose also that $\mu_0(z) \equiv 0$. We come now to the main formula that we shall apply. If $\mu_0 \equiv 0$, and if for small t the Beltrami coefficient is given by

$$\mu_t(z) = t\hat{\nu}(z) + o(t), \text{ where } \hat{\nu} \in L^\infty(U), \tag{2}$$

then one has an important integral formula expressing the solutions of the family of Beltrami equations, as a perturbation of the identity homeomorphism

$$w_{\mu_t}(z) = z + tw_1(z) + o(t), z \in U.$$

Indeed, the crucial first variation term, $w_1 = \dot{w}$, for real t is given by

$$w_1(z) = -\frac{1}{\pi}\iint_U [\hat{\nu}(\zeta)R(\zeta, z) + \overline{\hat{\nu}(\zeta)}R(\bar{\zeta}, z)]\mathrm{d}\xi\mathrm{d}\eta,$$

$$R(\zeta, z) = \frac{z(z-1)}{\zeta(\zeta-1)(\zeta-z)}, \text{ and } \zeta = \xi + i\eta.$$

This perturbation formula (see Ahlfors [A], or section 1.2.13, 1.2.14, as well as page 175, eq. (1.21) of Nag [N]), will be fundamental for us. We shall apply it to the family of quasiconformal mappings Φ_t (§3) standing for the family w_{μ_t}.

Since in our set up t is a *complex* parameter we may as well deduce the form of the variational terms for general t complex – which follows by simply applying the real t formula above appropriately. We show this:

If t is complex, write in polar form: $t = |t|\mathrm{e}^{i\alpha}$ then put $\tau = \mathrm{e}^{-i\alpha}t = |t|$. Then it is straight forward to see that

$$w_1(z) = -\frac{1}{\pi}\iint_U [\hat{\nu}(\zeta)R(\zeta, z) + \mathrm{e}^{-2i\alpha}\,\overline{\hat{\nu}(\zeta)}R(\bar{\zeta}, z)]\mathrm{d}\xi\mathrm{d}\eta,$$

where $\alpha = \arg(t)$. But $t\mathrm{e}^{-2i\alpha}$ is the conjugate of t. Therefore, this last formula says that for complex t we have the final important formulae:

$$w_{\mu_t}(z) = z + tw_1(z) + \bar{t}w_1^*(z) + o(t), \quad z \in U \tag{3}$$

where

$$(w_1(z), w_1^*(z)) = \left(-\frac{1}{\pi}\iint_U [\hat{\nu}(\zeta)R(\zeta, z)]\mathrm{d}\xi\mathrm{d}\eta, -\frac{1}{\pi}\iint_U [\overline{\hat{\nu}(\zeta)}R(\bar{\zeta}, z)]\mathrm{d}\xi\mathrm{d}\eta\right). \tag{4}$$

Equation (4) will be manipulated to produce the chief formulae of §4.

Let $\Gamma \equiv G_0 \subset PSL(2, \mathbf{R})$ denote the uniformizing Fuchsian group acting as deck transformations for the covering π. Then there is a biholomorphic equivalence:

$$X_0 = U/G_0. \tag{5}$$

It follows from the standard Ahlfors-Bers deformation theory of Fuchsian groups (see Nag [N]) that the quasiconformal homeomorphism Φ_t is *compatible with the Fuchsian group* G_0, in the sense that $g_t = \Phi_t \circ g \circ \Phi_t^{-1}$ is again a Möbius transformation in

$PSL(2,\mathbf{R})$ for every $g \in G_0$, and the new Fuchsian group (which evidently remains bstractly isomorphic to G_0) is the Fuchsian group:

$$G_t = \Phi_t \circ G_0 \circ {\Phi_t}^{-1}. \tag{6}$$

This is the group of deck transformations for the covering π_t, so that X_t is biholomorphically equivalent to U/G_t. We shall write

$$g_t = \Phi_t \circ g \circ {\Phi_t}^{-1} \in G_t \tag{7}$$

or any fixed $g \in G_0 \equiv \Gamma$.

In this notation, the central problem of our work is to determine explicit and applicable ormulae for the variation of g_t – or, equivalently, to compute the t-derivative: $\dot{g}_t$ at $t = 0$. As g varies over any generating set of elements for the group G_0, we shall then obtain, up o first order approximation, a corresponding set of generating elements for the deformed groups G_t.

.2 *The Beltrami coefficient μ_t of Φ_t*

Notational set up. Let us, for notational convenience, denote as x_* the meromorphic unction on U given by $x \circ \pi$, (this is, of course, a holomorphic branched covering of the Riemann sphere by the upper half plane). Clearly, x_* is automorphic with respect to the Fuchsian group Γ, since x_* descends onto the surface X_0 as the meromorphic function x hereon. In particular, let us note the well-known fact that this function, x_*, can be expressed in terms of the standard Poincaré theta-series on U with respect to the group Γ.

Now recall from the previous section that the mapping ϕ_t was, by our very definition, a piecewise affine quasiconformal mapping. So the Beltrami coefficient of ϕ_t was a complex constant on each triangle of the triangulation of the domain rectangle R. (The Beltrami coefficient need only be specified almost everywhere – therefore we will ignore t on the edges and vertices of the triangulation.)

Moreover we know that the vertices of the triangulation (in the image plane) depend holomorphically on t – since the ramification points $\zeta_j(t)$ were holomorphic functions of . Here is the main proposition we require.

PROPOSITION

The Beltrami coefficient of Φ_t is

$$\mu_t(z) = t\hat{\nu}(z) + o(t), z \in U, \hat{\nu}(z) \in L^\infty(U),$$

where

$$\hat{\nu}(z) = \nu(w)\frac{\overline{(x \circ \pi)'(z)}}{(x \circ \pi)'(z)}, \quad \textit{where} \quad w = (x \circ \pi)(z) = x_*(z) \in \mathbf{CP}^1. \tag{8}$$

Here the Beltrami coefficient for the piecewise-affine mappings ϕ_t on the Riemann w-sphere has been expanded up to first order in t as below:

$$\frac{\phi_{t,\bar{w}}(w)}{\phi_{t,w}(w)} = t\nu(w) + o(t), \nu \in L^\infty(\mathbf{CP}^1). \tag{9}$$

Further note that $\nu(w)$ is a *constant on each triangle* of the first (domain) triangulation of R, and it is zero for all w outside R.

Note. The Γ invariant Beltrami coefficient $\hat{\nu}$ above, represents the *tangent vector* to the one parameter family of Beltrami coefficients μ_t which arise from the one parameter family of quasiconformal mappings Φ_t.

Proof. From the above commutative diagram for the liftings we have

$$(x_t \circ \pi_t) \circ \Phi_t = \phi_t \circ (x \circ \pi). \tag{10}$$

Taking the ∂ and $\bar{\partial}$ derivatives in (10), and remembering that all the vertical maps are *holomorphic* coverings (possibly branched as we know), we obtain the Beltrami coefficient of Φ_t on U:

$$\mu_t(z) \equiv \frac{\Phi_{t,\bar{z}}}{\Phi_{t,z}} = \frac{\phi_{t,\bar{w}}(w)\overline{(x \circ \pi)'(z)}}{\phi_{t,w}(w)(x \circ \pi)'(z)}, \quad z \in U. \tag{11}$$

Clearly then the statements in the Proposition follow because the $\phi_t(w)$ are a family of piecewise affine quasiconformal homeomorphisms on the w-sphere which vary holomorphically in t. Thus, remembering that ϕ_0 is the identity, we see that the Beltrami coefficients of the family ϕ_t indeed must have an expression as in (9) with $\nu(w)$ being piecewise-constant. □

Remark. Note that the Beltrami coefficient of ϕ_t is a *holomorphic* function of the parameter t in the neighborhood of $t = 0$. The map

$$t \mapsto \frac{\phi_{t,\bar{z}}}{\phi_{t,z}}$$

takes values in the complex Banach space $L^\infty(\mathbf{CP}^1)$, – and the holomorphy is as a map into this Banach space.

Beltrami coefficients automorphic with respect to Γ. We must remember from the general theory (see §1.3.3 of Nag [N]) one further fundamental fact. Since the quasiconformal maps Φ_t are compatible with Γ their Beltrami coefficients are $(-1, 1)$ forms on U with respect to Γ. (We called them Γ-invariant Beltrami coefficients.)

Indeed, if μ is the complex dilatation of a quasiconformal mapping that conjugates Γ into any group of Möbius transformations, then

$$(\mu \circ g)(\bar{g}'/g') = \mu, a.e., \quad \text{for all} \quad g \in \Gamma. \tag{12}$$

We denote the Banach space of complex valued L^∞ functions on U that satisfy equation (12) for every $g \in \Gamma$, by the notation: $L^\infty(U, \Gamma)$. See p. 49 of [N]. Thus, μ_t belongs to the open unit ball of this Banach space for all small t , and also therefore $\hat{\nu}$ belongs to this Banach space of automorphic objects.

4.3 *The variational formula for* Φ_t

We come to the chief application of the perturbation formula (eq. (4)) in our specific context of varying algebraic curves.

Let F denote a closed fundamental domain, with boundary of two-dimensional measure zero, for the action of Γ on U; (for instance, we may choose F as any standard Dirichlet fundamental polygon for the Fuchsian group Γ). Thus π maps F onto X_0, and π is one-to-one when restricted to the interior of F.

Recall that x was itself a meromorphic function of degree N on the compact Riemann surface X_0, (see §2, 3). Consequently, when restricted to the interior of F the mapping x_* is a N-to-1 branched holomorphic covering map onto the Riemann sphere – missing only a set of areal measure zero. Since this is a finite covering space situation (aside from a measure zero set of branch points which we may discard to start with), we may choose a decomposition of F into N regions:

$$F = D_1 \cup D_2 \cup \cdots \cup D_N. \tag{13}$$

Here the D_j are mutually disjoint domains (except for boundary contact, as usual in choice of fundamental regions), partitioning F, with the basic property that each D_j maps, via x_*, in a one-to-one fashion onto the entire Riemann sphere (missing atmost a measure zero subset). (Recall that the compact Riemann surface X_0 was described as an N-sheeted branched cover of the sphere – by the degree N meromorphic function x.)

A kernel function associated to Γ. We introduce as an useful matter of notation, the following function of two variables: $z \in U$, $\tau \in \mathbf{C}$ (not lying on the Γ orbit of z):

$$K_\Gamma(z,\tau) = \sum_{g\in\Gamma} \frac{[g'(z)]^2}{g(z)(g(z)-1)(g(z)-\tau)}. \tag{14}$$

We are now in a position to state a main result.

Theorem. *On variation of* Φ_t. *The lifted quasiconformal maps* Φ_t *on* U *satisfy the following first order expansion for small* t,

$$\Phi_t(z) = z + tw_1(z) + \bar{t}w_1^*(z) + o(t), z \in U, \tag{15}$$

where

$$w_1(z) = \frac{z(z-1)}{2\pi\sqrt{-1}} \sum_{k=1}^{N} \int\int_{\mathbf{CP}^1} \left\{ \nu(w) K_\Gamma(x_{*,k}^{-1}(w), z) \left[\frac{\partial x_{*,k}^{-1}}{\partial w}(w) \right]^2 \right\} \mathrm{d}w \wedge \mathrm{d}\bar{w},$$

$$w_1^*(z) = \frac{z(z-1)}{2\pi\sqrt{-1}} \sum_{k=1}^{N} \int\int_{\mathbf{CP}^1} \left\{ \overline{\nu(w)} K_\Gamma(\overline{x_{*,k}^{-1}(w)}, z) \left[\overline{\frac{\partial x_{*,k}^{-1}}{\partial w}(w)} \right]^2 \right\} \mathrm{d}w \wedge \mathrm{d}\bar{w}.$$

Here we have denoted by $x_{*,k}$ *the restriction of the projection* $x_* = x \circ \pi$ (*which is a meromorphic and* Γ*-automorphic function on* U), *to the region* $D_k \subset F$, $k = 1, \ldots, N$. *Here* ν *denotes the function on the w-sphere appearing in formula* (9) *of the Proposition in sub-section* 4.2 *above.* (*Recall that* ν *is simply a constant assigned on each triangle in the triangulation of* R, *with* ν *being identically zero outside* R.)

Note furthermore, that since x_* *is a meromorphic function on* U, *we may replace in the above formula the derivative of its inverse by the reciprocal of its own derivative, as shown below:*

$$\frac{\partial x_{*,k}^{-1}}{\partial w}(w) = 1 \Big/ \frac{\mathrm{d}x_{*,k}}{\mathrm{d}z}(z), \quad w = x_*(z), \quad z \in D_k.$$

These derivatives can therefore be calculated from the expression for x_* *which will be available in terms of the standard Poincaré theta series on* U *with respect to* γ. (*Therefore we see that if* $\gamma \in G_0$ *then the variational formula for* $\gamma_t = \Phi_t \circ \gamma \circ \Phi_t^{-1} \in G_t$ *is*

$$\gamma_t = \gamma + t\dot{\gamma} + \bar{t}\dot{\gamma}^* + o(t), \tag{16}$$

where

$$\dot{\gamma} = w_1 \circ \gamma - \gamma' w_1,$$
$$\dot{\gamma}^* = w_1^* \circ \gamma - \gamma' w_1^*.$$

For this, see Nag [*N*].)

During the course of the proof we shall show that all integrals and summations appearing in sight are absolutely convergent. For facts regarding Poincaré theta series and their utilization in expressing meromorphic functions on U/Γ, see [Kra, Kr].

Proof. We shall have to manipulate the variational formula (4) which said:

$$w_1(z) = \frac{1}{2i\pi}\int\int_U [\hat{\nu}(w)R(w,z) + \overline{\hat{\nu}(w)}R(\bar{w},z)]\mathrm{d}w \wedge \mathrm{d}\bar{w}$$

with $R(w,z) = \frac{z(z-1)}{w(w-1)(w-z)}$.

By general theory quoted above, the integrals involved in (4) are necessarily absolutely convergent.

To obtain the final result for w_1 and w_1^*, there are several chief ideas which we first explain in words:

(i) Write each of the two integrals over U as a sum of integrals over all the tiles in the Γ-tessellation of U – obtained by decomposing U as the union of the fundamental domain F and its translates: i.e., $U = \cup_{g\in\Gamma}(g(F))$.

(ii) Utilizing the Γ-automorphic nature of the Beltrami coefficient $\hat{\nu}$ (see eq. (12) above), and making a change of variables by $w = g(z)$, we can transform the integral over $g(F)$ to an integral again over F itself.

(iii) Consequently, the original expression for w_1 becomes simply an integration over F of a certain expression on F, after interchanging summation and integration. (The validity of the interchange is guaranteed by the absolute convergence of the result, together with the dominated convergence theorem. The main details of this critical interchange of sum and integral are spelled out in the remarks attached at the end of the proof.)

(iv) Finally we decompose F itself into the N pieces $D_1, \ldots, D_N$ (as explained with eq. (13) above) – and hence we may eliminate $\hat{\nu}$ by replacing it with occurrences of ν itself, and thus express the final result as integrations over the Riemann sphere $\mathbf{CP}^1$, as desired.

The first three of the above steps are carried out e.g. in [A]. Let us now get down to the main business of showing the exact nature of how these transformations come about in the expression for w_1. First of all note:

$$\int\int_U \frac{\hat{\nu}(w)}{w(w-1)(w-\tau)}\mathrm{d}w \wedge \mathrm{d}\bar{w}$$

$$= \sum_{g\in\Gamma}\int\int_{g(F)} \frac{\hat{\nu}(w)}{w(w-1)(w-\tau)}\mathrm{d}w \wedge \mathrm{d}\bar{w}, \quad F = \text{fundamental}$$

region of Γ in U.

Perform a change of variables on $g(F)$ by $w = u + iv = g(z)$

$$= \sum_{g\in\Gamma} \iint_F \frac{\hat{\nu}(z)\frac{g'(z)}{\overline{g'(z)}} \mid g'(z) \mid^2}{g(z)(g(z)-1)(g(z)-\tau)} \mathrm{d}z \wedge \mathrm{d}\bar{z}$$

$$= \sum_{g\in\Gamma} \iint_F \frac{\hat{\nu}(z)[g'(z)]^2}{g(z)(g(z)-1)(g(z)-\tau)} \mathrm{d}z \wedge \mathrm{d}\bar{z}.$$

For convergence arguments we note that since

$$\sum_{g\in\Gamma} \iint_F \frac{\mid \hat{\nu}(z) \mid \frac{|g'(z)|}{|\overline{g'(z)}|} \mid g'(z) \mid^2}{\mid g(z) \mid\mid g(z)-1 \mid\mid g(z)-\tau \mid} \mathrm{d}x\mathrm{d}y$$

$$= \sum_{g\in\Gamma} \iint_{g(F)} \frac{\mid \hat{\nu}(w) \mid}{\mid w \mid\mid w-1 \mid\mid w-\tau \mid} \mathrm{d}u\mathrm{d}v < \infty.$$

This demonstrates that the series

$$\sum_{g\in\Gamma} \iint_F \frac{\hat{\nu}(z)[g'(z)]^2}{g(z)(g(z)-1)(g(z)-\tau)} \mathrm{d}z \wedge \mathrm{d}\bar{z} = \sum_{g\in\Gamma} \iint_F \psi_g(z) \mathrm{d}z \wedge \mathrm{d}\bar{z} \tag{17}$$

is absolutely convergent. Note that, for convenience, we have written ψ_g here for the following frequently recurring expression:

$$\psi_g(z) = \frac{\hat{\nu}(z)[g'(z)]^2}{g(z)(g(z)-1)(g(z)-\tau)}.$$

We shall show by a measure-theoretic lemma in the remarks appended to the bottom of this proof, that we are allowed to change summation and integration in the summation (17). We shall utilize crucially this interchange immediately in what follows. Returning therefore to the actual expression for the variational term w_1, we now obtain:

$$w_1(z) = -\frac{1}{2i\pi} \iint_U \hat{\nu}(\zeta) R(\zeta, z) \mathrm{d}\zeta \wedge \mathrm{d}\bar{\zeta}$$

$$= \frac{z(z-1)}{2\pi i} \iint_U \frac{\hat{\nu}(\zeta)}{\zeta(\zeta-1)(\zeta-z)} \mathrm{d}\zeta \wedge \mathrm{d}\bar{\zeta}$$

$$= \frac{z(z-1)}{2\pi i} \sum_{g\in\Gamma} \iint_F \frac{\hat{\nu}(\zeta)[g'(\zeta)]^2}{[g(\zeta)][g(\zeta)-1][g(\zeta)-z]} \mathrm{d}\zeta \wedge \mathrm{d}\bar{\zeta}$$

$$= \frac{z(z-1)}{2\pi i} \iint_F \sum_{g\in\Gamma} \frac{\hat{\nu}(\zeta)[g'(\zeta)]^2}{g(\zeta)(g(\zeta)-1)(g(\zeta)-z)} \mathrm{d}\zeta \wedge \mathrm{d}\bar{\zeta}$$

$$= \frac{z(z-1)}{2\pi i} \iint_F \hat{\nu}(\zeta) K_\Gamma(\zeta, z) \mathrm{d}\zeta \wedge \mathrm{d}\bar{\zeta}.$$

Similarly

$$w_1^*(z) = \frac{z(z-1)}{2\pi i} \iint_F \overline{\hat{\nu}(\zeta)} K_\Gamma(\bar{\zeta}, z) \mathrm{d}\zeta \wedge \mathrm{d}\bar{\zeta}.$$

That completes the manipulation of the formula to a point that already has points interest; we have carried out steps (i), (ii), (iii) – and now we are integrating over F (i. over X_0), rather than over U.

The final steps are for carrying out the program outlined in point number (iv) abo This goes as detailed below:

Let

$$(x \circ \pi)(D_i) = \mathbf{CP}^1, \quad \text{and denote } x \circ \pi|_{D_i} = x_{*,i}$$

for each $i = 1, \ldots, N$. Setting $(x \circ \pi)(\zeta) = w$, $\zeta \in U$ and $w \in \mathbf{CP}^1$, and using the relati (eq. (8)) between $\hat{\nu}$ and ν, we will have

$$\begin{aligned} w_1(z) &= \frac{z(z-1)}{2\pi i} \\ &\sum_{i=1}^{N} \iint_{\mathbf{CP}^1} \left[\sum_{g \in \Gamma} \left(\frac{\nu(w)([\overline{\mathrm{d}w/\mathrm{d}\zeta}]/[\mathrm{d}w/\mathrm{d}\zeta])[g'(x_{*,i}^{-1}(w))]^2}{g(x_{*,i}^{-1}(w))(g(x_{*,i}^{-1}(w))-1)(g(x_{*,i}^{-1}(w))-z)} \right) \right] \frac{\mathrm{d}w \wedge \mathrm{d}\bar{w}}{|\mathrm{d}w/\mathrm{d}\zeta|^2} \\ &= \frac{z(z-1)}{2\pi i} \\ &\quad \sum_{i=1}^{N} \iint_{\mathbf{CP}^1} \left[\sum_{g \in \Gamma} \frac{\nu(w)[g'(x_{*,i}^{-1}(w))]^2 [\partial x_{*,i}^{-1}/\partial w(w)]^2}{g(x_{*,i}^{-1}(w))(g(x_{*,i}^{-1}(w))-1)(g(x_{*,i}^{-1}(w))-z)} \right] \mathrm{d}w \wedge \mathrm{d}\bar{w} \\ &= \frac{z(z-1)}{2\pi i} \sum_{i=1}^{N} \iint_{\mathbf{CP}^1} \left[\nu(w) K_\Gamma(x_{*,i}^{-1}(w), z) \left[\frac{\partial x_{*,i}^{-1}}{\partial w}(w) \right]^2 \right] \mathrm{d}w \wedge \mathrm{d}\bar{w}. \end{aligned}$$

Similarly

$$w_1^*(z) = \frac{z(z-1)}{2\pi i} \sum_{i=1}^{N} \iint \left[\overline{\nu(w)} K_\Gamma(\overline{x_{*,i}^{-1}(w)}, z) \left[\overline{\frac{\partial x_{*,i}^{-1}}{\partial w}(w)} \right]^2 \right] \mathrm{d}w \wedge \mathrm{d}\bar{w}$$

That at last is exactly the expression desired and claimed in the Theorem and we through.

The interchange of summation and integration above in the series (17), follows f some straightforward facts of the theory of measure and integration. For instance, purposes are adequately served by the following result (see Rudin [R]):

Lemma. Suppose $\{f_n\}$ *is a sequence of complex measurable functions defined al everywhere on a complete measure space* (X, μ) *such that*

$$\sum_1^{\infty} \int_X |f_n| \, \mathrm{d}\mu < \infty.$$

Then the series $f(x) = \sum_1^{\infty} f_n(x)$ *converges absolutely for almost all x, and* $f \in L^1$ *moreover, the summation and integration can be interchanged, namely:*

$$\int_X f \mathrm{d}\mu = \sum_1^{\infty} \int_X f_n \mathrm{d}\mu.$$

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Limits of commutative triangular systems on locally compact groups

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Abstract. On a locally compact group G, if $\nu_n^{k_n} \to \mu$, $(k_n \to \infty)$, for some probability measures ν_n and μ on G, then a sufficient condition is obtained for the set $A = \{\nu_n^m | m \leq k_n\}$ to be relatively compact; this in turn implies the embeddability of a shift of μ. The condition turns out to be also necessary when G is totally disconnected. In particular, it is shown that if G is a discrete linear group over **R** then a shift of the limit μ is embeddable. It is also shown that any infinitesimally divisible measure on a connected nilpotent real algebraic group is embeddable.

Keywords. Embeddable measures; triangular systems of measures; infinitesimally divisible measures; totally disconnected groups; real algebraic groups.

. Introduction

Commutative triangular systems of probability measures on locally compact groups have been studied extensively and recently the embedding of the limit μ (or a translate $x\mu$, $x \in G$) have been shown on a large class of groups under certain conditions like infinitesimality of triangular system and/or 'fullness' of the limit μ (see [S4] for the latest results and the literature cited therein for earlier results). Generalizing the techniques developed in [S3,S4], we extend our earlier result to some particular triangular systems on algebraic groups. We also discuss special triangular systems of identical measures, i.e. limit theorems. In particular if $\nu_n^{k_n} \to \mu$ on G then we give a sufficient condition for the set $A = \{\nu_n^m \mid m \leq k_n\}$ to be relatively compact; this in turn would imply the embeddability of a shift of the limit μ. The condition turns out to be also necessary if G is totally disconnected. We hereby generalize our earlier results on limit theorems on Lie groups to general locally compact groups. We also show the embedding of a shift of the limit μ if G is a discrete linear group over **R**.

Let G be a locally compact (Hausdorff) group and let $M^1(G)$ be the topological semigroup of probability measures with weak topology and convolution as the semigroup operation. Let μ, ν be any measures in $M^1(G)$. Let the convolution product of μ and ν be denoted by $\mu\nu$. For any compact subgroup H of G let ω_H denote the normalized Haar measure of H. Let $M_H^1(G) = \omega_H M^1(G) \omega_H$, then $M_H^1(G)$ is a closed subsemigroup of $M^1(G)$ with identity ω_H. For any $x \in G$, let δ_x denote the Dirac measure at x and let $x\mu = \delta_x\mu$, (similarly, $\mu x = \mu\delta_x$). Let $I_\mu = \{x \in G \mid x\mu = \mu x\}$ and let $I(\mu) = \{x \in G \mid x\mu = \mu x = \mu\}$, then I_μ (resp. $I(\mu)$) is a closed (resp. compact) subgroup of G. Let $J_\mu = \{\lambda \in M^1(G) \mid \lambda\mu = \mu\lambda = \mu\}$. Clearly, J_μ is a compact semigroup and for any $\lambda \in M^1(G)$, $\lambda \in J_\mu$ if and only if $\operatorname{supp}\lambda \subset I(\mu)$. Let $G(\mu)$ be the smallest closed subgroup of G containing $\operatorname{supp}\mu$. Let $N(\mu)$ (resp. $Z(\mu)$) be the normalizer (resp. centralizer) of $G(\mu)$ in

G. Let $\tilde{\mu}$ denote the *adjoint of* μ, defined by $\tilde{\mu}(B) = \mu(B^{-1})$, for all Borel subsets B μ is said to be *symmetric* if $\mu = \tilde{\mu}$. Let G^0 denote the connected component of the ide in G. For a set $A \subset M^1(G)$ and a normal subgroup $C \subset G$, we denote $A/C = \pi(A)$, v $\pi : G \to G/C$ is the natural projection.

A measure $\mu \in M^1(G)$ is said to be *infinitely divisible* (resp. *weakly infinitely divi* if for every $n \in \mathbb{N}$, there exists $\mu_n \in M^1(G)$ such that $\mu_n^n = \mu$ (resp. $\mu_n^n x_n = \mu$ for $x_n \in G$); and it is said to be *embeddable* if there exists a continuous one-para convolution semigroup $\{\mu_t\}_{t \geq 0}$ such that $\mu_1 = \mu$. Since we aim to prove the embed lity of a given measure under various conditions, the reader is referred to [M2], a s article on the embedding problem of infinitely divisible measures.

Let S be a Hausdorff semigroup with identity e and let $s \in S$. Let T_s denote the two sided factors of s, that is, $T_s = \{t \in S \mid tr = rt = s \text{ for some } r \in S\}$. Elements s, are said to be *associates* if s and t are two sided factors of each other, i.e. $s \in T$ $t \in T_s$. A subset A of S is said to be *associatefree* if $s, t \in A$ are associates then $s =$ element h in S is said to be an *idempotent* if $h^2 = h$. An element s is said to be *bald* if e is the only idempotent contained in T_s. For a subset A of S, a decomposition o $s = s_1 \cdots s_n$, for some $n \in \mathbb{N}$, where $s_i \in A$ and $s_i s_j = s_j s_i$ for all i, j, is called *decomposition of s*. An element s (in S) is said to be *infinitesimally divisible* if s has decomposition for every neighbourhood U of e in S. A set $\Delta = \{s_{ij} \in S \mid i \in \mathbb{N}, 1$ $n_i, n_i \to \infty$ as $i \to \infty\}$ is said to be a *triangular system* in S; we will sometimes $\Delta = (s_{ij})_{i \in \mathbb{N}, j=1}^{n_i}$. Δ is said to be commutative if for every fixed i, s_{ij} commute with other, it is said to be *infinitesimal* if as $i \to \infty$, $s_{ij} \to e$ uniformly in j. We say t converges to μ if $s_{i1} \cdots s_{in_i} = s_i \to \mu$.

In § 2, we prove a limit theorem for general locally compact groups, (see Theorem In § 3, we show that if $\nu_n^{k_n} \to \mu$, $(k_n \to \infty)$, on a discrete linear group over **R**, then embeddable for some $x \in G$ (see Theorem 3.1). In § 4, we show that any infinitesi divisible probability measure μ on a connected nilpotent real algebraic group is er dable, (more generally see Theorem 4.1).

2. Limit theorems on locally compact groups

Theorem 2.1. *Let G be a locally compact group and let $\pi : G \to G/G^0$ be the n projection. Let $\{\nu_n\}$ be a relatively compact sequence in $M^1(G)$ such that for an point ν of it, $G(\pi(\nu))$ is a compact group in G/G^0 and $\nu_n^{k_n} \to \mu$ for some $\mu \in M^1($ for some unbounded sequence $\{k_n\} \subset \mathbb{N}$. Suppose that for some connected nil normal subgroup N of G, the closed subgroup generated by* supp μ *and N contai Then the set $A = \{\nu_n^m \mid m \leq k_n\}$ is relatively compact and there exists $x \in I_\mu$ such t is embeddable.*

Remarks. (1) The above theorem generalizes Theorem 1.7(1) of [S4]. (2) If G is disconnected then $G^0 = \{e\}$ and hence the above theorem implies that if $\nu_n^{k_n} \to \mu$ $\{\nu_n\}$ is relatively compact and for any limit point ν of it, $G(\nu)$ is compact the relatively compact. Conversely, if A is relatively compact then so are $\{\nu_n\}$ and $\{\nu_n^{k_n}$ for any limit point ν of $\{\nu_n\}$, $G(\nu)$ is compact as $\{\nu^n\} \subset A$. Thus, for disconnected groups we get a necessary and sufficient condition for the set A as ab be relatively compact.

We first prove a more general theorem for totally disconnected locally compact g

Theorem 2.2. *Let G be a totally disconnected locally compact group and let $\{\nu_n\} \subset M^1(G)$ be such that $\nu_n \to \nu$ where $G(\nu)$ is compact and $\nu_n^{k_n}\nu_n' \to \mu$ for some sequence $\{\nu_n'\}$ in $M^1(G)$ such that $\nu_n\nu_n' = \nu_n'\nu_n$ for all n. Then given any neighbourhood U of e and an $\epsilon > 0$ there exists an l, such that for all large n, $\nu_n^m(G(\nu)I(\mu)U) > (1-\epsilon)^l$, for all $m \leq k_n$. In particular $A = \{\nu_n^m \mid m \leq k_n\}$ and $\{\nu_n'\}$ are relatively compact.*

Proof. As $G(\nu)$ is compact and $\nu^m \in T_\mu$, for all m, $\operatorname{supp}\nu \subset xI(\mu) = I(\mu)x$, for all $x \in \operatorname{supp}\nu$ (cf. [S4], Theorem 2.4). Therefore $G(\nu)I(\mu)$ is a compact group.

Let V be an open compact subgroup of G such that V is normalized by $G(\nu)I(\mu)$, and $V \subset U$. Since $\nu_n \to \nu$, $\nu_n(G(\nu)I(\mu)V) > 1-\epsilon$, for all large n. Let $V' = \{\lambda \mid \lambda(G(\nu)I(\mu)V) > (1-\delta)^{1/2}\}$ and let $U' = \{\lambda \mid \lambda(G(\nu)I(\mu)V) \geq 1-\delta\}$ for some positive $\delta < \epsilon$. Then $V'V' \subset U'$. Let $J = \{\lambda \in M^1(G) \mid \operatorname{supp}\lambda \subset G(\nu)I(\mu)V\}$. Clearly, J is a compact semigroup and $JV' = V'$. Let $\lambda \in U' \setminus V'$. If possible, suppose that $\lambda^n \in T_\mu$ for all n, then by Theorem 2.4 of [S4], $\operatorname{supp}\lambda \subset xI(\mu) = I(\mu)x$, for all $x \in \operatorname{supp}\lambda$. Since $\lambda \in U'$, $\operatorname{supp}\lambda \subset G(\nu)I(\mu)V$, i. e. $\lambda \in J \subset V'$, a contradiction. Hence for $\lambda \in T_\mu \cap U' \setminus V'$, there exists $n = n(\lambda)$, such that $\lambda^n \notin T_\mu$. By Lemma 2.1 of [S4], $T_\mu \cap U' \setminus V'$ is compact. As in the proof of Lemma 2.5 in [S4], one can find l, such that for any $\lambda \in T_\mu \cap U' \setminus V'$, μ cannot be expressed as $\mu = \lambda^l\lambda'$, for any λ' which commutes with λ.

Since $\nu_n \to \nu$, $\nu_n \in V'$, for all large n. Let such a large n be fixed. Then there exists $a_n > 1$, such that $\nu_n^m \in V'$, for all $m < a_n$ and $\nu_n^{a_n} \notin V'$. Therefore, $\nu_n^{a_n} \in V'V' \setminus V' \subset U' \setminus V'$. Let $b_n = k_n - la_n$ if $la_n \leq k_n$, otherwise $b_n = 0$. If $b_n = 0$, then $\nu_n^m \in (U')^l$, for all $m \leq k_n$. Therefore, for all large n, $\nu_n^m(G(\nu)I(\mu)V) \geq (1-\delta)^l$, and hence $\nu_n^m(G(\nu)I(\mu)U) > (1-\epsilon)^l$ for all $m \leq k_n$, as $V \subset U$ and $\delta < \epsilon$.

We now show that $b_n = 0$, for all large n. If $b_n \neq 0$ for infinitely many n, then $\nu_n^{la_n}\nu_n^{b_n}\nu_n' \to \mu$. Since $\{\nu_n^{a_n}\} \subset U' \setminus V'$, by Lemma 2.1 of [S4], $\{\nu_n^{a_n}\}$ is relatively compact and it has a limit point (say) λ, such that $\mu = \lambda^l\lambda'$, for some λ', which is a limit point of $\{\nu_n^{b_n}\nu_n'\}$, i.e. $\lambda \in T_\mu \cap U' \setminus V'$ and $\lambda\lambda' = \lambda'\lambda$. This is a contradiction to the choice of l as above.

Now it is enough to show that A is relatively compact as this would also imply that $\{\nu_n'\}$ is relatively compact. Let $\mu_n = \nu_n^{k_n}\nu_n'$. Then $\mu_n \to \mu$ and for each n, $\nu_n^m \in T_{\mu_n}$ for all $m \leq k_n$. Let $F = G(\nu)I(\mu)V$ for V as above. Then F is compact. Let $A' = \{\lambda \in M^1(G) \mid \lambda(F) \geq (1-\epsilon)^l/2\}$. Then from above, $A \subset A'$. Since $\mu_n \to \mu$, for every $\delta > 0$ such that $\delta < (1-\epsilon)^l/2$, there exists a compact set K_δ such that $\mu_n(K_\delta) > 1-\delta$ (cf. [H], Properties 1.2.20(2)). Therefore, for every n, m as above, there exists $x_{n,m}$ such that $\nu_n^m(K_\delta x_{n,m}) > 1-\delta$. Now since $A \subset A'$, the above implies that $x_{n,m} \in K_\delta^{-1}F$ and hence $\nu_n^m(K_\delta') > 1-\delta$, where $K_\delta' = K_\delta K_\delta^{-1}F$ which is a compact set. In particular A is relatively compact (cf. [H], 1.2.20). This completes the proof.

We now prove several results which will be needed to prove Theorem 2.1.

Lemma 2.3. *Let G be a locally compact first countable group and let $\{\mu_n\}$, $\{\lambda_n\}$ and $\{\nu_n\}$ be sequences in $M^1(G)$ such that $\lambda_n\nu_n = \nu_n\lambda_n = \mu_n \to \mu$ for some $\mu \in M^1(G)$. Then there exists a sequence $\{x_n\}$ such that $x_n \in N(\mu_n)$ for each n and $\{\lambda_n x_n\}$ and $\{x_n\lambda_n\}$ are relatively compact and all its limit points are supported on* supp μ.

The proof is quite similar to Proposition 1.2 in [DM] and Theorem 2.2 in ch. III of [P].

Proof. For any integer $r > 0$ there exists a compact set $K_r \subset \operatorname{supp}\mu$ such that $\mu(K_r) > 1 - 4^{-(r+1)}$. Without loss of generality, we may assume that $K_r \subset K_{r+1}$ for all r. Let $\{U_r\}$ be a neighbourhood basis of e in G such that each U_r is relatively compact,

$U_{r+1} \subset U_r$ for all r and $\cap_r U_r = \{e\}$. Since $\mu_n \to \mu$, there exists $n_r \in \mathbf{N}$, such t $\mu_n(K_rU_r) > 1 - 2^{-r}$ and $\mu_n(U_rK_r) > 1 - 2^{-r}$, for all $n \geq n_r$. Let $E^r_n = \{x \in G \mid$ $(K_rU_rx^{-1}) > 1 - 2^{-r}\}$ and let $F_n = \cap_{\{r|n_r \leq n\}} E^r_n$.

A simple calculation as in Theorem 2.2 of ch. III in [P] shows that for $n \geq$ $\nu_n(G \setminus E^r_n) \geq 2^{-(r+2)}$ and hence $\nu_n(G \setminus F_n) \geq 1/4$. Similarly, we define $B^r_n = \{x \in$ $\lambda_n(x^{-1}U_rK_r) > 1 - 2^{-r}\}$ for any r and $C_n = \cap_{\{r|n_r \leq n\}} B^r_n$. Then $\nu_n(G \setminus C_n) \geq 1/4$.

Therefore $\nu_n(F_n \cap C_n) \geq 1/2$. For each n, we pick $x_n \in F_n \cap C_n \cap \operatorname{supp} \nu_n$ as it nonempty, $x_n \in \operatorname{supp} \nu_n \subset N(\mu_n)$. Then for any $r > 0$, $\lambda_n x_n(K_rU_r) > 1 - 2^{-r}$ and x $(U_rK_r) > 1 - 2^{-r}$ for all $n \geq n_r$ and hence by tightness criterion, $\{\lambda_n x_n\}$ and $\{x_n\lambda_n\}$ tight. Also, since $K_r \subset \operatorname{supp} \mu$ for all r, $\lambda_n x_n((\operatorname{supp} \mu)U_r) > 1 - 2^{-r}$, for all $n \geq n_r$. Si $\cap_r U_r = \{e\}$, it easily follows that for any limit point λ of $\{\lambda_n x_n\}$, $\operatorname{supp} \lambda \subset$ sup Similarly, the limit points of $\{x_n\lambda_n\}$ are also supported on $\operatorname{supp} \mu$.

Lemma 2.4. *Let G be a locally compact group and let $\mu_n \to \mu$ in $M^1(G)$. Let B b subgroup which centralizes an open subgroup H containing* supp μ. *Then the follow hold:*

1. *For any sequence $\{x_n\}$ in B, $\{x_n^{-1}\mu_n x_n\}$ is relatively compact and it converges to*
2. *Let $\mu_n = \lambda_n\nu_n = \nu_n\lambda_n$, for all n. If for sequences $\{x_n\}$ and $\{y_n\}$ in B, $\{x_n\lambda_n y_n$ relatively compact then its limit points belong to T_μ; in particular if $\lambda_n a_n \to \lambda$ some $\{a_n\} \subset B$, then $\lambda \in T_\mu$ and the limit points of $\{x_n\lambda_n y_n\}$ are of the form $z\lambda =$ for some $z, z' \in Z(\mu)$.*

Proof. Let U be any open set contained in H and let $K \subset \operatorname{supp} \mu$ be any compact set s that $\mu(K) > 0$. Then given $0 < \epsilon < \mu(K)$, there exists N such that $\mu_n(KU) > \mu(K) -$ all $n > N$. Since x_n centralizes KU, $x_n^{-1}\mu_n x_n(KU) = \mu_n(KU) > \mu(K) - \epsilon$. Since thi true for all K and U as above, $\{x_n^{-1}\mu_n x_n\}$ is relatively compact and it converges to μ. Le be a limit point of a relatively compact sequence $\{\lambda'_n = x_n\lambda_n y_n\}$, where $x_n, y_n \in B$. S $\{x_n\mu_n x_n^{-1}\}$ converges to μ, $\{y_n^{-1}\nu_n x_n^{-1}\}$ is relatively compact and there exists a limit p ν' of it such that $\lambda'\nu' = \mu$. Also, $\nu'\lambda'$ is a limit point of $\{y_n^{-1}\mu_n y_n\}$, which converges t Therefore, $\nu'\lambda' = \mu$ and hence $\lambda' \in T_\mu$. Now suppose $\lambda_n a_n \to \lambda$, $\{a_n\} \subset B$, then f above $\lambda \in T_\mu$. Therefore, $\lambda = x\beta$, for some $x \in N(\mu)$ and β supported on $G(\mu) \subset H$. T $x^{-1}\lambda_n a_n \to \beta$. Let K' be any compact subset in H such that $\beta(K') > 0$. Then for any subset U contained in H, $\lambda_n a_n(xK'U) = z_n\lambda'_n(xK'U)$, where $z_n = xy_n^{-1}a_n x^{-1}x_n^{-1} \in Z$ as $B \subset Z(\mu)$ and $x \in N(\mu)$ which normalizes $Z(\mu)$. Since this is true for all n an compact subsets K' of $\operatorname{supp} \beta$ it implies that $\{z_n\}$ is relatively compact in $Z(\mu)$. There $\lambda' = z\lambda$, for λ' as above, where z is a limit point of $\{z_n^{-1}\}$. Now since $\lambda \in T_\mu$ and $z \in Z$ $z\lambda = zx\beta = xz'\beta = x\beta z' = \lambda z'$, where $z' = x^{-1}zx \in Z(\mu)$.

PROPOSITION 2.5

Let G be a locally compact group and let C be a closed normal (real) vector subgrou G. Suppose that $\{\mu_n\} \subset M^1(G)$ be a sequence such that $\mu_n \to \mu$, the closed subg (say) H, generated by the centralizer $Z(C)$ of C and supp μ, *is open in G. Suppose there exists a sequence $\{x_n\}$ in C such that $\{x_n^{-1}\mu_n x_n\}$ is relatively compact. $\{x_n\}/(Z(\mu) \cap C)$ is relatively compact. In particular $I_\mu \cap C = Z(\mu) \cap C$.*

Proof. Suppose $C \subset Z(\mu)$ then there is nothing to prove. Now let $V = Z(\mu) \cap C$, w is a proper closed subgroup of C. Since C is normal in G; for any $x \in G$, $i_x : C$ -

$i_x(c) = xcx^{-1}$ for all $c \in C$, is a continuous homomorphism of C and hence it is a linear operator in $M(d, \mathbf{R})$, where d is such that C is isomorphic to $\mathbf{R}^d$. Now $V = Z(\mu) \cap C = \cap_{x \in \operatorname{supp} \mu} \ker(i_x)$ and hence V is a (possibly trivial) vector subspace and $C = V \times W$, a direct product. Now for each n, $x_n = z_n + y_n$, where $z_n \in V$ and $y_n \in W$. Let $\mu'_n = z_n^{-1} \mu_n z_n$, for each n. Since V centralizes $G(\mu)$ and hence H which is open, by Lemma 2.4, $\mu'_n \rightarrow \mu$.

Now it is enough to show that $\{y_n\}$ is relatively compact. If possible, suppose it has a subsequence, denote it by $\{y_n\}$ again, which is divergent, i.e. it has no convergent subsequence. We know that $\{y_n^{-1} \mu'_n y_n = x_n^{-1} \mu_n x_n\}$ is relatively compact. Passing to a subsequence if necessary, we get that $y_n / ||y_n|| \rightarrow y$ in W, where $||\ ||$ denotes the usual norm in the vector space C. Since $\mu'_n \rightarrow \mu$, arguing as in Proposition 9 in [M1], we get that $G(\mu) \subset Z(y)$, the centralizer of y in G, a contradiction as $y \notin Z(\mu) \cap C = V$, for $y \in W$ and $||y|| = 1$. Therefore, $\{y_n\}$ is relatively compact. If $x \in I_\mu$ then $x \mu x^{-1} = \mu$ therefore, $(I_\mu \cap C)/(Z(\mu) \cap C)$ is a compact group, but since C and $Z(\mu) \cap C$ are both vector groups so is $C/(Z(\mu) \cap C)$ and hence has no nontrivial compact subgroups. Therefore, $I_\mu \cap C = Z(\mu) \cap C$.

PROPOSITION 2.6

Let G and C be as above. Let $\{\nu_n\}$ be a relatively compact sequence in $M^1(G)$ such that $\nu_n^{k_n} \rightarrow \mu$ and the closed subgroup (say) H, generated by the centralizer $Z(C)$ of C and $\operatorname{supp} \mu$, *is open in G. Let $A = \{\nu_n^m \mid m \leq k_n\}$. If A/C is relatively compact then so is A.*

Proof. Let A/C be relatively compact. If possible, suppose that A is not relatively compact. That is, there exists a subsequence of $\{\nu_n\}$, denote it by same notation, such that $\{\nu_n^{l(n)}\}$ is divergent, where $l(n) < k_n$ for all n. Passing to a subsequence if necessary, we get that $\nu_n \rightarrow \nu$ (say). Let $\pi : G \rightarrow G/G^0$ be the natural projection. Since $\{\pi(\nu)^n \mid n \in \mathbf{N}\} \subset \overline{\pi(A)}$ which is compact, $G(\pi(\nu))$ is compact. Also, since $\{\nu^n \mid n \in \mathbf{N}\} \subset T_\mu$, by Theorem 2.4 of [S4], $\operatorname{supp} \nu \subset xI(\mu) = I(\mu)x$, for any $x \in \operatorname{supp} \nu$. Since A/C is relatively compact, there exists a sequence $\{x_{n,m}\}$ in C such that $\{\nu_n^m x_{n,m}\}$ is relatively compact and $\{x_{n,l(n)}\}$ is divergent. Also since $\nu_n^{k_n} \rightarrow \mu$ (resp. $\nu_n^{k_n+1} \rightarrow \mu\nu = \nu\mu$) the above implies $\{x_{n,m}^{-1} \nu_n^{k_n - m}\}$ (resp. $\{x_{n,m}^{-1} \nu_n^{k_n+1-m}\}$) and hence $\{x_{n,m}^{-1} \nu_n^{k_n} x_{n,m}\}$ (resp. $\{x_{n,m}^{-1} \nu_n^{k_n+1} x_{n,m}\}$) is relatively compact. Now by Proposition 2.5, $\{x_{n,m}\}/(Z(\mu) \cap C)$ (resp. $\{x_{n,m}\}/(Z(\mu\nu) \cap C)$) is relatively compact. As $Z(\mu) \cap Z(\mu\nu) = Z(\mu) \cap Z(\nu)$, the above implies that $\{x_{n,m}\}/(Z(\mu) \cap Z(\nu) \cap C)$ is relatively compact. Without loss of generality we may assume that $\{x_{n,m}\} \subset C' = Z(\mu) \cap Z(\nu) \cap C$, which is a vector group centralizing $G(\nu)$ and H. Therefore, $H' = Z(C')$ contains H and hence it is an open subgroup in G containing $\operatorname{supp} \mu$ and $\operatorname{supp} \nu$. We may also assume that $x_{n,1} = x_{n,k_n} = e$ for every n as $\{\nu_n\}$ and $\{\nu_n^{k_n}\}$ are relatively compact.

Let $n \in \mathbf{N}$ and let $1 \leq m \leq k_n$. From Theorem 2.2, $\nu_n^m(H') > \delta > 0$ and hence $\nu_n^m x_{n,m}(H') > \delta$. Since $\{\nu_n^m x_{n,m}\}$ is relatively compact, there exists a compact set $L \subset H'$, such that $\nu_n^m x_{n,m}(L) > \delta/2$. Let $0 < \epsilon < \min\{\delta/2, 1/4\}$. There exists a compact set $K \subset \operatorname{supp} \mu$ such that $\mu(K) > 1 - \epsilon$. Let $U \subset H'$ be such that U is open in G. Then there exists N, such that for all $n \geq N$, $\nu_n^{k_n}(KU) > 1 - \epsilon$. Let $n \geq N$ and let $1 \leq m \leq k_n$. Then there exists $\{y_{n,m}\} \subset G$, such that $\nu_n^m y_{n,m}(KU) > 1 - \epsilon$. Since $\epsilon < \delta/2$, $KUy_{n,m}^{-1} \cap Lx_{n,m}^{-1} \neq \emptyset$. That is, $y_{n,m}^{-1} \in K' x_{n,m}^{-1}$, where $K' = (KU)^{-1} L \subset H'$ and hence $\nu_n^m x_{n,m}(K_1) > 1 - \epsilon$ and each $x_{n,m}$ commutes with all the elements of $K_1 = KUK' \subset H'$. Now for $m, l < k_n$ such that $m + l \leq k_n$, we get that $\nu_n^{m+l}(K_1 x_{n,m}^{-1} K_1 x_{n,l}^{-1}) \geq (1 - \epsilon)^2$. Since

$\nu_n^{m+l}(K_1x_{n,m+l}^{-1}) > 1-\epsilon$ and $\epsilon < 1/4$, we get that $K_1x_{n,m+l}^{-1} \cap K_1x_{n,m}^{-1}K_1x_{n,l}^{-1} \neq \emptyset$. Therefe
$x_{n,m}x_{n,l}x_{n,m+l}^{-1} \in K_1^2K_1^{-1} \cap C'$. Since C' is a vector group, C' is strongly root compact
3.1.12 of [H] and hence by the definition of strong root compactness (see 3.1.10 of [H
there exists a compact subset K'' such that $x_{n,m} \in K''$, for all m, n. This is a contradict
to the fact that $\{x_{n,l(n)}\}$ is divergent. Therefore A is relatively compact. This completes
proof.

Let $\lambda \in M^1(G)$. For some $\alpha = (r_1, l_1, \ldots, r_m, l_m)$, where $m \in \mathbf{N}$, and $r_i, l_i \in \mathbf{N} \cup \{$
fixed, let $\alpha(\lambda) = \lambda^{r_1}\tilde{\lambda}^{l_1} \ldots \lambda^{r_m}\tilde{\lambda}^{l_m}$, where $\lambda^0 = \tilde{\lambda}^0 = \delta_e$. For any such α, the map $\lambda \mapsto \alpha$
on $M^1(G)$ is continuous. Also, $G(\lambda) = \overline{\cup_\alpha \operatorname{supp} \alpha(\lambda)}$ (over all possible choices of α
above).

Proof of theorem 2.1. Without loss of generality we may assume that $\{\nu_n\}$ is converge
that is $\nu_n \to \nu$ (say). From the hypothesis, $G(\pi(\nu))$ is compact, and hence by Theorem
$\pi(A)$ is relatively compact. It is enough to show that A is relatively compact as
Theorem 3.6 of [S1], there exists x such that $x\mu = \mu x$ is embeddable.

Step 1. Let K be the maximal compact normal subgroup of G^0, then K is characteristi
G^0 and hence normal in G. Since A is relatively compact if and only if its image on G
is relatively compact, without loss of generality we may assume that G^0 has no nontri
compact normal subgroups. In particular, G^0 is a Lie group. Let L be any open
projective subgroup of G. Let M be any compact normal subgroup of L such that L/M
Lie group, then $G^0M = G^0 \times M$, a direct product, as both G^0 and M are normal in L
$G^0 \cap M = \{e\}$. Moreover $H = G^0M$ is an open subgroup in G. Since $I(\mu)$ is comp
without loss of generality, we may assume that $I(\mu)$ normalizes H.

Step 2. Now we prove the assertion by induction on the dimension of the Lie group
Let $\dim G^0 = 0$. Then G is totally disconnected and the assertion follows from abo
Now suppose that for any $k > 1$, the assertion holds for G such that $\dim G^0 < k$. Now
$\dim G^0 = k$.

Step 3. Suppose that there exists a subsequence of $\{\nu_n\}$, denote it by $\{\nu_n\}$ again, s
that $\{\nu_n^{l_n}\}$ is divergent. By Theorem 1.2.21 of [H], there exists a sequence $\{x_n\} \subset G$, s
that $\{\nu_n^{l_n}x_n\}$ and hence $\{x_n^{-1}\nu_n^{k_n-l_n}\}$ and $\{x_n^{-1}\nu_n^{k_n}x_n\}$ are relatively compact and we
assume that $\{x_n\}$ is divergent. Since $\pi(A)$ is relatively compact, $\{\pi(x_n)\}$ is relati
compact in G/G^0 and hence we may choose $\{x_n\}$ to be in G^0.

Without loss of generality we may assume that the subgroup N, as in the hypothesi
the nilradical. Suppose that N is trivial. Then G^0 is a connected semisimple gro
Suppose that the center of G is trivial. Then G^0 is an almost algebraic subgrou
$GL_n(\mathbf{R})$. By Propositions 4–6 of [M1], there exists a proper closed subgroup G' of
such that given any relatively compact sequence $\{z_n\} \subset G^0$, the limit points of $\{x_nz_nx$
are contained in G'. Now since $G^0 \subset G(\mu)$, there exists an $x \in G(\mu) \cap (G^0 \setminus G')$. Si
$G^0 \setminus G'$ is open in G^0, there exists a set U which is open in G^0 such that $x \in$
$\overline{U} \subset G^0 \setminus G'$ and $\overline{U}$ is compact. Then for some $\alpha = (r_1, l_1, \ldots, r_m, l_m)$, we have
$\alpha(\mu)(UM) = \delta > 0$, as $UM = U \times M$ is open in G, for a compact group M as ab
Since $\alpha(\nu_n^{k_n}) \to \alpha(\mu)$, $\alpha(\nu_n^{k_n})(UM) > \delta/2$ for all large n. Now since $\{x_n^{-1}\nu_n^{k_n}x_n$
relatively compact, so is $\{x_n^{-1}\alpha(\nu_n^{k_n})x_n\}$. Therefore, there exists a compact set K such
$(x_n^{-1}\alpha(\nu_n^{k_n})x_n)(K) = \alpha(\nu_n^{k_n})(x_nKx_n^{-1}) > 1-\delta/4$ for all n. From the above equa
$UM \cap x_nKx_n^{-1} \neq \emptyset$, for all large n. Therefore, there exists a sequence $\{a_n\} \subset K$, such

for all large n, $x_na_nx_n^{-1} = u_nv_n$, where $u_n \in U$ and $v_n \in M$ and hence $x_na_nv_n^{-1}x_n^{-1} = u_n$. For each n, put $z_n = a_nv_n^{-1}$, then since $x_n, u_n \in G^0$, $z_n \in G^0$. Also $\{z_n\} \subset KM$ is relatively compact. Therefore the limit points of $\{x_nz_nx_n^{-1} = u_n\}$ belong to G'. But $\{u_n\} \subset U$ and $\overline{U} \subset G^0 \setminus G'$, a contradiction. Therefore, A is relatively compact.

Step 4. Now suppose G^0 is a semisimple group with nontrivial center Z. Then Z is a discrete group normal in G and $Z = \mathbf{Z}^n$, for some n, as we have assumed that G^0 has no nontrivial compact subgroups normal in G. The action of G on $\mathbf{Z}^n$ extends to the action of G on $\mathbf{R}^n$. Therefore, we can form a semidirect product $G_1 = G \cdot \mathbf{R}^n$. Let $D = \{(z,z) \mid z \in \mathbf{Z}^n\}$. Then D is normal in G_1. Now G can be embedded as a closed subgroup in $G_2 = G_1/D$ and $G_2^0 = (G^0 \times \mathbf{R}^n)/D$. It is easy to see that the center C of G_2^0 is isomorphic to $\mathbf{R}^n$. Also, C is normal in G_2 and G_2^0/C is a semisimple group with trivial center. Let $\psi : G_2 \to G_2/C$ be the natural projection. It is easy to see that $G(\psi(\mu))$ contains G_2^0/C, the connected component in G_2/C, and hence by the above argument, $\psi(A)$ is relatively compact. Since H centralizes G^0 in G, $H' = H \times \mathbf{R}^n = G^0 \times M \times \mathbf{R}^n$ is open in G_1 and hence H'/D is an open subgroup in G_2 which centralizes C. Now the assertion in this case follows from Proposition 2.6.

Step 5. Now suppose the nilradical N of G is nontrivial. Let C be the center of N. Since G^0 does not contain any compact subgroups normal in G, C is a vector group, i.e. C is isomorphic to $\mathbf{R}^n$, for some n. Since N is normal in G, so is C. Let $\psi : G \to G/C$ be the natural projection. Then since $\dim G^0/C < k$, we have that $\psi(A)$ is relatively compact. Now since C centralizes $N \times M$, M as above, and $\operatorname{supp}\mu$ and N generate a subgroup containing G^0, the assertion follows from Proposition 2.6.

Remark. Theorem 2.1 continues to hold if the conditions in it are replaced by the following: $\nu_n^{k_n} \to \mu$, the closed subgroup generated by $\operatorname{supp}\mu$ and N is whole of G (where N is as in the hypothesis of the theorem), $\{\nu_n\}/G^0$ is relatively compact and for any limit point ν of it, $G(\nu)$ is compact in G/G^0. For the proof, A/G^0 is relatively compact by Theorem 2.2 and the first three steps of the proof of the above theorem will apply word for word. Also, for a normal subgroup C in steps 4 and 5 above, $Z(\mu) \cap C$ is a central vector group in G by the above condition and hence by Proposition 2.5, the relative compactness of A/C implies that of $A/(Z(\mu) \cap C)$. Therefore A is relatively compact by Lemma 3.2 of [S1]. The above variation of Theorem 2.1 generalizes Theorem 3.1 of [S1].

3. Limit theorems on discrete linear groups over R

Theorem 3.1. *Let G be a discrete linear group over* $\mathbf{R}$. *Let $\{\nu_n\}$ be a sequence in $M^1(G)$ such that $\nu_n^{k_n} \to \mu$, for some $\mu \in M^1(G)$ and some unbounded sequence $\{k_n\}$ in* $\mathbf{N}$. *Then there exists $x \in I_\mu$, such that $x\mu$ is embeddable.*

Remark. So far, in the limit theorems on discrete groups, one had either the support condition or the infinitesimality condition imposed (see [S4] and Theorem 2.2 above). The above theorem gives a generalization of Theorems 1.5, 1.7(1) of [S4] for this special class of discrete groups. It also generalizes Theorem 1.2 of [DM3]. One cannot get an embedding of μ itself or an element x as above to be infinitely divisible as in $G = GL(1, \mathbf{Z}) = \{-1, 1\}$, for $x = -1$, $\delta_x = \delta_x^{2n+1}$, for all n, but δ_x is clearly not infinitely divisible and hence not embeddable.

To prove the theorem, we need preliminary results.

Lemma 3.2. *Let V be a finite dimensional vector space over* $\mathbf{R}$. *Let* $\{\tau_n\}$ *be a diverg sequence in* $GL(V)$ *such that for some* $b > 0$, $|\det(\tau_n)| \geq b$ *for all n. Then there exist proper subspace W of V such that the following holds: if* $\{\mu_n\} \subset M^1(V)$ *is such t* $\mu_n \to \mu$ *and* $\{\tau_n(\mu_n)\}$ *is relatively compact, then* $\operatorname{supp}\mu \subset W$.

The proof of the Lemma is exactly same as the proof of Proposition 3.2 in [S2] usi Proposition 1.4 in [DM1]. We will not repeat it here.

PROPOSITION 3.3

Let G be a discrete linear group over $\mathbf{R}$ *and let* $\{\mu_n\}$ *be a sequence converging to* μ $M^1(G)$. *Let* $\lambda_n \in T_{\mu_n}$ *for each n. Then there exist sequences* $\{z_n\}$ *and* $\{z'_n\}$ *in* $Z(\mu)$ *s that* $\{\lambda_n z_n\}$ *and* $\{z'_n\lambda_n\}$ *are relatively compact and all their limit points belong to* T

Proof. There exists a sequence $\{\lambda'_n\}$ in $M^1(G)$, such that $\lambda_n\lambda'_n = \lambda'_n\lambda_n = \mu_n \to \mu$. Lemma 2.3, there exists a sequence $\{x_n\}$ in G such that $\{\lambda_n x_n\}$ and $\{x_n\lambda_n\}$ are relativ compact and all its limit points are supported on $\operatorname{supp}\mu$. Therefore, by Theorem 1.2.21 [H], $\{x_n^{-1}\lambda'_n\}$, $\{\lambda'_n x_n^{-1}\}$ and hence $\{x_n^{-1}\mu_n x_n\}$ and $\{x_n\mu_n x_n^{-1}\}$ are all relatively compact ν is a limit point of $\{x_n^{-1}\lambda'_n\}$ then there exists a limit point λ of $\{\lambda_n x_n\}$ such that $\lambda\nu =$ Since $\operatorname{supp}\lambda \subset \operatorname{supp}\mu = \overline{\operatorname{supp}\lambda\operatorname{supp}\nu}$, $\operatorname{supp}\nu \subset G(\mu)$. Therefore all the limit points $\{x_n^{-1}\lambda'_n\}$ and also of $\{x_n^{-1}\mu_n x_n\}$ are supported on $G(\mu)$.

Similarly, the limit points of $\{x_n\mu_n x_n^{-1}\}$ are also supported on $G(\mu)$, and $\{x_n^{-1}\alpha(\mu_n)$ and $\{x_n\alpha(\mu_n)x_n^{-1}\}$ are relatively compact and their limit points are supported on $G(\mu)$, any α (where α and $\alpha(\mu_n)$ are defined as in § 2). Also, for any $\epsilon > 0$, there exist compact set K such that $(x_n^{-1}\mu_n x_n)(K) > 1-\epsilon$ for all n. Now for any limit point γ $\{x_n^{-1}\mu_n x_n\}$, $\gamma(K \cap G(\mu)) > 1-\epsilon$. Therefore it is easy to see that $(x_n^{-1}\mu_n x_n)(K') > 1$ for all large n, where $K' = K \cap G(\mu)$.

We know that $G \subset GL(n,\mathbf{R}) \subset M(n,\mathbf{R})$. Let V_μ be the vector space generated $G(\mu)$ in $M(n,\mathbf{R})$. There exists a finite set $\{y_1,\ldots,y_m\} \subset G(\mu)$ such that $\{y_1,\ldots,$ generates V_μ. Since $G(\mu) = \cup_\alpha \operatorname{supp}\alpha(\mu)$, where α and $\alpha(\mu)$ are as defined in § 2, th exist $\alpha_1,\ldots,\alpha_m$ such that $y_i \in \operatorname{supp}\alpha_i(\mu)$, for each i. Therefore, as G is discrete, some $\delta > 0$, $\alpha_i(\mu)\{y_i\} > \delta$ for all i. Since $\alpha_i(\mu_n) \to \alpha_i(\mu)$, there exists N such $\alpha_i(\mu_n)\{y_i\} > \delta/2$, for all $n > N$, for all i.

Now since $\{x_n^{-1}\alpha_i(\mu_n)x_n\}$ is relatively compact and all its limit points are supported $G(\mu)$, arguing as above we can get a compact set $K_1 \subset G(\mu)$, such that $(x_n^{-1}\alpha_i(\mu_n)$ $(K_1) > 1-\delta/2$ for all i, for all large n. That is, $\alpha_i(\mu_n)(x_nK_1x_n^{-1}) > 1-\delta/2$ for all i, all large n. Therefore, $y_i \in x_nK_1x_n^{-1}$, or $x_n^{-1}y_ix_n \in K_1 \subset G(\mu) \subset V_\mu$, for all large n. Si V_μ is generated by $\{y_1,\ldots,y_m\}$, the above implies that $x_n^{-1}V_\mu x_n = V_\mu$, for all large

Let $\tilde{G}$ be the Zariski closure of G in $GL(d,\mathbf{R})$ and let $N(V_\mu)$ (resp. $Z(V_\mu)$) be normaliser (resp. centraliser) of V_μ in $\tilde{G}$. Then $Z(V_\mu)$ and $N(V_\mu)$ are algebraic subgro of $\tilde{G}$ and $Z(V_\mu)$ is normal in $N(V_\mu)$. Now $N(V_\mu)$ acts on V_μ linearly and the $\rho : N(V_\mu) \to GL(V_\mu)$ is a rational morphism, as in the proof of Theorem 3.2 in [DM Therefore, the image of ρ, $\operatorname{Im}(\rho)$ is closed in $GL(V_\mu)$ and since $\ker\rho = Z($ $\rho' : N(V_\mu)/Z(V_\mu) \to \operatorname{Im}\rho$ is a topological isomorphism.

We know that $\{x_n\} \subset N(V_\mu)$. Now if possible, suppose that $\{x_n\}/Z(V_\mu)$ is relatively compact. Going to a subsequence if necessary, without loss of generality, may assume that $\{x_n\}/Z(V_\mu)$ is divergent; i.e. it has no convergent subsequence, and for some $\delta > 0$, either $|\det\rho'(x_nZ(V_\mu))| = |\det\rho(x_n)| > \delta$ or $|\det\rho'(x_n^{-1}Z(V_\mu))| > \delta$.

Suppose $|\det\rho'(x_nZ(V_\mu))| = |\det\rho(x_n)| > \delta$ for all n. By Lemma 3.2, there exists a proper subspace W of V_μ such that $\operatorname{supp}\alpha(\mu) \subset W$ for all α, as $\alpha(\mu_n) \to \alpha(\mu)$ and $\{(\rho'(x_nZ(V_\mu)))(\alpha(\mu_n)) = x_n\alpha(\mu_n)x_n^{-1}\}$ is relatively compact. This implies that $G(\mu) = \cup_\alpha \operatorname{supp}\alpha(\mu) \subset W$, a contradiction as $G(\mu)$ generates V_μ and W is a proper subspace.

Now suppose $|\det\rho'(x_n^{-1}Z(V_\mu))| > \delta$. Now using the fact that for every α, $\{(\rho'(x_n^{-1}Z(V_\mu)))(\alpha(\mu_n)) = x_n^{-1}\alpha(\mu_n)x_n\}$ is relatively compact and replacing $\{x_n\}$ by $\{x_n^{-1}\}$ in the above argument we arrive at a contradiction. Therefore, $\{x_n\}/Z(V_\mu)$ is relatively compact.

Clearly, $N(V_\mu) \cap G$ normalizes $Z(V_\mu)$. Let $H = (N(V_\mu) \cap G)Z(V_\mu)$ and let $x \in H$. Then $x(V_\mu \cap G)x^{-1} = V_\mu \cap G$. Let G_μ be the closed subgroup generated by $V_\mu \cap G$ in G. Then $G(\mu) \subset G_\mu$ and x normalizes G_μ. Therefore $\overline{H}$ is a closed subgroup (in $\tilde{G}$) normalizing G_μ. Since G_μ is discrete, the connected component $\overline{H}^0$ of $\overline{H}$, centralizes G_μ and hence $\overline{H}^0 \subset Z(V_\mu) \subset H$ as V_μ is generated by $G(\mu)$ and $G(\mu) \subset G_\mu$. Since $\overline{H}^0$ is open in $\overline{H}$, it follows that H is open in $\overline{H}$. That, is $\overline{H} = H$ and H is a closed subgroup. This implies that $((N(V_\mu) \cap G)Z(V_\mu))/Z(V_\mu)$ is isomorphic to $(N(V_\mu) \cap G)/(Z(V_\mu) \cap G)$. Therefore $\{x_n\}/Z(\mu)$ is relatively compact as $Z(V_\mu) \cap G = Z(\mu)$. Therefore $x_n = z_na_n = a_nz_n'$, for some relatively compact sequence $\{a_n\}$ in G and some sequences $\{z_n\}$ and $\{z_n'\}$ in $Z(\mu)$. Also, since $\{\lambda_nx_n\}$ and $\{x_n\lambda_n\}$ are relatively compact, so are $\{\lambda_nz_n\}$ and $\{z_n'\lambda_n\}$, and all their limit points belong to T_μ by Lemma 2.4.

Proof of Theorem 3.1. Since $\nu_n^{k_n} \to \mu$, by Proposition 3.3, for any m, there exists a sequence $\{z_{m,n}\} \subset Z(\mu)$ such that $\{\nu_n^mz_{m,n}\}$ is relatively compact. Passing to a subsequence if necessary, without loss of generality, we may assume that $\{\nu_nz_{1,n}\}$ is convergent, with the limit ν. Then $\nu \in T_\mu$ by Lemma 2.4. Also, for any m, $\{\nu_{m,n} = z_{m,n}^{-1}\nu_nz_{m+1,n}\}$ is relatively compact and its limit points are of the form $z\nu = \nu z'$, for some $z, z' \in Z(\mu)$ (cf. Lemma 2.4).

Suppose for any fixed m, the limit points of $\{\nu_n^mz_{m,n}\}$ are of the form ν^mz_m for some $z_m \in Z(\mu)$. Then combining the above two statements, we get that the limit points of $\{\nu_n^{m+1}z_{m+1,n}\}$ have the form $\nu^mz_mz\nu = \nu^{m+1}z_{m+1}$, for some $z_{m+1} \in Z(\mu)$. By induction, for any m, the limit points of $\{\nu_n^mz_{m,n}\}$ are of the form ν^mz_m, for some $z_m \in Z(\mu)$. Moreover, by Lemma 2.4, $\nu^m \in T_\mu$, as it is a limit point of $\{\nu_n^mz_{m,n}z_m^{-1}\}$, for each m. Also $\operatorname{supp}\nu \subset N(\mu)$.

Now by Proposition 3.3, $\{\nu^n\}/Z(\mu)$ is relatively compact. Therefore $G(\nu)Z(\mu)/Z(\mu)$ is compact and hence finite of order (say) s, as G is discrete. Let $x \in \operatorname{supp}\nu$, then $x^s \in Z(\mu)$. Let $\beta = \nu^sz = z\nu^s$ for $z = x^{-s} \in Z(\mu)$. Then $e \in \operatorname{supp}\beta$ and $\beta^n \in T_\mu$ for all n. Therefore by Theorem 2.4 of [S4], $\operatorname{supp}\beta \subset I(\mu)$ and, furthermore, $\beta^n \to \omega_H$, where $H = G(\beta) \subset I(\mu)$. Hence $\operatorname{supp}\nu \subset xH \cap Hx$. Therefore $x\mu = \nu\mu = \mu\nu = \mu x$, and hence $x \in I_\mu$, for all $x \in \operatorname{supp}\nu$.

Now we show that μ has a shift which is infinitely divisible. Let $l \in \mathrm{N}$ be fixed. Let $a_n = [k_n/l]$ and $b_n = k_n - la_n$. Then for any, $m \leq l$, $\nu_n^{ma_n}\nu_n^{k_n-ma_n} \to \mu$ and hence there exist sequences $\{z_{m,n}'\}$ in $Z(\mu)$ such that $\{\nu_n^{ma_n}z_{m,n}'\}$ are relatively compact. Arguing as above, we get that the limit points of $\{\nu_n^{la_n}z_{l,n}'\}$ are of the form λ_l^lz, for some $z \in Z(\mu)$ and some limit point λ_l of $\{\nu_n^{a_n}z_{1,n}'\}$. Let $r \in \mathrm{N}$ be fixed. Since $a_n \to \infty$, for large n such that $a_n > r$, $\nu_n^{a_n}z_{1,n}' = \nu_n^rz_{r,n}\gamma_n$, where $\{\gamma_n = z_{r,n}^{-1}\nu_n^{a_n-r}z_{1,n}'\}$ which is relatively compact and hence $\lambda_l = \nu^r\gamma$ for some γ. Also $\nu_n^{a_n}z_{1,n}' = \nu_n^{a_n-r}\nu_n^rz_{1,n}'$. By Proposition 3.3, there exists $\{y_n\}$ in $Z(\mu)$ such that $\{\nu_n^{a_n-r}y_n\}$ is relatively compact and hence so is $\{y_n^{-1}\nu_n^rz_{1,n}'\}$

and all its limit points are of the form $z'\nu^r$ for some $z' \in Z(\mu)$ (cf. Lemma 2.4). T
is, $\lambda_l = \gamma'\nu^r$, for some γ' and hence for $\beta = \nu^s z = z\nu^s$ defined as above, $\lambda_l =$
$= \beta''\beta^r$ for some β', β''. Since this is true for all r, $\omega_H \in T_{\lambda_l}$. That is, $\lambda_l\omega_H = \omega_H\lambda_l$
for all l.

For each n, let $z_n = (z'_{l,n})^{-1}$. Then the sequence $\{z_n\nu_n^{b_n}\}$ is relatively compact. Cle
$b_n < l$ for all n. Let $r < l$ be such that $r = b_{n_k}$ for infinitely many n_k. Then clearly
limit points of $\{z_n\nu_n^{b_n}\}$ are contained in $\{g\nu^r \mid r < l, g \in Z(\mu)\}$ and hence if ρ_l is
such limit point then $\operatorname{supp}\rho_l \subset G(\nu)Z(\mu) \subset I_\mu$ and $\rho_l\omega_H = x_l\omega_H$ (resp. $\omega_H\rho_l = \omega$
where $x_l^s \in Z(\mu)$, where s is the cardinality of $G(\nu)Z(\mu)/Z(\mu)$.

Combining the above we get that $\mu = \lambda_l^l\rho_l = \lambda_l^l\omega_H\rho_l = \lambda_l^l x_l (= x_l\lambda_l^l)$ for s
$x_l \in \operatorname{supp}\rho_l \subset I_\mu$, for each l. That is, μ is weakly infinitely divisible. As $\lambda_l \in$
$\operatorname{supp}\lambda_l \subset y_lG(\mu)$ for some $y_l \in \operatorname{supp}\lambda_l \subset N(\mu)$. Since for each l, $\mu = \lambda_l^l x_l$ and $x_l \in$
$Z(\mu)$, we get that $y_l^l \in G(\nu)Z(\mu)G(\mu)$. Hence $(y_l)^{ls} \in G(\mu)Z(\mu)$, as $G(\nu)Z(\mu)/Z(\mu)$
finite group of order s. Since $T_\mu/Z(\mu)$, is relatively compact, arguing as in Theorem 3
[DM3], we get that $F = T_\mu/G(\mu)Z(\mu)$ is finite and it obviously consists of
measures. Also, the above implies that the image of λ_l on $G' = N(\mu)/G(\mu)Z(\mu)$ is
where $\bar{y}_l = y_lG(\mu)Z(\mu)$ in G' and $\bar{y}_l^{ls} = \bar{e}$, the identity in G'. Let $B = \{\gamma \in F \mid \gamma^r$
for some $r \in \mathrm{N}\}$. Since F is finite, so is B and there exists an element of maximal
in B; let i be the maximal order. Then $\gamma^{i!} = \delta_{\bar{e}}$ for all $\gamma \in B$. Since the image of
$N(\mu)/G(\mu)Z(\mu)$ belongs to B, we have that $\operatorname{supp}\lambda_l^{i!} \subset G(\mu)Z(\mu)$, for all l. Now for
m, let $\beta_m = \lambda_{i!m}^{i!}$, where $\mu = \lambda_{i!m}^{i!m}x$, for some $x \in I_\mu$. Then $\mu = \beta_m^m x$ and $\operatorname{supp}\beta_m \subset$
$Z(\mu)$. Also, since $\operatorname{supp}\beta_m \subset yG(\mu)$ for some $y \in \operatorname{supp}\beta_m$, $y = zy' = y'z$, for s
$y' \in G(\mu), z \in Z(\mu)$. Then $\beta'_m = z^{-1}\beta_m = \beta_m z^{-1}$ is supported on $G(\mu)$. Also, $\mu = \beta$
$(\beta'_m)^m z^m x = (\beta'_m)^m x'$, where $x' = z^m x \in I_\mu \cap G(\mu)$ as $\operatorname{supp}\beta'_m \subset G(\mu)$. That is,
weakly infinitely divisible on $G(\mu)$. Moreover, from the above equation, we have
$\{\beta'_m\}/Z_\mu$ is relatively compact, where $Z_\mu = G(\mu) \cap Z(\mu)$ is the center of $G(\mu)$
[DM3], Theorem 2.1). In fact, $\{\beta'_m z_m\}$ is relatively compact for some sequence $\{z$
Z_μ. Let $\gamma'_m = \beta'_m z_m$. Then $(\gamma'_m)^m = (\beta'_m)^m z_m^m$ and hence $\mu = (\gamma'_m)^m x_m$ for some
$I_\mu \cap G(\mu)$, for all m. Now if γ' is a limit point of $\{\gamma'_m\}$ then $(\gamma')^n \in T_\mu$ for all n and h
as earlier, $\operatorname{supp}\gamma' \subset xI(\mu) = I(\mu)x$, for some $x \in I_\mu \cap G(\mu)$. Since $(I_\mu \cap G(\mu))/$
finite (cf. [DM3], Theorem 2.1), if a is its cardinality then $\operatorname{supp}(\gamma')^a \subset zI(\mu) = I(\mu)$
some $z \in Z_\mu$. Therefore limit points of $\{(\gamma'_m)^a\}$ are supported on $zI(\mu) = I(\mu)z$, z
Let $\gamma_m = (\gamma'_{am})^a$. Then $\mu = \gamma_m^m x_{am}$, where $x_{am} \in I_\mu \cap G(\mu)$. Let $\{\gamma_{c_m}\}$ be a conve
subsequence of $\{\gamma_{m!}\}$ converging to γ. Then from above, $\operatorname{supp}\gamma \subset zI(\mu) = I(\mu)$
some $z \in Z_\mu$. Therefore, for each m, replacing γ_{c_m} by $\gamma_{c_m}z^{-1}$ (and using the same
tion), we get that $\mu = \gamma_{c_m}^{c_m}y_m$, $y_m \in I_\mu \cap G(\mu)$ and $\gamma_{c_m} \to \gamma$ and $G(\gamma) \subset I(\mu)$, whi
compact. Also $\{y_m\}/Z_\mu$ is finite, and hence passing to a subsequence again, we
assume that $y_m = az'_m = z'_m a$, where $a \in I_\mu \cap G(\mu)$ and $z'_m \in Z_\mu$. There
$\gamma_{c_m}^{c_m}z'_m = a^{-1}\mu = \mu a^{-1}$. Now applying Theorem 2.2, we get that $A = \{\gamma_{c_m}^n \mid n \leq$
and $\{z'_m\}$ are relatively compact. Now if β is a limit point of $\{\gamma_{c_m}^{c_m}\}$ then $a^{-1}\mu = \beta$
some $z' \in Z_\mu$. Since for all m, $c_m = l_m!$, where $l_m \to \infty$, any n divides c_m for all lar
Also since A is relatively compact, it is easy to see that β has an n-th root in $\overline{A}$, na
any limit of the sequence $\{\gamma_{c_m}^{c_m/n}\}$. Therefore, $y\mu = \beta$ is infinitely divisible in the co
set $\overline{A}$, where $y = (z')^{-1}a^{-1} \in I_\mu \cap G(\mu)$. Now as in the proof of Theorem 3.1.32 o
$y\mu$ is rationally embeddable, i.e. there exists a homomorphism $f : \mathbf{Q}_+^* \to M^1(G)$ suc
$f(]0,1[\cap\mathbf{Q}_+^*) \subset \overline{A}$ is relatively compact and $f(1) = \mu$. Now since G is discrete
compact connected subgroup of G has to be $\{e\}$. Therefore, as in the proof of The
3.5.4 of [H], f extends to $\mathbf{R}_+$ and hence $y\mu$ is embeddable.

4. Infinitesimally divisible measures on algebraic groups

We first recall that an element s, in a Hausdorff semigroup S with identity e, is said to be infinitesimally divisible if for every neighbourhood U of e in S, s has a U-decomposition, i.e. there exist $s_1, \dots, s_n \in U$ such that s_i's commute and $s = s_1 \cdots s_n$. The following theorem generalizes Theorem 1.2 of [S3] in a certain sense.

Theorem 4.1. *Let G be a real algebraic group and let $\mu \in M^1(G)$ be infinitesimally divisible in $M^1(G)$. Then there exist a closed semigroup $S \subset M^1_H(G)$, with identity ω_H for some compact subgroup H of $I(\mu)$, and an equivalence relation $\sim$, such that $\mu \in S$ and if $\rho : S \to S^* = S/\sim$ is the natural map then $\rho(\mu)$ is bald and infinitesimally divisible in S^*, and $T_{\rho(\mu)}$ is compact and associatefree in S^*. Moreover, if G is connected and nilpotent then μ is embeddable.*

Before proving the above theorem, we define an equivalence relation on a certain kind of subsemigroup of $M^1(G)$, for any locally compact (Hausdorff) group G. For a $\mu \in M^1(G)$, let S_μ be the closed subsemigroup generated by T_μ in $M^1(G)$. Since $T_\mu \subset M^1(N(\mu))$, $S_\mu \subset M^1(N(\mu))$. In fact, for any $\lambda \in T_\mu$, $\operatorname{supp} \lambda \subset xG(\mu)$, for some $x \in \operatorname{supp} \lambda \subset N(\mu)$. Therefore, it easily follows that for any $\beta \in S_\mu$, $\operatorname{supp} \beta \subset xG(\mu)$, for any $x \in \operatorname{supp} \beta \subset N(\mu)$. We also know that $Z(\mu) \subset T_\mu \subset S_\mu$ and $Z(\mu)T_\mu = T_\mu Z(\mu) = T_\mu$. Let us define an equivalence relation '$\approx$' on S_μ as follows: for any

$$\beta, \lambda \in S_\mu, \beta \approx \lambda \text{ if } \quad \beta = z\lambda \text{ for some } \quad z \in Z(\mu).$$

For $\{\beta_n\}, \{\lambda_n\} \subset S_\mu$, suppose $\beta_n \approx \lambda_n$, i.e. $\beta_n = z_n\lambda_n$ for some $z_n \in Z(\mu)$, for each n. Now if $\beta_n \to \beta$ and $\lambda_n \to \lambda$, then we have that $\{z_n\}$ is relatively compact and for any limit point z of it, $z \in Z(\mu)$ and $\beta = z\lambda$. Therefore, $\beta \approx \lambda$.

Now for $\lambda \in S_\mu$, for any fixed $x \in \operatorname{supp} \lambda$, $\operatorname{supp} (\lambda x^{-1}) \subset G(\mu)$. For any $z \in Z(\mu)$, $z' = xzx^{-1} \in Z(\mu)$ as $Z(\mu)$ is normal in $N(\mu)$ and hence

$$\lambda z = (\lambda x^{-1})xz = (\lambda x^{-1})z'x = z'(\lambda x^{-1})x = z'\lambda.$$

Similarly, one can also show that $z\lambda = \lambda z''$, for some $z'' \in Z(\mu)$.

Now for $i \in \{1,2\}$, $\beta_i, \lambda_i \in S_\mu$, let $\beta_i \approx \lambda_i$, i.e. there exist $z_i \in Z(\mu)$, such that $\beta_i = z_i\lambda_i$, Then from the above equation, $\beta_1\beta_2 = z_1\lambda_1 z_2\lambda_2 = z_1 z_2'\lambda_1\lambda_2$ for some $z_2' \in Z(\mu)$. That is, $\beta_1\beta_2 \approx \lambda_1\lambda_2$. Let $\psi : S_\mu \to S^*_\mu = S_\mu/\approx$ be the natural projection. Then ψ is a continuous open homomorphism and it is also easy to show that S^*_μ is Hausdorff.

In case of a real algebraic group G, we define an analogous equivalence relation $\approx'$ with respect to $Z^0(\mu)$, the connected component of the identity in $Z(\mu)$, i.e. for $\beta, \lambda \in S_\mu$, $\beta \approx' \lambda$ if $\beta = z\lambda$, for some $z \in Z^0(\mu)$. It is easy to verify as above that this is an equivalence relation using the fact that $Z^0(\mu)$ is normal in $N(\mu)$.

Proof of Theorem 4.1. Let G be a real algebraic group and let μ be infinitesimally divisible in $M^1(G)$. Since G is metrizable, so is $M^1(G)$.

Step 1. Let S_μ, $\approx'$, S^*_μ and $\psi : S_\mu \to S^*_\mu$ be as above. Clearly, S_μ and S^*_μ are second countable and $\psi(\mu)$ is infinitesimally divisible in S^*_μ.

Since G is algebraic, by Theorem 3.2 of [DM2], $T_\mu/Z^0(\mu)$ is relatively compact. Clearly, $\psi(T_\mu) \subset T_{\psi(\mu)}$. Now for any $\{\lambda_n\} \subset T_\mu$, there exists a sequence $\{z_n\} \subset Z^0(\mu)$, such that $\{\lambda_n z_n\}$ is relatively compact and hence $\{\psi(\lambda_n) = \psi(\lambda_n z_n)\}$ is also relatively compact. Since $T_\mu Z(\mu) = T_\mu$, $\{\lambda_n z_n\} \subset T_\mu$ and the above implies that $\psi(T_\mu)$ is compact in S^*_μ.

Since μ is infinitesimally divisible so is $\psi(\mu)$ in S^*_μ. We can choose a neighbourhood basis $\{U_i\}_{i\in\mathrm{N}}$ of δ_e in $M^1(G)$. For any i, there exist $\mu_{i1}, \ldots, \mu_{in_i} \in U_i \cap T_\mu$, such that μ_{ij}s commute and $\mu = \mu_{i1} \cdots \mu_{in_i}$. Therefore $\psi(\mu) = \psi(\mu_{i1}) \cdots \psi(\mu_{in_i})$ is a $\psi(U_i)$-decomposition of $\psi(\mu)$ in $\psi(T_\mu)$. Let $\Delta = (\mu_{ij})^{n_i}_{i\in\mathrm{N},j=1}$ and $\psi(\Delta) = (\psi(\mu_{ij}))^{n_i}_{i\in\mathrm{N},j=1}$. Then Δ (resp. $\psi(\Delta)$) is a commutative infinitesimal triangular system in S_μ (resp. in S^*_μ) converging to μ (resp. $\psi(\mu)$). In fact, $\mu = \Pi^{n_i}_{j=1}\mu_{ij}$ and $\psi(\mu) = \Pi^{n_i}_{j=1}\psi(\mu_{ij})$.

Step 2.. Since I^0_μ is open in I_μ, one can choose U and W to be neighbourhoods of $I^0_\mu J_\mu$ such that $U = \{\nu \in M^1(G) \mid \nu(I^0_\mu I(\mu)V) > \delta\}$, for some $\delta > 0$, $\overline{U} \cap I_\mu J_\mu = I^0_\mu J_\mu$, for some relatively compact neighbourhood V of e in G^0 and $WW \subset U$. Now let $\lambda \in S_\mu \cap \overline{U} \setminus W$ be such that $\psi(\lambda^n) \in T_{\psi(\mu)}$ in S^*_μ for all n, then $\mu = \lambda^n \nu_n = \nu'_n \lambda^n$, for some ν_n, ν'_n in S_μ for all n. Then the concentration functions of both λ and $\tilde{\lambda}$ do not converge to zero. Since λ commutes with μ, as in the proof of Theorem 2.4 of [S4], $\operatorname{supp}\lambda \subset xI(\mu) = I(\mu)x$, for some $x \in \operatorname{supp}\lambda \subset I_\mu \cap \overline{U}$. i.e. $\lambda \in I^0_\mu J_\mu$, a contradiction as $\lambda \notin W$. Now as in the proof of Lemma 2.5 in [S4], there exists n such that for any $m \geq n$, $\psi(\mu)$ cannot be expressed as $\psi(\mu) = \psi(\lambda_1) \cdots \psi(\lambda_m)\psi(\nu)$, where $\psi(\lambda_j)$s commute with each other and also with $\psi(\nu)$ for any $\lambda_j \in S_\mu \cap \overline{U} \setminus W$, for all j.

Since $I_\mu \subset T_\mu$, $\psi(I_\mu)$ is compact. Let $K = \psi(I^0_\mu J_\mu)$. Then K is a compact semigroup and $\psi(U \cap S_\mu)$ and $\psi(W \cap S_\mu)$ are neighbourhoods of K in S^*_μ.

Since $\psi(\mu)$ is a limit of a triangular system as above, as in Lemma 2.6 of [S4], given any neighbourhood U' of K in S^*_μ, one can choose small neighbourhoods U and W as above such that $\psi(U \cap S_\mu) \subset U'$ and show that there exists a U'-decomposition of $\psi(\mu)$ in $\psi(T_\mu)$, namely, $\psi(\mu) = \psi(\mu_1) \cdots \psi(\mu_n)$, where each $\psi(\mu_i) \in U'$ is a limit of a subsystem of $\psi(\Delta)$.

Step 3. Let $\{U'_n\}$ be a neighbourhood basis of K in S^*_μ such that $U'_{n+1} \subset U'_n$ for all n and $\cap_{n\in\mathrm{N}} U'_n = K$. Now let $\psi(\mu) = \gamma_1 \cdots \gamma_n$ be a U'_1-decomposition of $\psi(\mu)$ in $\psi(T_\mu)$ obtained as above. Given any U'_k-decomposition of $\psi(\mu)$ as $\psi(\mu) = \nu_1 \cdots \nu_r$, $\nu_l = \Pi_{j\in J_{il}} \psi(\mu_{k(i)j})$, where $\cup_l J_{il} = \{1, \ldots, n_{k(i)}\}$ we get U'_{k+1}-decomposition of each ν_l in such a way that $\nu_l = \nu_{l1} \cdots \nu_{ln_l}$, $\nu_{lm} \in U'_{k+1}$, where $\nu_{lm}\nu_{pq} = \nu_{pq}\nu_{lm}$, for all l, m, p, q, and all the ν_{lm} are limits of a subsystem of $(\psi(\mu_{(k+1)(i)j}))$, where $\{(k+1)(i)\}$ is a subsequence of $\{k(i)\}$. Clearly $\psi(\mu) = \Pi_{l,m}\nu_{lm}$ is a U'_{k+1}-decomposition for $\psi(\mu)$.

For each $k \in \mathrm{N}$, let M_k be the subsemigroup of S^*_μ generated by U'_k-decomposition obtained in above manner. Then each M_k is abelian, $\mu \in M_k$ and $M_k \subset M_{k+1}$. Let $M = \overline{\cup_k M_k}$ and let $K' = K \cap M = \psi(I^0_\mu J_\mu) \cap M$. Then M (resp. K') is a closed (resp. compact) abelian semigroup. Also, given any neighbourhood U' of K' in M, there exists a neighbourhood U'' of K in S^*_μ, such that $U'' \cap M \subset U'$. Hence μ has a U'-decomposition in M for every neighbourhood U' of K'.

Step 4. We now show that $T_{\psi(\mu)}$ is compact in M. Let U, W and V be as in Step 2. Let $\nu \in S_\mu$ be such that $\psi(\nu) \in T_{\psi(\mu)}$ in M. Now $\mu = \nu\nu' = \nu''\nu$ for some $\nu', \nu'' \in S_\mu$. Arguing as in Step 2, there exists n (which does not depend on the choice of $\psi(\nu) \in T_{\psi(\mu)}$) such that for any $m \geq n$, $\psi(\nu)$ cannot be expressed as $\psi(\nu) = \psi(\lambda_1) \cdots \psi(\lambda_m)\psi(\beta)$ in M for $\lambda_j \in S_\mu \cap \overline{U} \setminus W$, for all j, and $\psi(\lambda_j)$s commute and they also commute with $\psi(\beta)$. Here, $\psi(\nu)$ is a limit of a commutative K'-infinitesimal triangular system in M, i.e.

$\psi(\nu) = \lim_{i\to\infty} \Pi_{j=1}^{n_i} \psi(\nu_{ij})$ for some $\nu_{ij} \in S_\mu$. Again arguing as in Step 2, $\psi(\nu) = \psi(\nu_1)\cdots\psi(\nu_n)$ for $\psi(\nu_i) \in T_{\psi(\mu)} \cap \psi(\overline{U} \setminus W)$. That is, $\psi(\nu) \in (\psi(\overline{U} \setminus W)^n$. Since n does not depend on the choice of $\psi(\nu)$ in $T_{\psi(\mu)}$, $T_{\psi(\mu)} \subset (\psi(\overline{U} \setminus W))^n$. Hence it is easy to show as in the proof of Lemma 2.1 of [S4] that $T_{\psi(\mu)}$ is relatively compact.

Step 5. Let $J = \psi(J_\mu) \cap M$. Then J is a compact semigroup and there exists a maximal idempotent h_1 in J. Then $J' = Jh_1$ is a group. Let $H = \{x \in I(\mu) \mid \psi(x)h_1 \in J'\}$. It is easy to check that H is a compact group. Let $h = \omega_H$ and let $h^* = \psi(\omega_H)$. Then $Jh^* = J'h^* = h^*$ and $K'' = K'h^* = (\psi(I_\mu^0) \cap M)h^*$, which is a compact group. Let $M^* = Mh^*$. M^* is a closed abelian semigroup with identity h^* and $K'' \subset M^* \subset M$. Now if U is a neighbourhood of K'' in M^* then there exists a neighbourhood U' of K' in M such that $U'h^* \subset U$, and hence if $\psi(\mu) = \lambda_1 \cdots \lambda_n$ is a U'-decomposition of $\psi(\mu)$ in M, then since $\mu = \mu h = \mu h^n$, $\psi(\mu) = \lambda_1 \cdots \lambda_n \psi(h^n)$ and hence $\psi(\mu) = \lambda_1 h^* \cdots \lambda_r h^*$ is a U-decomposition of $\psi(\mu)$ in M^*. Now we define an equivalence relation $\sim'$ on M^* as follows: For

$$\lambda, \nu \in M^*, \lambda \sim' \nu \text{ if } \quad \lambda = k\nu \text{ for some } \quad k \in K''.$$

Let $S^* = M^* / \sim'$ and let $\phi : M^* \to S^*$ be the natural projection and let $\rho = \phi \circ \psi$. Then S^* is a Hausdorff abelian semigroup with identity $\phi(h^*)$, $\rho^{-1}(S^*) = S$ is a closed semigroup in $M_H^1(G)$, the relation $\sim$ is defined by ρ on S, each $\rho(\lambda)$ in $T_{\rho(\mu)}$ is inifinitesimally divisible in S^* and by step 4, $T_{\rho(\mu)}$ is compact. Now if $a, b \in T_{\rho(\mu)}$ are associates then $a = a'b$ and $b = b'a$. Let $\beta, \beta' \in S_1$ be such that $\rho(\beta) = b$ and $\rho(\beta') = b'$ and $\rho(\gamma) = a'$, then since $b = b'a'b$, $\psi(\beta) = k\psi(\beta')\psi(\gamma)\psi(\beta)$ for some $k \in K''$ and hence $\psi(\beta')^n \in T_{\psi(\beta)}$ for all n. As in step 2, supp $\beta' \subset xI(\beta) = I(\beta)x$, for some $x \in I_\mu$, and since $\rho(\beta') = b'$ is infinitesimally divisible, it is easy to show that $x \in I_\mu^0$. Therefore b' is identity in S^* and $b = a$, i.e. $T_{\rho(\mu)}$ is associatefree.

Now if $\beta \in S_\mu$ be such that $\rho(\beta) \in T_{\rho(\mu)}$ is an idempotent then $\psi(\beta)^n \in T_{\psi(\mu)}$ for all n and hence as in step 2, supp$\,\beta \subset xI(\mu) = I(\mu)x$ for some $x \in I_\mu$. Since $\rho(\beta)$ is also infinitesimally divisible in S^* one can easily show that $x \in I_\mu^0$ and $\beta = x\omega_{H'} = \omega_{H'}x$ for some $H' \subset I(\mu)$ and hence $\psi(\beta) \in K''$ and hence $\rho(\beta)$ is identity in S^*. Therefore $\rho(\mu)$ is bald.

Step 6. Now let G be connected and nilpotent and let Z be the center of G. Then G/Z is simply connected and hence so are $N(Z(\mu))/Z$ and $N(Z(\mu))/Z(\mu)$, where $N(Z(\mu))$ is the normaliser of $Z(\mu)$, and both of them are connected. Therefore, $I_\mu = Z(\mu)$ as $I_\mu/Z(\mu)$ is compact. Hence, in the above equation $K'' = h^*$ and $\sim'$ is a trivial relation, i.e. $S^* = M^*$ and also $\rho = \psi$.

Now we show that for $s \in T_{\psi(\mu)} \setminus \psi(h)$ in S^*, there exists a continuous s-norm f_s on T_s (in S^*) such that $f_s(s) > 0$, (an s-norm on T_s (in S^*) is a map $f_s : T_s \to \mathbf{R}_+$ which is continous at the identity and it is a partial homoporphism, i.e. $f_s(s_1s_2) = f_s(s_1) + f_s(s_2)$ if $s_1, s_2, s_1s_2 \in T_s$). This would imply the embedding of $\psi(\mu)$ in a continuous real one-parameter semigroup $\{\gamma_t\}_{t\in\mathbf{R}_+}$ in S^* (cf. [S3], Theorem 2.3 or [S4], Theorem 4.1) and in particular, $\mu = \lambda_n^n x_n$ $x_n \in Z(\mu) = Z^0(\mu)$.

Let $\lambda \in S$ be such that $\psi(\lambda) = s$. If λ is not a translate of an idempotent then as in the proof of Theorem 5.1 in [S3], there exists a continuous λ-norm f_λ on S such that $f_\lambda(\lambda) > 0$, (it is easy to see that one does not need the underlying semigroup to be abelian in that proof). Moreover, if $\psi(\nu_1) = \psi(\nu_2)$ then $\nu_1 = \nu_2 x$ for some $x \in Z(\mu)$. Then $\nu_1\tilde{\nu}_1 = \nu_2\tilde{\nu}_2$ and $f_\lambda(\nu_1) = f_\lambda(\nu_2)$ (see the proof of Theorem 5.1 in [S3]). Therefore, we can define a s-norm f_s on T_s in S^* such that $f_s(\psi(\nu)) = f_\lambda(\nu)$. Now if λ is indeed a translate of an idempotent, i.e. $\lambda = x\omega_K = \omega_K x$ for some compact group $K \subset I(\mu) \subset Z$, then clearly $x \in I_\mu = Z(\mu)$ and hence $s = \psi(\lambda)$ is an idempotent. Now since $\psi(\mu)$ is bald $s = \psi(h)$, a contradiction.

The embeddability of $\psi(\mu)$ in particular implies that $\psi(\mu) = \psi(\lambda_n)^n$, and hence $\mu = \lambda_n^n x_n$, $x_n \in Z(\mu)$ for all n. Therefore, $\operatorname{supp} \lambda_n^n \subset G(\mu)Z(\mu)$. Here, $\operatorname{supp} \lambda_n \subset y_n G(\mu)$ for some $y_n \in \operatorname{supp} \lambda_n \subset N(\mu)$. Therefore, $y_n^n \in G(\mu)Z(\mu) \subset \tilde{G}(\mu)Z(\mu)$, where $\tilde{G}(\mu)$ is the Zariski closure of $G(\mu)$. Since $N(\mu)/\tilde{G}(\mu)Z(\mu)$ is simply connected, $y_n \in \tilde{G}(\mu)Z(\mu)$ for all n. That is, for each n, $\operatorname{supp} \lambda_n \subset \tilde{G}(\mu)Z(\mu)$ and hence $\lambda_n = \beta_n z_n$ for some $z_n \in Z(\mu)$ and $\operatorname{supp} \beta_n \subset \tilde{G}(\mu)$ and $\mu = \beta_n^n z_n'$, where $z_n' = z_n^n x_n \in Z(\mu)$. Now we have that $z_n' \in C = \tilde{G}(\mu) \cap Z(\mu)$, which is the center of $\tilde{G}(\mu)$. Therefore, $CZ \subset Z(\mu)$ is an abelian algebraic subgroup containing the center Z of G. Therefore CZ is connected, and hence it is divisible. In particular, each z_n' is infinitely divisible in CZ, and hence μ is infinitely divisible on G which is a connected nilpotent Lie group, therefore μ is embeddable (cf. [BM]).

Remark. As remarked in [S4], Theorem 4.1 also holds for $\mu \in M_H^1(G)$ which is infinitesimally divisible in $M_H^1(G)$.

We now state the following theorem for maximally almost periodic groups without a proof. A locally compact group G is said to be *maximally almost periodic* if its irreducible finite dimensional unitary representations separate points of G.

Theorem 4.2. *Let G be a maximally almost periodic first countable group. Let Δ be a commutative infinitesimal triangular system of probability measures converging to μ in $M^1(G)$. Then there exists an $x \in G^0$ such that $x\mu = \mu x$ is embeddable.*

If G is as above then there exists a normal vector subgroup V, such that G^0/V is compact and V centralises an open subgroup of finite index in G (cf. [RW], Theorems 1, 2]. The above theorem can be proven using the above fact, Proposition 2.5, Lemma 2.4, Proposition 3.3 and Theorem 4.2 of [S4] and the techniques developed above.

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Topological *-algebras with C^*-enveloping algebras II

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Abstract. Universal C^*-algebras $C^*(A)$ exist for certain topological *-algebras called algebras with a C^*-enveloping algebra. A Frechet *-algebra A has a C^*-enveloping algebra if and only if every operator representation of A maps A into bounded operators. This is proved by showing that every unbounded operator representation π, continuous in the uniform topology, of a topological *-algebra A, which is an inverse limit of Banach *-algebras, is a direct sum of bounded operator representations, thereby factoring through the enveloping pro-C^*-algebra $E(A)$ of A. Given a C^*-dynamical system (G,A,α), any topological *-algebra B containing $C_c(G,A)$ as a dense *-subalgebra and contained in the crossed product C^*-algebra $C^*(G,A,\alpha)$ satisfies $E(B)=C^*(G,A,\alpha)$. If $G=\mathbb{R}$, if B is an α-invariant dense Frechet *-subalgebra of A such that $E(B)=A$, and if the action α on B is m-tempered, smooth and by continuous *-automorphisms: then the smooth Schwartz crossed product $S(\mathbb{R},B,\alpha)$ satisfies $E(S(\mathbb{R},B,\alpha))=C^*(\mathbb{R},A,\alpha)$. When G is a Lie group, the C^∞-elements $C^\infty(A)$, the analytic elements $C^\omega(A)$ as well as the entire analytic elements $C^{e\omega}(A)$ carry natural topologies making them algebras with a C^*-enveloping algebra. Given a non-unital C^*-algebra A, an inductive system of ideals I_α is constructed satisfying $A=C^*\text{-ind}\lim I_\alpha$; and the locally convex inductive limit $\text{ind}\lim I_\alpha$ is an m-convex algebra with the C^*-enveloping algebra A and containing the Pedersen ideal K_A of A. Given generators G with weakly Banach admissible relations R, we construct universal topological *-algebra $A(G,R)$ and show that it has a C^*-enveloping algebra if and only if (G,R) is C^*-admissible.

Keywords. Frechet *-algebra; topological *-algebra; C^*-enveloping algebra; unbounded operator representation; O^*-algebra; smooth Frechet algebra crossed product; Pedersen ideal of a C^*-algebra; groupoid C^*-algebra; universal algebra on generators with relations.

1. Statements of the results

In [5], a functor E has been considered that associates C^*-algebras $E(A)$ with certain topological *-algebras A, called algebras with a C^*-enveloping algebra. By a classic construction due to Gelfand and Naimark, a Banach *-algebra A admits a C^*-enveloping algebra $C^*(A)=E(A)$ ([14], 2.7, p. 47). By ([15], Theorem 2.1), a complete locally m-convex *-algebra has a C^*-enveloping algebra if and only if it admits a greatest continuous C^*-seminorm. The following extrinsic characterization of such algebras has been motivated by the simple observation that any *-homomorphism from a Banach *-algebra into the *-algebra of linear operators on an inner product space maps the algebra into bounded operators.

Theorem 1.1. *Let A be a Frechet *-algebra. Then A is an algebra with a C^*-enveloping algebra if and only if every *-representation of A is a bounded operator representation.*

The above theorem is false without the assumption that A is metrizable (see Remark 4.4). By a **-representation* $(\pi, \mathcal{D}(\pi), H)$ of a *-algebra A [37] is meant a homomorphism π from A into linear operators (not necessarily bounded) all defined on a common dense invariant subspace $\mathcal{D}(\pi)$ of a Hilbert space H such that for all x in A, $\pi(x^*) \subset \pi(x)^*$. In the general theory of *-algebras, following Palmer [24], A is called a *BG^*-algebra* if every *-homomorphism from A into linear operators on a pre-Hilbert space maps A into bounded operators. The absence of a complete algebra norm on a non-Banach *-algebra A indicates that A may contain elements that fail to be bounded in any natural sense. Hence an appropriate framework for the representation theory of A is that of unbounded operator representations. However, this natural point of view was developed rather late, following [30, 20]. Prior to (and later, in spite of) this, bounded operator representations of A have been investigated in detail, especially when A is a locally m-convex *-algebra, i.e., $A = \text{proj} \lim A_\alpha$, the inverse limit (also called the projective limit) of Banach *-algebras [9, 15], (see [16] for a summary of bounded operator representations of A). In fact, such an A, when *-semisimple, admits sufficiently many continuous irreducible bounded operator representations [9]. Then the enveloping pro-C^*-algebra (projective limit of C^*-algebras) $E(A)$ of A, discussed in [10], [19] and [15], turns out to be $E(A) = \text{proj} \lim E(A_\alpha)$, $E(A_\alpha) = C^*(A_\alpha)$ being the enveloping C^*-algebra of the Banach *-algebra A_α ([15], Theorem 4.3). Thus A has a *C^*-enveloping algebra* if $E(A)$ is a C^*-algebra. By the construction, $E(A)$ is universal for norm-continuous bounded operator representations of A. Theorem 1.2, to be used to prove Theorem 1.1, shows desirably that $E(A)$ is also universal for representations into unbounded operators. The uniform topology ([37], p. 77, 78) on an unbounded operator algebra is defined at the end of this section.

Theorem 1.2. *Let A be complete locally m-convex *-algebra. Let $(\pi, D(\pi), H)$ be a closed *-representation of A continuous in the uniform topology on $\pi(A)$. Then there exists a unique *-representation $(\sigma, D(\sigma), H_\sigma)$ of $E(A)$ such that the following hold.*

(1) *$H_\sigma = H$ and $D(\sigma) = D(\pi)$.*

(2) *As a representation of $E(A)$, σ is closed and continuous in the uniform topology on $\sigma(E(A))$.*

(3) *σ is an 'extension' of π to $E(A)$ in the sense that for all x in A, $(\sigma \circ j)(x) = \pi(x)$, $j: A \to E(A)$ being the natural map, $j(x) = x + \text{srad}(A)$, $\text{srad}(A)$ denoting the star radical of A.*

(4) *On the unbounded operator algebra $\pi(A)$, the uniform topology $\tau_D^{\pi(A)}$ is a (not necessarily complete) pro-C^*-topology which coincides with the relative uniform topology $\tau_D^{\sigma(E(A))}$ from $\sigma(E(A))$.*

COROLLARY 1.3

Let π be a closed irreducible *-representation of a complete locally m-convex *-algebra A continuous in the uniform topology on $\pi(A)$. (In particular, let A be Frechet and π be irreducible). Then π maps A into bounded operators.

AO^*-algebras (abstract O^*-algebras) [36, 37] provide the unbounded operator algebra analogues of C^*-algebras. Starting with a topological (not necessarily m-convex)

*-algebra A, one can construct an enveloping AO^*-algebra $O(A)$ universal for *-representations continuous in the uniform topology, and declare A to have a *C^*-enveloping algebra* if the uniform topology on $O(A)$ is normable. On the other hand, by modifying the construction in [15], the pro-C^*-algebra $E(A)$ can also be considered as the universal object for norm-continuous bounded operator *-representations of more general locally convex, non-m-convex, *-algebras A. In general, the completion of $O(A)$ differs from $E(A)$. For a barrelled A, $O(A)$ is normable implies that $E(A)$ is a C^*-algebra, but the converse does not hold. In the present context, the following shows that both the approaches are consistent in the metrizable case.

Theorem 1.4. *Let A be a Frechet *-algebra. Then the pro-C^*-algebra $E(A)$ is the completion of the AO^*-algebra $O(A)$. Thus $O(A)$ is normable if and only if A is an algebra with a C^*-enveloping algebra.*

There are several situations in C^*-algebra theory in which topological *-algebras arise naturally [27]. Enveloping C^*-algebras provide a standard method of constructing C^*-algebras; and frequently, lurking behind such a construction is a topological *-algebra B such that $E(B) = A$. Let α be a strongly continuous action of a locally compact group G by *-automorphisms of a C^*-algebra A. The crossed product C^*-algebra $C^*(G,A,\alpha)$ is the enveloping C^*-algebra of the L^1-crossed product Banach *-algebra $L^1(G,A,\alpha)$. If B is a topological *-algebra such that $C_c(G,A) \subseteq B \subseteq C^*(G,A,\alpha)$ and $C_c(G,A)$ is dense in B, then $E(B) = C^*(G,A,\alpha)$. Let G be a Lie group. Then the *-subalgebra $C^\infty(A)$ of C^∞-elements of A is a Frechet *-algebra with an appropriate topology such that $E(C^\infty(A)) = A$. The *-algebras $C^\omega(A)$ and $C^{e\omega}(A)$ consisting of analytic elements and entire elements of A are shown to carry natural topologies making them algebras with C^*-enveloping algebras. We also consider the smooth crossed product [29, 34]. For simplicity, we take $G = \mathbb{R}$, and prove the following.

Theorem 1.5. *Let α be a strongly continuous action of $\mathbb{R}$ by *-automorphisms of a C^*-algebra A. Suppose that B is a dense Frechet *-subalgebra of A satisfying the following.*

(a) *A has a bounded approximate identity contained in B and which is a bounded approximate identity for B.*
(b) *$E(B) = A$.*
(c) *B is α-invariant; and the action α of $\mathbb{R}$ on B is smooth, m-tempered and by continuous *-automorphisms of B.*

Then the smooth Schwartz crossed product $S(\mathbb{R},B,\alpha)$ is a Frechet *-algebra with a C^*-enveloping algebra, and $E(S(\mathbb{R},B,\alpha)) = C^*(\mathbb{R},A,\alpha)$. Further, if the action of $\mathbb{R}$ on B is isometric (see § 5), then the L^1-crossed product $L^1(\mathbb{R},B,\alpha)$ is also a Frechet *-algebra with a C^*-enveloping algebra, and $E(L^1(\mathbb{R},B,\alpha)) = C^*(\mathbb{R},A,\alpha)$.

It follows that $E(S(\mathbb{R},C^\infty(A),\alpha) = C^*(\mathbb{R},A,\alpha)$. In particular, if α is a smooth action of $\mathbb{R}$ on a C^∞-manifold M, then $E(S(\mathbb{R},C^\infty(M),\alpha) = C^*(\mathbb{R},C(M),\alpha)$, the covariance C^*-algebra of the $\mathbb{R}$-space M.

For a locally compact Hausdorff space X, let $\mathcal{K}$ be the directed set consisting of all compact subsets of X. For $K \in \mathcal{K}$, let $C_K(X) = \{f \in C_c(X) : \operatorname{supp} f \subseteq K\}$, $C_c(X)$ denoting the compactly supported continuous functions on X. It is well known that $\{C_K(X) : K \in \mathcal{K}\}$ forms an inductive system; and $C_0(X) = C^*$-ind $\lim C_K(X)$ (C^*-inductive limit), $C_c(X) = \text{ind} \lim C_K(X)$ (locally convex inductive limit). Further, $C_c(X)$ with the locally convex inductive limit topology is a complete locally m-convex Q-algebra and $E(C_c(X)) = C_0(X)$. The following provides a non-commutative analogue of this. We refer to the last paragrapgh in this section for the relevant definitions pertaining to topological algebras.

Theorem 1.6. *Let A be a non-unital C^*-algebra. Let K_A denote its Pedersen ideal. For $a \in K_A^+$, let I_a denote the closed two sided ideal of A generated by aa^*. Let $K_A^{nc} = \mathrm{U}\{I_a : a \in K_A^+\}$. Then the following hold.*

1. *$\{I_a : a \in K_A^+\}$ forms an inductive system, $A = C^* - \text{ind}\lim\{I_a : a \in K_A^+\}$, and $K_A^{nc} = \text{ind}\lim\{I_a : a \in K_A^+\}$.*
2. *K_A^{nc} with the locally convex inductive limit topology t is a locally m-convex Q-algebra satisfying $E(K_A^{nc}) = E(K_A) = A$.*
3. *If A has a countable bounded approximate identity, then (K_A^{nc}, t) is an LFQ-algebra.*

In general $K_A \neq K_A^{nc}$, though $K_A \subseteq K_A^{nc}$. Now K_A has been interpreted as a non-commutative analogue of $C_c(X)$. Then K_A^{nc} may be interpreted as continuous functions on a non-commutative space vanishing at infinity in 'commutative directions' and having compact supports in 'non-commutative directions'. This interpretation is suggested by the remarks preceeding ([28], Theorem 8).

The universal C^*-algebra $C^*(G,R)$ on a C^*-admissible set of generators G with relations R provides another method of constructing C^*-algebras. Motivated by some problems in C^*-algebras, Phillips introduced more general weakly C^*-admissible generators with relations (G,R) leading to the construction of the universal pro-C^*-algebra $C^*(G,R)$ on (G,R) [27]. In § 8, we construct a universal topological *-algebra $A(G,R)$ on (G,R) with weakly Banach admissible relations R, and prove the following.

Theorem 1.7. *Let (G,R) be weakly Banach admissible.*

1. *$E(A(G,R)) = C^*(G,R)$.*
2. *$A(G,R)$ has a C^*-enveloping algebra if and only if (G,R) is C^*-admissible.*

The paper is organized as follows. Proofs of Theorems 1.1, 1.2 and 1.4 are presented in § 3. The preliminary lemmas and constructions in the locally convex, non-m-convex set up more general than in [5], are discussed in § 2. Section 4 contains a couple of remarks including some corrections in [5]. The smooth crossed product is discussed in § 5 culminating in the proof of Theorem 1.5. Section 6 contains the proof of Theorem 1.6. This is followed by a brief discussion on the C^*-algebra of a groupoid in § 7. Universal C^*-algebras on generators with relations are discussed in § 8. In what follows, we briefly recall the relevant ideas in unbounded operator representations.

For the basic theory of unbounded operator *-representations $(\pi, \mathcal{D}(\pi), H)$ of a *-algebra A, we refer to [37, 30]. Let A^1 denote the unitization of A. The graph topology $t_\pi = t_{\pi(A^1)}$ on $\mathcal{D}(\pi)$ is defined by seminorms $\xi \rightarrow ||\xi|| + ||\pi(x)\xi||$, where $x \in A$. The *closure* $\bar{\pi}$ of π is the *-representation $(\bar{\pi}, D(\bar{\pi}), H)$, where $D(\bar{\pi}) = \bigcap\{D(\overline{\pi(x)}) : x \in A^1\}$, $D(\overline{\pi(x)})$ being the domain of the closure $\overline{\pi(x)}$ of $\pi(x)$; and $\bar{\pi}(x) = \overline{\pi(x)}|_{D(\bar{\pi})}$ for all x in A^1. Throughout, π is assumed *non-degenerate*, i.e., the norm closure $(\pi(A)H)^- = H$ and the t_π-closure $\overline{(\pi(A)\mathcal{D}(\pi))}^{t_\pi} = \mathcal{D}(\bar{\pi})$. If $\pi = \bar{\pi}$, then π is *closed*. The *hermitian adjoint* π^* of π is the representation (not necessarily a *-representation) $(\pi^*, D(\pi^*), H)$, where $D(\pi^*) = \bigcap\{D(\pi(x)^* : x \in A^1\}$, and $\pi^*(x) = \pi(x^*)^*|_{D(\pi^*)}$ for all $x \in A^1$. If $\pi = \pi^*$, then π is *self-adjoint*. Further, π is *standard* if $\pi(x^*)^* = \overline{\pi(x)}$ for all x in A^1. If each $\pi(x)$ is a bounded operator, then π is *bounded.* If π is a direct sum of bounded representations, then π is *weakly unbounded.* An O^*-*algebra* is a collection $\mathcal{U}$ of linear operators T all defined on a dense subspace D of a Hilbert space H such that for all $T \in \mathcal{U}$, one has $TD \subseteq D$, and $T^*D \subseteq D$; and $\mathcal{U}$ is a *-algebra with the pointwise linear operations, composition as the multiplication, and $T \to T^+ := T^*|_D$ as the involution. Given a *-representation $(\pi, D(\pi), H)$ of a *-algebra A, the *uniform topology* [20], ([37], p. 77–78) $\tau_D = \tau_{D(\pi)}^{\pi(A)}$ on the O^*-algebra $\pi(A)$ is the locally convex topology defined by the seminorms $\{q_K : K$ is a bounded subset of $(D(\pi), t_\pi)\}$, where

$$q_K(\pi(x)) = \sup\{|\langle \pi(x)\xi, \eta\rangle| : \xi, \eta \text{ in } K\}.$$

A vector ξ in $D(\pi)$ is *strongly cyclic* [30] (called cyclic in [37]) if $D(\pi) = (\pi(A)\xi)^{-t_\pi}$ the closure of $(\pi(A)\xi)$ in $(D(\pi), t_\pi)$. By a *cyclic vector*, we mean ξ in $D(\pi)$ such that the norm closure $(\pi(A)\xi)^- = H$. For topological *-algebras, we refer to [21]. A *Q-algebra* is a topological algebra whose quasi-regular elements form an open set. An *LFQ-algebra* is a Q-algebra which is an LF-space [41]. The topology of a locally convex (respectively locally m-convex) *-algebras A is determined by the family $K(A)$ (respectively $K_s(A)$), or a separating subfamily $\mathcal{P}$ thereof, consisting of continuous *-seminorms (repsectively continuous submultiplicative *-seminorms) p. If A has a bounded approximate identity (e_i), then it is assumed that $p(e_i) \leq 1$ for all i and all p. A *pro-*C^*-*algebra* is a complete locally m-convex *-algebra whose topology is determined by a family of C^*-seminorms. A *Frechet* *-*algebra* (respectively *locally convex* F^*-*algebra*) is a complete metrizable locally m-convex (respectively locally convex) *-algebra. A σ-C^*-*algebra* means a Frechet pro-C^*-algebra. For pro-C^*-algebras, we refer to [26, 27].

2. Preliminary constructions and lemmas

Let A be a *-algebra, not necessarily having an identity element. Let f be a positive linear functional on A. Then f is *representable* if there exists a closed strongly cyclic *-representation $(\pi, D(\pi), H)$ of A having a strongly cyclic vector $\xi \in D(\pi)$ such that $f(x) = \langle \pi(x)\xi, \xi\rangle$ for all $x \in A$. If π can be chosen to be a bounded operator representation, then f is *boundedly representable.* The first half of the following is an unbounded representation theoretic analogue of ([39], Theorem 1), whereas the remaining half improves a part of ([39], Theorem 1) even in the bounded case. The proof exhibits the unbounded analogue of the GNS construction in the case of non-unital algebras. This provides a useful supplement to ([37], § 8.6). It is well-known that a representable functional is boundedly representable if and only if it is *admissible* in the sense that for each $x \in A$, there exists $k > 0$ such that $f(y^*x^*xy) \leq kf(y^*y)$ for all $y \in A$. In the following, Lemma 2.1(3) is very close to ([39], Theorem 1) in which a C^*-seminorm p is taken.

Lemma 2.1. *Let f be a positive linear functional on a *-algebra A. The following are equivalent.*

(1) *f is representable.*

(2) *There exists $m > 0$ such that $|f(x)|^2 \leq mf(x^*x)$ for all $x \in A$.*

Further, f is boundedly representable if and only if f satisfies (2) *above and the following.*

(3) *There exists a submultiplicative *-seminorm p on A and $M > 0$ such that $|f(x)| \leq Mp(x)$ for all $x \in A$.*

When A is a Banach *-algebra, Lemma 2.1 is given in ([7], Theorem 37.11, p. 199). In the framework of unbounded representation theory, it is discussed in [2]. There is a gap in the proof in ([7], Theorem 37.11) in that hermiticity of f has been implicitly used. Regrettably it remained unnoticed in [2]. This was rectified in [39] in the formalism of bounded representations. The following proof provides an analogous correction in the context of unbounded representations.

Proof. Suppose (1) holds with $f(x) = \langle \pi(x)\xi, \xi\rangle$ for all $x \in A$. Then for all x in A.

$$\begin{aligned}|f(x)|^2 &\leq ||\pi(x)\xi||^2||\xi||^2 \leq ||\xi||^2\langle \pi(x)\xi, \pi(x)\xi\rangle \\ &= ||\xi||^2\langle \pi(x)^*\pi(x)\xi, \xi\rangle = ||\xi||^2\langle \pi(x^*)\pi(x)\xi, \xi\rangle\end{aligned}$$

as $\pi(A)\xi \subseteq D(\pi) = D(\pi(x)) = D(\pi(x^*))$ and $\pi(x^*) \subseteq \pi(x)^*$. Thus

$$|f(x)|^2 \leq ||\xi||^2\langle \pi(x^*x)\xi, \xi\rangle = ||\xi||^2 f(x^*x)$$

for all $x \in A$, giving (2).

Conversely, assume (2). We adopt the GNS construction. Let $N_f = \{x \in A : f(x^*x) = 0\}$. By the Cauchy-Schwarz inequality, N_f is a left ideal of A. Let $X_f = A/N_f$, and $\lambda_f : A \longrightarrow X_f$ be $\lambda_f(x) = x + N_f$. Then $\langle \lambda_f(x), \lambda_f(y)\rangle = f(y^*x)$ defines an inner product on X_f. Let H_f be the Hilbert space obtained by completing X_f. Let $\varphi : X_f \rightarrow \mathbb{C}$ be $\varphi(\lambda_f(x)) = f(x)$, a linear functional. Then for all $x \in A$,

$$|\varphi(\lambda_f(x))|^2 = |f(x)|^2 \leq mf(x^*x) = m\langle \lambda_f(x), \lambda_f(x)\rangle = m||\lambda_f(x)||^2.$$

Thus φ extends uniquely to H_f as a bounded linear functional; and by Riesz theorem, there exists a $\xi \in H_f$ such that for all $x \in A, f(x) = \varphi(\lambda_f(x)) = \langle \lambda_f(x), \xi\rangle$. Further, if m is the minimum possible constant in the assumed inequality, then $||\xi|| = m^{1/2}$. The idea of using Riesz theorem at this stage is borrowed from [39]. Define a *-representation $(\pi_0, D(\pi_0), H_f)$ of A by: $D(\pi_0) = X_f$; and for any x in A, $\pi_0(x)\lambda_f(y) = \lambda_f(xy)$ for all y in A. Let π be the closure of π_0. Then for all x, y in A,

$$\langle \lambda_f(x), \lambda_f(y)\rangle = f(y^*x) = \langle \lambda_f(y^*x), \xi\rangle = \langle \pi_0(y^*)\lambda_f(x), \xi\rangle. \tag{i}$$

Assertion 1. $X_f = \pi_0^*(A)\xi$.

Let $x \in A$. For all $y \in A$,

$$\begin{aligned}|\langle \pi_0(x)\lambda_f(y), \xi\rangle| &= |\langle \lambda_f(xy), \xi\rangle| = |f(xy)| \leq f(xx^*)^{1/2}f(y^*y)^{1/2} \\ &= f(xx^*)^{1/2}||\lambda_f(y)||\end{aligned}$$

showing that the linear functional $\lambda_f(y) \rightarrow \langle \pi_0(x)\lambda_f(y), \xi\rangle$ on $D(\pi_0)$ is $||\ \ ||$-continuous. Hence $\xi \in D(\pi_0(x)^*)$ for all $x \in A$. It follows, by the definition of $D(\pi_0^*)$, that $\xi \in D(\pi_0^*)$. Now (i) becomes $\langle \lambda_f(x), \lambda_f(y)\rangle = \langle \lambda_f(x), \pi_0(y^*)^*\xi\rangle$ for all $x \in A$. Since X_f is dense in H_f, we obtain $\lambda_f(y) = \pi_0(y^*)^*\xi = \pi_0^*(y)\xi$ for all y in A. Thus $X_f = \pi_0^*(A)\xi$.

Assertion 2. $\xi \in D(\pi)$.

Since $\overline{\pi_0(x)} = \pi_0(x)^{**}$, we show that $\xi \in D(\pi_0(x)^{**})$ for all $x \in A$, i.e., for all x, the functional on $D(\pi_0(x)^*)$ given by $\eta \rightarrow \langle \pi_0(x)^*\eta, \xi\rangle$ is $||\ \ ||$-continuous. Fix an $x \in A$. Now $\xi \in D(\pi_0^*)$, hence $\xi \in D(\pi_0(x^*)^*)$ so that the functional g on $D(\pi_0(x^*)) = X_f$ defined by $g(\eta) = \langle \pi_0(x^*)\eta, \xi\rangle$ is $||\ \ ||$-continuous, and extends continuously to H_f. Now let $\psi \in D(\pi_0(x)^*)$. Let (η_k) be a sequence in X_f such that $\eta_k \rightarrow \psi$ in $||\ \ ||$. Then $\xi \in D(\pi_0^*)$ implies that for any $x \in A$,

$$\begin{aligned}\langle \pi_0(x)^*\psi, \xi\rangle &= \langle \psi, \pi_0(x^*)^*\xi\rangle = \langle \psi, \pi_0^*(x)\xi\rangle \\ &= \lim\langle \eta_k, \pi_0^*(x)\xi\rangle = \lim\langle \pi_0(x^*)\eta_k, \xi\rangle = g(\psi)\end{aligned}$$

showing that $\psi \to \langle \pi_0(x)^*\psi, \xi\rangle$ is $\|\ \|$-continuous on $D(\pi_0(x)^*)$. This proves the assertion 2.

Now by the proof of assertions 1 and 2 above, it follows that for any $x \in A$,

$$\begin{aligned} f(x) = \varphi(\lambda_f(x)) = \langle \lambda_f(x), \xi\rangle = \langle \pi_0(x^*)^*\xi, \xi\rangle = \langle \pi_0^*(x)\xi, \xi\rangle \\ = \langle \bar{\pi}_0(x)\xi, \xi\rangle = \langle \pi(x)\xi, \xi\rangle. \end{aligned}$$

Clearly ξ is a strongly cyclic vector for π. Thus (2) implies (1).

Now assume (2) and (3). Let $N_p = \{x \in A : p(x) = 0\}$, a *-ideal in A. Let A_p be the Banach *-algebra obtained by completing A/N_p in the norm $\|x_p\|_p = p(x)$ where $x_p = x + N_p$. By (3), $F(x_p) = f(x)$ gives a well-defined continuous positive functional on A_p. By standard Banach *-algebra theory, for all x, y in A,

$$\begin{aligned} \|\pi(x)\pi(y)\xi\|^2 = \langle \pi(y^*x^*xy)\xi, \xi\rangle = f(y^*x^*xy) = F(y_p^*x_p^*x_py_p) \\ \leq \|x_p^*x_p\|F(y_p^*y_p) \leq p(x)^2 f(y^*y) = p(x)^2\|\pi(y)\xi\|^2. \end{aligned}$$

Since $\pi(A)\xi$ is dense in H_f, π is a bounded operator representation.

COROLLARY 2.2

Let A be a *-algebra.

(1) *A positive functional f on A is representable if and only if f is extendable as a positive functional on the unitization A^1 of A.*
(2) *A representable positive functional on A satisfies $f(x^*) = f(x)^-$ for all x in A.*
(3) *Let A be a topological *-algebra having a bounded approximate identity. Then every continuous positive functional on A is representable.*

COROLLARY 2.3

Let A be a complete locally m-convex *-algebra with a bounded approximate identity (e_γ) satisfying $p(e_\gamma) \leq 1$ for all p in a defining family of seminorms.

(1) *Let f be a continuous positive functional on A. Then f is boundedly representable and there exists $p \in K_s(A)$ such that $|f(x)| \leq (\limsup f(e_\gamma e_\gamma^*))p(x)$ for all $x \in A$.*
(2) *Let $(\pi, \mathcal{D}(\pi), H)$ be a *-representation of A. Then each $\pi(e_\gamma)$ is a bounded operator and $\|\pi(e_\gamma)\| \leq 1$ for all γ. Further, if π is strongly continuous (in particular, if π is continuous in the unifrom topology, which is the case if A is locally convex F^** ([37], *Theorem 3.6.8, p. 99)), then $\|\pi(e_\gamma)\xi - \xi\| \to 0$ for each ξ.*

Proof. (1) By continuity, there exist $p \in K_s(A)$ and $m > 0$ such that $|f(x)| \leq mp(x)$ for all $x \in A$. Now Lemma 2.1 applies by Corollary 2.2(3). Let $l = \limsup f(e_\gamma e_\gamma^*)$, which is finite. Let $c = \sup\{|f(x)| : p(x) = 1\}$. Choose a sequence (x_n) in A such that $f(x_n) \to c$ and $p(x_n) = 1$ for all n. Then, by the Cauchy-Schwarz inequality,

$$|f(x_n)|^2 = \lim_\gamma |f(e_\gamma x_n)|^2 \leq (\limsup f(e_\gamma e_\gamma^*))f(x_n^*x_n) \leq lc,$$

as $p(x_n^*x_n) \leq p(x_n)^2 = 1$. Hence $c^2 \leq lc$, i.e. $c \leq l$, and the assertion follows.

(2) Let $P = (p_\alpha)$ be a cofinal subset of $K_s(A)$ determining the topology of A. Let $A_p = \{x \in A : \sup_\alpha p_\alpha(x) < \infty\}$. Then A_p is a *-subalgebra of A containing each e_γ. As A is complete, A_p is a Banach *-algebra with norm $p(x) = \sup_\alpha p_\alpha(x)$. For any $\xi \in H$,

consider the positive functional $\omega_\xi(x) = \langle \pi(x)\xi, \xi\rangle$ on A. Then for all $x \in A$, $|\omega_\xi(x)|^2 \leq ||\xi||^2\ \omega_\xi(x^*x)$. By Lemma 2.1, ω_ξ is representable, hence extends as a positive functional ω on the unitization A^1 of A. In view of the inclusion map $(A_p)^1 \to A^1$, ω is a positive functional on $(A_p)^1$. By ([7], Corollary 37.9, p. 198), ω is continuous in the norm of $(A_p)^1$. It follows that ω_ξ restricted to A_p is continuous in the norm of A_p and $||\omega_\xi|| \leq ||\xi||^2$. For each γ,

$$||\pi(e_\gamma)\xi||^2 = \omega_\xi(e_\gamma e_\gamma^*) \leq ||\omega_\xi|| p(e_\gamma)^2 \leq ||\xi||^2$$

showing that $||\pi(e_\gamma)|| \leq 1$. Now suppose that π is strongly continuous. Let $\eta \in \mathcal{D}(\pi)$ and $\varepsilon > 0$. There exists $x \in A$ and $\eta' \in \mathcal{D}(\pi)$ such that $||\pi(x)\eta' - \eta|| \leq \varepsilon/3$. Since $e_\gamma x \to x$, there exists γ_0 such that for all $\gamma \geq \gamma_0$,

$$\begin{aligned} ||\eta - \pi(e_\gamma)\eta|| \leq ||\eta - \pi(x)\eta'|| &+ ||\pi(x)\eta' - \pi(e_\gamma x)\eta'|| \\ &+ ||\pi(e_\gamma)||\ ||\pi(x)\eta' - \eta|| < \varepsilon \end{aligned}$$

showing that $\pi(e_\gamma)\eta \to \eta$ for each $\eta \in \mathcal{D}(\pi)$. This completes the proof of Corollary 2.3.

The enveloping pro-C^*-algebra $E(A)$

We construct the enveloping pro-C^*-algebra $E(A)$ for a locally convex *-algebra A with jointly continuous multiplication. This extends the consideration in [10, 15, 19] in which A is additionally assumed m-convex. The added generality will include several constructions relevant in C^*-algebra theory (like the C^*-algebra of a groupoid). Let $R(A)$ denote the set of all continuous bounded operator *-representations $\pi : A \to B(H_\pi)$ of A into the C^*-algebras $B(H_\pi)$ of all bounded linear operators on Hilbert spaces H_π. Let $R'(A) = \{\pi \in R(A) : \pi \text{ is topologically irreducible}\}$. For $p \in K(A)$, let

$$R_p(A) = \{\pi \in R(A) :\ \text{for some } k > 0, ||\pi(x)|| \leq kp(x) \text{ for all } x\},$$

and $R_p'(A) = R_p(A) \cap R'(A)$. Then

$$R(A) = \bigcup\{R_p(A) : p \in K(A)\}, \quad R'(A) = \bigcup\{R_p'(A) : p \in K(A)\}.$$

Let $r_p(x) = \sup\{||\pi(x)|| : \pi \in R_p(A)\}$.

Lemma 2.4. *Let A be as above, $p \in K(A)$. Then $r_p(\ )$ is a continuous C^*-seminorm on A satisfying $r_p(x) \leq p(x^*x)^{1/2}$. If $p \in K_s(A)$, then $r_p(x) = \sup\{||\pi(x)|| : \pi \in R_p'(A)) \leq p(x)$ for all $x \in A$.*

Proof. Let $s_p(x) = p(x^*x)^{1/2}$. Let $h = h^* \in A$ and $\pi \in R_p(A)$. Then $||\pi(h^n)|| \leq kp(h^n)$ for all $n \in \mathbb{N}$. By standard Banach algebra arguments, the spectral radius satisfies

$$r(\pi(h)) = \lim\inf||\pi(h^n)||^{1/n} = \inf||\pi(h^n)||^{1/n} \leq \inf p(h^n)^{1/n} \leq p(h).$$

Hence, for any $x \in A$,

$$||\pi(x)||^2 = ||\pi(x^*x)|| = r(\pi(x^*x)) \leq p(x^*x),$$

so that $r_p(x) \leq s_p(x)$. We use the joint continuity of multiplication to conclude the continuity of the C^*-seminorm $x \to r_p(x)$. Now suppose $p \in K_s(A)$. Then

$$r_p(x) \leq s_p(x) \leq (p(x^*)p(x))^{1/2} \leq p(x).$$

Further, let $N_p = \{x \in A : p(x) = 0\}$, a closed *-ideal in A. Let A_p be the Banach *-algebra obtained by completing A/N_p in the norm $||x + N_p||_p = p(x)$. Then $R_p(A)$ (respectively $R'_p(A)$) can be identified with $R(A_p)$ (respectively $R'(A_p)$). The assertion follows from the fact that for all $z \in A_p$,

$$\sup\{||\varphi(z)|| : \varphi \in R(A_p)\} = \sup\{||\varphi(z)|| : \varphi \in R'(A_p)\}$$

([14], 2.7, p. 47). This completes the proof of the lemma.

Define the *star radical* to be

$$\begin{aligned}\operatorname{srad}(A) &= \{x \in A : r_p(x) = 0 \text{ for all } p \in K(A)\}\\ &= \{x \in A : \pi(x) = 0 \text{ for all } \pi \in R(A)\}.\end{aligned}$$

For each $p \in K(A)$, $q_p(x + \operatorname{srad}(A)) = r_p(x)$ defines a continuous C^*-seminorm on the quotient locally convex *-algebra $A/\operatorname{srad}(A)$ with the quotient topology. Let τ be the Hausdorff topology on $A/\operatorname{srad}(A)$ defined by $\{q_p : p \in K(A)\}$. The *enveloping pro-C^*-algebra* $E(A)$ of A is the completion of $(A/\operatorname{srad}(A), \tau)$. When A is metrizable, $E(A)$ is metrizable. In view of Corollary 2.2, when A is m-convex, this coincides with the enveloping l.m.c. *-algebra defined in [10, 19, 15].

Lemma 2.5. *Let A be a locally convex *-algebra with jointly continuous multiplication.*

(a) *Let $\bar{A}$ be the completion of A. Then $E(\bar{A}) = E(A)$.*
(b) $E(A^1) = E(A)^1$.

Proof. Since A has jointly continuous multiplication, $\bar{A}$ is a complete locally convex *-algebra. The map $i : A/\operatorname{srad}(A) \to \bar{A}/\operatorname{srad}(\bar{A})$, where $i(x + \operatorname{srad}(A)) = x + \operatorname{srad}(\bar{A})$, is a well defined *-isomorphism into $E(\bar{A})$. Note that for any $p \in K(\bar{A})$, $R_p(A) = R_p(\bar{A})$ via the restriction (in fact, also $K(\bar{A}) = K(A)$), hence $\operatorname{srad}(A) = A \cap \operatorname{srad}(\bar{A})$. For any $p \in K(A)$, let $\tilde{p} \in K(\bar{A})$ be the unique extension of p. Then, for any $x \in A$,

$$q_p(x + \operatorname{srad}(A)) = r_p(x) = q_{\tilde{p}}(x + \operatorname{srad}(\bar{A}));$$

and for any $\tilde{p} \in K(\bar{A})$, $q_{\tilde{p}}(x + \operatorname{srad}(A)) = q_{\tilde{p}|A}(x + \operatorname{srad}(A))$. Thus i is a homeomorphism for the respective pro-C^*-topologies. On the other hand, i has dense range in $\bar{A}/\operatorname{srad}(\bar{A})$. Indeed, let $z \in \bar{A}$. Choose a net (x_i) in A such that $x_i \to z$ in the topology t of $\bar{A}$. Then

$$\begin{aligned}q_{\tilde{p}}(x_i - z + \operatorname{srad}(A)) &= r_{\tilde{p}}(x_i - z) = \sup\{||\pi(x_i - z)|| : \pi \in R_{\tilde{p}}(\bar{A})\}\\ &\leq k\tilde{p}(x_i - z) \to 0\end{aligned}$$

for all $\tilde{p} \in K(\bar{A})$. Thus $E(\bar{A})$, which is the completion of $\bar{A}/\operatorname{srad}(\bar{A})$, coincides with the completion $E(A)$ of $A/\operatorname{srad}(A)$. This completes the proof of (a). We omit the proof of (b).

A representation $(\pi, D(\pi), H)$ of A is *countably dominated* if there exists a countable subset B of A such that for any $x \in A$, there exists $b \in B$ and a scalar $k > 0$ such that $||\pi(x)\xi|| \leq k||\pi(b)\xi||$ for all $\xi \in D(\pi)$ ([22], p. 419).

Lemma 2.6. (a) *Let A be a locally convex *-algebra. Let $j : A \to E(A)$, $j(x) = x + \operatorname{srad}(A)$.*

(1) *If $\pi : A \to B(H)$ is a continuous bounded operator *-representation, then there exists a unique continuous *-representation $\sigma : E(A) \to B(H)$ such that $\pi = \sigma \circ j$. Further, π is irreducible if and only if σ is irreducible.*

(2) *Let $(\pi, \mathcal{D}(\pi), H)$ be a closed *-representation of A continuous in the unifor topology. Let π be weakly unbounded. Then there exists a closed weakly unboune *-representation $(\sigma, \mathcal{D}(\sigma), H)$ of $E(A)$ such that $\pi = \sigma \circ j$ and $\mathcal{D}(\sigma)$ is dense in th locally convex space $(\mathcal{D}(\pi), t_\pi)$.*

(3) *Let A be unital and symmetric. Assume that A is separable or nuclear (as a locall convex space). Let $(\pi, \mathcal{D}(\pi), H)$ be a separably acting, countably dominate *-representation of A continuous in the uniform topology. Then there exists a close *-representation $(\sigma, \mathcal{D}(\sigma), H)$ of $E(A)$ such that $\pi = \sigma \circ j$.*

(b) (1) *There exists a unital, locally convex, non-m-convex, F^*-algebra A such that admits a faithful family of unbounded operator *-representations, but admits n non-zero bounded operator *-representation.*

(2) *There exists a unital non-locally-convex F^*-algebra that admits no non-zer *-representation.*

Proof. (a) (1) follows by the definition of $E(A)$.
(2) Let $\pi = \oplus\pi_i$, where each π_i is a norm continuous bounded operator *-representatio $\pi_i : A \to B(H_i)$ on a Hilbert space H_i. We take $D(\pi_i) = H_i$. Let $E_i : H \to H_i$ be th orthogonal projection. By (1), there exist continuous *-homomorphisms $\sigma_i : E(A)$ – $B(H_i)$, $\sigma_i \circ j = \pi_i$. Let $\sigma = \oplus\sigma_i$ on the Hilbert direct sum $\oplus H_i = H$ having the domai

$$\mathcal{D}(\sigma) = \{\eta = \Sigma E_i\eta \in H : \Sigma||\sigma_i(z)E_i\eta||^2 < \infty \text{ for all } z \in E(A)\}$$
$$\subset \mathcal{D}(\pi) = \{\eta = \Sigma E_i\eta \in H : \Sigma||\pi_i(x)E_i\eta||^2 < \infty \text{ for all } x \in A\}.$$

On $\mathcal{D}(\sigma)$, the σ-graph topology $t_{\sigma(E(A))}$ is finer than the relativized π-graph topolog $t_\pi|_{\mathcal{D}(\sigma)}$. Being closed and weakly unbounded, both σ and π are standard representation Hence, for all $h = h^*$ in A, the operators $\sigma(j(h))$ having domain $\mathcal{D}(\sigma)$ and $\pi(h)$ wi domain $\mathcal{D}(\pi)$ are essentially self-adjoint. Since self-adjoint operators are maximal symmetric, $\mathcal{D}(\sigma)$ is dense in $\mathcal{D}(\overline{\pi(h)})$ for the graph topology defined by $\xi \to ||\xi||$ $||\overline{\pi(h)}\xi||$. Thus $\mathcal{D}(\sigma)$ is dense in the locally convex space $\mathcal{D}(\pi) = \cap\{\mathcal{D}(\overline{\pi(h)}) : h =$ in $A\}$.
(3) By ([22], Theorem 3.2 and remark on p. 422) and ([37], Theorem 12.3.5, p. 343 there exists a compact Hausdorff Z with a positive measure μ such that

$$\pi = \int_Z^\oplus \pi_\lambda \mathrm{d}\mu(\lambda), \quad \mathcal{D}(\pi) = \int_Z^\oplus \mathcal{D}(\pi_\lambda)\mathrm{d}\mu(\lambda), \quad H = \int_Z^\oplus H_\lambda \mathrm{d}\mu(\lambda)$$

and each π_λ is irreducible. Since A is symmetric, each $\bar{\pi}$ and $\bar{\pi}_\lambda$ are standard ([37 Corollary 9.1.4, p. 237) (the commutativity assumption in this reference is not require as the arguments in ([2], Theorem 3.5) shows); and by [3], each π_λ is a bounded operat representaion, being irreducible. Then we can proceed as in (2).
(b) (1) Take $A = L^\omega[0,1] = \bigcap_{1 \le p < \infty} L^p[0,1]$ (the Arens algebra) with pointwise operation complex conjugation, and the topology of L^p-convergence for each p, $1 \le p < \infty$. T algebra A is a unital, symmetric, locally convex F^*-algebra, admitting a faithful standa *-representation $(\pi, \mathcal{D}(\pi), H)$ such that $\overline{\pi(A)}$ is an extended C^*-algebra with a comm dense domain [13]. However, there exists no non-zero bounded operator representation A, as A admits no non-zero multiplicative linear functional; and hence no non-ze submultiplicative *-seminorm. Thus srad $(A) = A$ and $E(A) = (0)$. (2) Take $A = \mathcal{M}$ [0, the algebra of all Lebesgue measurable functions on [0,1] with the topology

convergence in measure. It admits no non-zero positive linear functional, and hence no non-zero $*$-representation.

Remark. 2.7. We call a $*$-representation $(\pi, \mathcal{D}(\pi), H)$ of a $*$-algebra A *boundedly decomposable* if it can be disintegrated as $\pi = \int_Z^{\oplus} \pi_\lambda \mathrm{d}\mu(\lambda)$ with each π_λ a bounded operator $*$-representation. One may show that $E(A)$ is universal for all closed boundedly decomposable $*$-representations of a locally convex F^*-algebra A. We do not know whether in (2) and (3) of Corollary (2.4) (a), σ is continuous in the uniform topology.

The *bounded vectors* [4] for a $*$-representation π of a $*$-algebra A are $B(\pi) = \bigcap\{B(\pi(x)) : x \in A\}$, where, for an operator T, the bounded vectors for T are

$$B(T) = \{\xi \in \mathcal{D}(T) : \text{ there exists a } > 0, c > 0 \text{ such that } \\ ||T^n\xi|| \leq ac^n \text{ for all } n \in \mathbb{N}\}.$$

The following is motivated by [35]. It shows that unbounded representations of locally m-convex $*$-algebras cannot be wildly unbounded,

Lemma 2.8. *Let $(\pi, \mathcal{D}(\pi), H)$ be a closed $*$-representation of a complete locally m-convex $*$-algebra A continuous in the uniform topology on $\pi(A)$. Then the following hold.*

1. *$\mathcal{D}(\pi) = B(\pi)$; and π is a direct sum of norm-continuous cyclic bounded operator $*$-representations.*
2. *π is standard. For commuting normal elements x, y of A, the normal operators $\overline{\pi(x)}$ and $\overline{\pi(y)}$ have mutually commuting spectral projections.*
3. *The uniform topology $\tau_\mathcal{D}$ on $\pi(A)$ is a pro-C^*-topology, i.e., it is determined by a family of C^*-seminorms.*
4. *If A is Frechet, then $\tau_\mathcal{D}$ is metrizable and π is direct sum of a countable number of cyclic bounded-operator $*$-representations.*

Proof. Let $\xi \in \mathcal{D}(\pi)$. Let ω_ξ on A be the positive functional $\omega_\xi(x) = \langle \pi(x)\xi, \xi \rangle$ for $x \in A$. By Lemma 2.1, ω_ξ is representable and admissible. Hence the closed GNS representation $(\pi_{\omega_\xi}, \mathcal{D}(\pi_{\omega_\xi}), H_{\omega_\xi})$ associated with ω_ξ is a cyclic, norm-continuous bounded operator $*$-representation with $\mathcal{D}(\pi_{\omega_\xi}) = H_{\omega_\xi}$. Let ξ_ω denote the cyclic vector for π_{ω_ξ}. Let $\mathcal{D}(\pi_\xi) = (\pi(A)\xi)^{-t_\pi}$ and $H_\xi = [\pi(A)\xi]^-$. Since π is closed, $\mathcal{D}(\pi_\xi) \subset \mathcal{D}(\pi)$. The π-invariant subspace $\mathcal{D}(\pi_\xi)$ defines a closed subrepresentation $\langle \pi_\xi, \mathcal{D}(\pi_\xi), H_\xi \rangle$ of π as $\pi_\xi(x) = \pi(x)|_{\mathcal{D}(\pi_\xi)}$. Since $\langle \pi_{\omega_\xi}(x)\xi_{\omega_\xi}, \xi_{\omega_\xi} \rangle = \omega_\xi(x) = \langle \pi_\xi(x)\xi, \xi \rangle$ for all $x \in A$, it follows that π_{ω_ξ} and π_ξ are unitarily equivalent. Thus π_ξ is a bounded operator representation, and $\mathcal{D}(\pi_\xi) = H_\xi \subset B(\pi)$. This also implies that H_ξ is reducing in the sense of ([37], § 8.3). Thus the following is established.

Assertion I. For any ξ in $\mathcal{D}(\pi)$, $[\pi(A)\xi]^{-t_\pi} = [\pi(A)\xi]^- \subset B(\pi)$.

It follows that $\pi(A)\mathcal{D}(\pi) \subset B(\pi)$, hence $B(\pi)$ is dense in $(\mathcal{D}(\pi), t_\pi)$ and norm dense in H. Since $B(\pi)$ forms a set of common analytic vectors for $\pi(A)$, the conclusion (2) follows, using ([40], Theorem 2). Also, a standard Zorn's lemma argument gives $\pi = \oplus\pi_i$, with each π_i a cyclic, continuous, bounded operator representation.

Assertion II. For each bounded subset M of $(\mathcal{D}(\pi), t_\pi)$, there exists $p \in K_s(A)$ such that $||\pi(x)\eta|| \leq ||\eta||p(x)$ for all $x \in A$, $\eta \in M$.

By continuity, given M as above, there is $k > 0$ and $p \in K_s(A)$ such that $q_M(\pi(x))$ $kp(x)$ for all $x \in A$. Hence, for each $\eta \in M$ and $x \in A$, $||\pi(x)\eta||^2 \leq kp(x^*x) \leq kp(x)^2$. Corollary 2.3, $||\pi(x)\eta||^2 \leq lp(x)^2$, where $l = \limsup \omega_\eta(e_\gamma e_\gamma^*) \leq ||\eta||^2$. Hence $||\pi(x)\eta||$ $||\eta|| p(x)$ for all $x \in A$, all $\eta \in M$.

Now let $\xi \in \mathcal{D}(\pi)$. By (II) above, there exists $p \in K_s(A)$ such that for all $n \in \mathbb{N}$,

$$||\pi(x)^n\xi||^2 = \langle \pi(x^{*^n}x^n)\xi, \xi\rangle \leq ||\xi||^2 p(x)^{2n}$$

showing that $\xi \in B(\pi(x))$. Thus $\mathcal{D}(\pi) = B(\pi)$ proving (1).

The proof of (3) is based on arguments in ([35], Theorem 1). Let $\mathcal{F}$ be the collection all subspaces (linear manifolds) K of $\mathcal{D}(\pi)$ such that K is π-invariant, and $\pi|_K$ is bounded operator *-representation. For $K \in \mathcal{F}$, let s_K be the C^*-seminorm

$$s_K(\pi(x)) = \sup\{||\pi(x)\eta|| : \eta \in K, ||\eta|| \leq 1\}.$$

Let τ_1 be the topology on $\pi(A)$ defined by $\{s_K : K \in \mathcal{F}\}$. We show that $\tau_{\mathcal{D}} = \tau_1$. Clea $\tau_1 \leq \tau_{\mathcal{D}}$. Let M be a bounded subset of $(\mathcal{D}(\pi), t_\pi)$. Choose k and p as in assertion (above. By Corollary 2.3, $|\omega_\xi(x)| \leq ||\xi|| p(x)$ for all $x \in A$, all $\xi \in M$. Thus

$$M \subset \mathcal{D}_p := \{\eta \in \mathcal{D}(\pi) : |\langle \pi(x)\eta, \eta\rangle| \leq ||\eta||^2 p(x) \text{ for all } x \text{ in} A\}.$$

Then $\mathcal{D}_p \in \mathcal{F}$; $||\pi(x)\eta|| \leq ||\eta||^2 p(x)^2$ for all $\eta \in \mathcal{D}_p$; and, as π is closed, ([35], Lemma implies that $\mathcal{D}_p$ is $||\ \ ||$-closed. Let $S = \{\xi \in \mathcal{D}_p : ||\xi|| \leq 1\}$. As M is also $||\cdot||$ bound $||\eta|| \leq r$ for all $\eta \in M$; and $M \subset rS$. Then, for all $x \in A$, $q_M(\pi(x)) \leq r^2 s_{\mathcal{D}_p}(\pi(x))$. T $\tau_{\mathcal{D}} \leq \tau$. This gives (3). Finally (4) is consequence of the fact that the topology o metrizable A is determined by a countable cofinal subfamily of $K_s(A)$. This completes proof of Lemma 2.8.

Now let A be commutative. Let $\mathcal{M}(A)$ be the Gelfand space consisting of all non-z continuous multiplicative linear functionals on A. Let $\mathcal{M}^*(A) = \{\varphi \in \mathcal{M}(A) : \varphi = \varphi$ and $\varphi^*(x) = \overline{\varphi(x^*)}$. For each $x \in A$, let $\hat{x} : \mathcal{M}^*(A) \longrightarrow \mathbb{C}$ be the map $\hat{x}(\varphi) = \varphi(x)$. T following, which incorporates the spectral theorem for unbounded normal operato describes all unbounded *-representations of A. The proof can be constructed us Lemma 2.8 and ([9], Theorem 7.3), in which all bounded *-representations of A h been realized.

COROLLARY 2.9

*Let A be a commutative complete locally m-convex *-algebra. Let $(\pi, \mathcal{D}(\pi), H)$ b closed *-representation of A continuous in the uniform topology. Then there exis positive regular Borel measure μ on $\mathcal{M}^*(A)$ and a spectral measure E on the Borel set $\mathcal{M}^*(A)$ with values in $B(H)$ such that the following hold.*

(1) *π is a unitarily equivalent to the representation $(\sigma, \mathcal{D}(\sigma), H_\sigma)$ by multiplicat operators in $H_\sigma = L^2(\mathcal{M}^*(A), \mu)$ with domain*

$$\mathcal{D}(\sigma) = \{f \in H_\sigma : \varphi \rightarrow \hat{x}(\varphi)f(\varphi) \text{is in} H_\sigma \text{ for all } x \in A\}$$

defined as $(\sigma(x)f)(\varphi) = \hat{x}(\varphi)f(\varphi)$.

(2) *For each $x \in A$, $\pi(x) = \int_{\mathcal{M}^*(A)} \hat{x}(\varphi) \mathrm{d}E(\varphi)$.*

We say that a locally convex *-algebra A is an *algebra with a C^*-enveloping algebr* the pro-C^*-algebra $E(A)$ is a C^*-algebra. In view of Lemma 2.5, we do not nee assume A to be complete or unital. In [5], A is further assumed to be m-convex.

following extends the main results in ([5], § 2) to the present more general set up, and can be proved as in [5]. A is called an *sQ-algebra* if for some $k > 0$, $p \in K(A)$, the spectral radius r satisfies $r(x^*x)^{1/2} \leq kp(x)$ for all $x \in A$; A is *-*sb* if $r(x^*x) < \infty$ for each x, equivalently, $r(h) < \infty$ for all $h = h^*$. Thus $Q => sQ => {}^*$-*sb*.

Lemma 2.10. *Let A be a complete locally convex *-algebra with jointly continuous multiplication.*

(1) *A is an algebra with a C^*-enveloping algebra if and only if A admits greatest continuous C^*-seminorm.*
(2) *If A is sQ, then A admits a greatest C^*-seminorm, which is also continuous.*
(3) *Let A be an F^*-algebra. If A is *-sb, then A has a C^*-enveloping algebra; but the converse does not hold* (see ([5], Example 2.4)).

The enveloping AO^*-algebra $O(A)$

For a locally convex *-algebra (A, t) (t denoting the topology of A), let $P_c(A, t)$ (respectively $P_{ca}(A, t)$) be the set of all continuous (respectively continuous admissible) representable positive functionals on A. For each f in $P_c(A, t)$, let $(\pi_f, \mathcal{D}(\pi_f), H_f)$ denote the strongly cyclic GNS representation defined by f as in Lemma 2.1. Let $I = \cap\{\ker \pi_f : f \in P_{ca}(A, t)\}$ and $J = \cap\{\ker \pi_f : f \in P_c(A, t)\}$. Then I and J are closed *-ideal of A, $J \subset I$, and $I = \operatorname{srad}(A)$ in view of the cyclic decomposability of any $\pi \in R(A)$. The *universal representation of* (A, t) is $\pi_u = \oplus\{\pi_f : f \in P_c(A, t)\}$. This is a slight variation of ([37], p. 228). Then $\sigma_u(x + J) = \pi_u(x)$ define a one–one *-homomorphism of A/J into the maximal O^*-algebra $\mathcal{L}^+(\mathcal{D}(\pi_u))$. Let $\sigma_u(t)$ be the topology on A/J induced by the uniform topology on $\pi_u(A)$; viz. $\sigma_u(t)$ is determined by the seminorms $\{q_M : M$ is a bounded subset of $(\mathcal{D}(\pi_u), t_{\pi_u})\}$, where $q_M(x + J) = \sup\{|\langle\pi_u(x)\xi, \eta\rangle| : \xi, \eta \text{ in } M\}$. Then $(A/J,\ \sigma_u(t))$ is an AO^*-algebra [36] in the sense that it is algebraically and topologically *-isomorphic to an O^*-algebra with uniform topology [37]. We call $(A/J,\ \sigma_u(t)$ the *enveloping AO^*-algebra* of A, denoted by $O(A)$.

Lemma 2.11. *Let A be as above.*

(1) *Every *-representation of A which is continuous in the uniform topology and which is a direct sum of strongly cyclic representations factors through $O(A)$. When A is either complete and m-convex, or is countably dominated, every *-representation of A continuous in the uniform topology factors through $O(A)$.*
(2) *Let A be barrelled. Then $\sigma_u(t)$ is coarser than the quotient topology t_q on A/J.*
(3) *There exists a continuous *-homomorphism from $O(A)$ into the pro-C^*-algebra $E(A)$.*
(4) *The following are equivalent.*
    (i) *$\sigma_u(t)$ is normable.*
    (ii) *$\sigma_u(t)$ is C^*-normable.*
    (iii) *There exists a linear norm on A/J defining a topology finer than $\sigma_u(t)$.*

When any of these conditions hold, and if A is barrelled, then A has a C^*-enveloping algebra; but the converse does not hold.

Proof. (1) follows from the construction of $O(A)$ and Lemma 2.8. (2) Let A be barrelled. Since J is closed. $(A/J, t_q)$ is barrelled ([32], ch. II, §7, Corollary 1, p. 61). Further, σ_u is

weakly continuous. Hence, σ_u is continuous in the uniform topology ([20], The
(3) Since $J \subset \mathrm{srad}\,A$, the map

$$\phi : A/J \to A/\mathrm{srad}\,A \to E(A), \phi(x + J) = x + \mathrm{srad}\,A$$

is a well defined *-homomorphism. Now, as $E(A)$ is a pro-C^*-algebra, $E(A)$
C^*-algebra for any $p \in K(A)$, denoted by $E_p(A)$, with the norm $\|z + \ker q_p$
$E(A) = \text{proj lim}\, E_p(A)$, inverse limit of C^*-algebras [26]. Let $\varphi_p : E(A)$
$\varphi_p(z) = z + \ker q_p$. For the continuity of $\phi : (O(A), \sigma_u(t)) \to (E(A), \tau)$, it is su
show the continuity of the *-homomorphism $\phi_p = \varphi_p \circ \phi : O(A) \to E_p(A)$. Now

$$\psi : A \to A/\mathrm{srad}\,(A) \to E(A) \to E_p(A), \psi(x) = (x + \mathrm{srad}\,(A)) + \ker$$

is a continuous bounded operator *-representation; and $\psi = \phi_p \circ j_u, j_u(x) = x$
ϕ_p is continuous for each $p \in K(A)$.
(4) (i) if and only if (ii) if and only if (iii) follows from ([20], Theorems 3.2, 3.
barrelled. Let $|\ |$ be a norm on A/J determining $\sigma_u(t)$. Since $t_q \to \sigma_u(t)$, p,
defines a continuous C^*-seminorm on A. Let p be any continuous C^*-seminorm
A_p be the completion of $A/\ker p$ in the C^*-norm $|x + \ker p| = p(x)$. Then π_p
$\pi_p(x) = x + \ker p$ defines a continuous bounded operator *-representation. By
exists a continuous *-homomorphism σ_p such that $\sigma_p \circ j_u = \pi_p$. Since the
topology on A_p is the $|\ |_p$-topology, and since $\sigma_u(t)$ is determined by $|\ |$, it follo
some $k > 0$, $|\sigma_p(z)| \leq k|z|$ for all $z \in A/J$. Thus $p(x) \leq kp_\infty(x)$; and so $p(x)$
all $x \in A$, both being C^*-seminorms. Thus p_∞ is the greatest continuous C^*-sem
A. By Lemma 2.8, $E(A)$ is a C^*-algebra. That the converse does not hold is ill
Arens' algebra $A = L^\omega[0, 1]$, wherein $E(A) = (o)$, $O(A) = A$ topologically as

3. Proofs of theorems 1.1, 1.2 and 1.4

Proof of Theorem 1.2. First we prove the following.

Assertion I. Given a bounded subset M of $(\mathcal{D}(\pi), t_\pi)$, there exists $p \in K_s(A$
such that $q_M(\pi(x)) \leq kr_p(x)$ for all $x \in A$.

By the continuity of π, given M, there exists $k > 0$ and $p \in K_s(A$
$q_M(\pi(x)) \leq kp(x)$ for all $x \in A$. Let $\xi \in M$. Then

$$|\omega_\xi(x)| = |\langle \pi(x)\xi, \xi \rangle| \leq q_M(\pi(x)) \leq kp(x)$$

for all x. Since ω_ξ is representable, it is extendable to A^1. The arguments in th
Corollary 2.3(1) applied to the extension of ω_ξ to A^1 give

$$\omega_\xi(x^*x) \leq ||\xi||^2 p(x^*x) \leq ||\xi||^2 p(x)^2$$

for all x in A. Thus $||\pi_{\omega_\xi}(x)\xi|| \leq ||\xi|| p(x)$: and by the definition of r_p,
$||\xi|| r_p(x)$ for all x in A. Since M is $||\ \ ||$-bounded, there exists $l > 0$ such that
M, all x in A,

$$|\omega_\xi(x^*x)| = ||\pi_{\omega_\xi}(x)\xi||^2 \leq l^2 r_p(x)^2.$$

It follows that for all x in A, and all ξ, η in M,

$$|\langle \pi(x)\xi, \eta \rangle| \leq ||\eta|| \omega_\xi(x^*x)^{1/2} \leq l^2 r_p(x).$$

Thus $q_M(\pi(x)) \leq l^2 r_p(x)$ for all x in A.

Now, by Lemma 2.8, $\pi = \oplus \pi_i$, with each $\pi_i : A \to B(H_i)$ norm continuous. By Lemma 2.6, there exists a closed representation $(\sigma', \mathcal{D}(\sigma'), H)$ $\sigma' = \oplus \sigma_i$ of $E(A)$, with each $\sigma_i : E(A) \to B(H_i)$ norm continuous, $\sigma_i \circ j = \pi_i$ for all i. We shall eventually show $\mathcal{D}(\pi) = \mathcal{D}(\sigma')$.

On the other hand, consider the *-representation $(\sigma, \mathcal{D}(\sigma), H)$ of $A/\mathrm{srad}\,(A)$ having domain $\mathcal{D}(\sigma) = \mathcal{D}(\pi)$, and given by $\sigma(j(x)) = \pi(x)$ for all $x \in A$. By ([37], Proposition 2.2.3, p. 39), on $\mathcal{D}(\pi)$, $t_\pi = t_{\mathcal{L}^+(\mathcal{D}(\pi))}$ which is the graph topology on $\mathcal{D}(\pi)$ due to the maximal O^*-algebra $\mathcal{L}^+(\mathcal{D}(\pi))$. Hence, on $\pi(A)$, the uniform topology $\tau_{\mathcal{D}}^{\pi(A)} = \tau_{\mathcal{D}}^{\mathcal{L}^+(\mathcal{D}(\pi))}$ $|_{\pi(A)} = \tau_1$ (say), which, by lemma 2.8, is a pro-C^*-topology. By ([37], Proposition 3.3.20, p. 85), $\sigma(A/\mathrm{srad}\,(A))$ is contained in a $\tau_{\mathcal{D}}^{\mathcal{L}^+(\mathcal{D}(\pi))}$-complete *-subalgebra of $\mathcal{L}^+(\mathcal{D}(\pi))$; and σ can be extended as a continuous *-homomorphism $\sigma(E(A), \tau) \to [\mathcal{L}^+(\mathcal{D}(\pi)), \tau_{\mathcal{D}}^{\mathcal{L}^+(\mathcal{D}(\pi))}]$ giving a closed *-representation σ of $E(A)$ on H with domain $D(\sigma) = D(\pi)$. Next we prove the following.

Assertion II. As representations of $E(A)$, $\sigma = \sigma'$.

This, we do, in the following steps.

(a) σ is an extension of σ'.

Clearly, $\mathcal{D}(\sigma') \subset \mathcal{D}(\pi) = \mathcal{D}(\sigma)$. We show $\sigma(z)'|_{\mathcal{D}(\sigma')} = \sigma'(z)$ for all $z \in E(A)$. Fix $z \in E(A)$. Let $\eta \in \mathcal{D}(\pi)$. Choose a net (x_r) in A such that for all $p \in K_s(A)$, $q_p(j(x_r) - z) \to 0$. Choose an appropriate p by (I) above. Then

$$\begin{aligned}
||\sigma(j(x_r))\eta - \sigma(j(x_{r'}))\eta||^2 &= ||\pi(x_r)\eta - \pi(x_{r'})\eta||^2 \\
&= \omega_\eta((x_r - x_{r'})^*(x_r - x_{r'})) \\
&\leq k\, r_p(x_r - x_{r'}) \\
&= k\, q_p(j(x)_r) - j(x_{r'})) \to 0.
\end{aligned}$$

Hence $\pi(x_r)\eta$ is norm Cauchy in $\mathcal{D}(\pi)$; and similarly, $\pi(x)\pi(x_r)\eta$ is norm Cauchy in $\mathcal{D}(\pi)$ for all $x \in A$. Thus $\pi(x_r)\eta$ is Cauchy in $(\mathcal{D}(\pi), t_\pi)$, which is complete as π is closed. Thus there exists $\xi \in \mathcal{D}(\pi)$ such that $\lim(x_r)\eta = \xi$ in t_π. This defines $\sigma(z)$ as $\sigma(z)\eta = \xi$, which gives $\sigma(z)|_{\mathcal{D}(\sigma')} = \sigma'(z)$.

(b) σ is a closed representation of $E(A)$.

Indeed, as π is closed.

$$\begin{aligned}
\mathcal{D}(\sigma) = \mathcal{D}(\pi) &= \bigcap\{\mathcal{D}(\overline{\pi_e(x)}) : x \in A^1\} \\
&= \bigcap\{\mathcal{D}(\overline{\sigma_e(j(x))} : j(x) \in j(A^1) = (j(A)^1)\} \\
&\supset \{\mathcal{D}(\overline{\sigma_e(z)}) : z \in (E(A))^1\} \\
&= \mathcal{D}(\bar{\sigma}) \supset \mathcal{D}(\sigma),
\end{aligned}$$

hence $\mathcal{D}(\sigma) = \mathcal{D}(\sigma')$. This also follows from the fact that π is closed: on $\mathcal{D}(\sigma) = \mathcal{D}(\pi)$, $t_\pi = t_{\mathcal{L}^+(\mathcal{D}(\pi))} = t_{\sigma((E(A)^1))}$; as well as $\pi(A) \subset \sigma(E(A)) \subset \mathcal{L}^+(\mathcal{D}(\pi))$. This further implies $\tau_{\mathcal{D}}^{\mathcal{L}^+(\mathcal{D}(\pi))}|_{\sigma(E(A))} = \tau_{\mathcal{D}}^{\sigma(E(A))}$; which, in turn gives the following.

(c) σ' is continuous in the uniform topology as a *-representation of $(E(A), \tau)$.

Now, by (c), Lemma 2.8 implies that the closed representation σ' is standard; hence self-adjoint, and so maximal hermitian ([31], (I), Lemma 4.2). Then (a) gives $\sigma' = \sigma$, thereby verifying (II). This completes the proof of Theorem 1.2.

If π is irreducible, then σ is irreducible, hence is a bounded operator representation by [3], ([6], Theorem 4.7). This gives Corollary 1.3.

Proof of Theorem 1.1. Let A be Frechet. Then $A = \text{proj lim} A_n$, an inverse limit of a sequence of Banach *-algebras A_n. Assume that each *-representation (and hence the universal representation π_u) of A is a bounded operator representation. Since A is Frechet, π_u is continuous. Let σ be the representation of $E(A)$ defined by Theorem 1.2 corresponding to π_u. Then σ is also a bounded operator *-representation. Further, as A is Frechet, $E(A) = \text{proj lim}\, C^*(A_n)$ is also Frechet. Thus σ is continuous and there exists a continuous C^*-seminorm $q_\circ$ on $E(A)$ such that $\|\sigma(z)\| \leq q_\circ(z)$ for all $z \in E(A)$. Now the bounded part of $E(A)$

$$b(E(A)) = \{z \in E(A) : q(z) < \infty \text{ for all continuous } C^*\text{-seminorm } q\}$$

is a C^*-algebra with the norm

$$\begin{aligned} \|z\|_\infty &= \sup\{q(z) : q \text{ is a continuous } C^*\text{-seminorm on } E(A)\} \\ &= \sup\{q_p(z) : p \in K_s(A)\}. \end{aligned}$$

Since σ is one–one, the restriction $\sigma' = \sigma|_{b(E(A))}$ is a *-isomorphism of the C^*-algebra $b(E(A))$ into $B(H_\sigma)$. Hence, for all $z \in b(E(A))$,

$$\|\sigma'(z)\| = \|z\|_\infty \geq q_\circ(z) \geq \|\sigma(z)\|.$$

It follows that $b(E(A)) = E(A)$. As $E(A)$ is Frechet, the continuous inclusion map $(b(A), \|\ \|_\infty) \to (E(A), \tau)$ is a homeomorphism. The converse follows from Theorem 1.2.

Proof of Theorem 1.4. By Corollary 2.3, $I = J = \text{srad}\,(A)$ in the notations of Lemma 2.9. Let $K = A/J$, a Frechet *-algebra in the quotient topology from A. By Lemma 2.8, the uniform topology $\tau_{\mathcal{D}}$ on $\pi_u(A)$ is a σ-C^*-topology; and the topology $\sigma_u(t)$ on K is determined by the (continuous) C^*-seminorms $\{s_G(\cdot) : G \in \mathcal{F}\}$, where $\mathcal{F}$ is the collection of all subspaces $\mathcal{D}$ of $\mathcal{D}(\pi_u)$ such that $\mathcal{D}$ is π_u-invariant and $\pi_u|_{\mathcal{D}}$ is a bounded operator *-representation; and $s_G(z) = \|\pi_u|_G(x)\|$ for all $z = x + J$, $x \in A$. Thus $\sigma_u(t) \leq \tau$ where τ is the relative topology from $E(A)$ defined by all C^*-seminorms on $E(A)$. To show that $\tau \leq s_u(t)$, let $z_n = x_n + J \in K$, $z_n \to 0$ in $\sigma_u(t)$. Let q be any C^*-seminorm on A. There exists $\pi \in R(A)$ such that $q(x) = \|\pi(x)\|$, and $\pi = \oplus\{\pi_f | f \in F_\pi\}$ for a suitable $F_\pi \subset P_c(A, t)$. Now $H_\pi = \oplus_{f \in F_\pi} H_f \subset \mathcal{D}(\pi_u)$, $H_\pi \in \mathcal{F}$, and $\|\pi(x_n)\| = s_{H_\pi}(z_n) \to 0$. Hence $z_n \to 0$ in τ. Thus $\tau = \sigma_u(t)$, and $E(A) = (O(A), \sigma_u(t))$, the completion. The remaining assertion follows from Lemma 2.11.

4. Remarks

PROPOSITION 4.1

*Let A be a *-sb Frechet *-algebra. If A is hermitian, then A is a Q-algebra.*

Proof. We can assume that A is unital. Let P be a sequence of submultiplicative *-seminorms defining the topology of A. Let $A = \text{proj lim} A_q$ be the Arens–Michael decomposition expressing A as an inverse limit of a sequence of Banach *-algebras; where, for $q \in P$, A_q is the Banach *-algebra obtained by completing $A/\ker q$ in the norm $\|x + \ker q\| = q(x)$. Let $\pi_q : A \to A_q$ be $\pi_q(x) = x + \ker q$.

Case 1. Assume that A is commutative. By hermiticity, $sp_A(h) = \{\phi(h) : \phi \in \mathcal{M}(A)\} \subset \mathbb{R}$ for all $h = h^* \in A$. Note that since A is hermitian. $\mathcal{M}(A) = \mathcal{M}^*(A)$. Using ([23], Proposition 7.5), it follows that for each q, $\mathcal{M}(A_q) = \mathcal{M}^*(A_q)$; hence by ([7], Theorem 35.3, p. 188), each A_q is hermitian. Now by ([17], Lemma 41.2, p. 225), for each $z \in A_q$, the spectral radius satisfies

$$r_{A_q}(z) \leq r_{A_q}(z^*z)^{1/2} = |z_q|_q,$$

$|\ |_q$ denoting the Gelfand–Naimark pseudonorm on A_q. Then $m_q(x) = |\pi_q(x)|_q$ defines a continuous C^*-seminorm on A. By Lemma 2.10, there exists a greatest continuous C^*-seminorm $p_\infty(\ \ )$ on A. By ([23], Corollary 5.3), for each $x \in A$,

$$r_A(x) = \sup\{r_{A_q}(\pi_q(x))\} \leq \sup\{m_q(x)\} \leq p_\infty(x).$$

By the continuity of p_∞, there exists a $p \in K_s(A)$ and $k > 0$ such that for all x in A, $r(x) \leq p_\infty(x) \leq kp(x)$. It follows from ([23], Proposition 13.5) that A is a Q-algebra.

Case 2. Let A be non-commutative. Let M be a maximal commutative *-subalgebra of A containing the identity of A. Since M is spectrally invariant in A, M is also hermitian. By *-spectral boundedness and hermiticity, each positive functional on M can be extended to a positive functional on A ([17], Theorem 9.3, p. 49). It follows from ([15], Corollary 2.8) and the continuity of positive functionals on unital Frechet *-algebras, that for all $z \in M$, $p_\infty(z) = p_\infty^M(z) \leq r_M(z^*z)^{1/2}$, p_∞^M being the greatest C^*-seminorm on M and $r_M(\cdot)$ denoting the spectral radius in M. Thus M is a commutative hermitian algebra with a C^*-enveloping algebra. By case 1, M is a Q-algebra. Further, M being hermitian, the Ptak's function $x \longrightarrow r_M(x^*x)^{1/2}$ is a C^*-seminorm on M ([17], Corollary 8.3, p. 38; Theorem 8.17, p. 45).

Now let $x \in A$, and take M to be the maximal commutative *-subalgebra containing x^*x. Let $r_K(\cdot)$ denote the spectral radius in an algebra K. Then by Ptak's inequality in hermitian Frechet *-algebras ([17], Theorem 8.17, p. 45)

$$r_A(x) \leq r_A(x^*x)^{1/2} = r_M(x^*x)^{1/2} = p_\infty^M(x^*x)^{1/2} = p_\infty(x^*x)^{1/2} \leq q(x),$$

q being a *-algebra seminorm on A depending on p_∞ only. It follows from ([23], Proposition 13.5) that A is a Q-algebra.

(4.2) (i) It is claimed in ([5], Corollary 2.4) that a complete hermitian m-convex *-algebra with a C^*-enveloping algebra is a Q-algebra. Regrettably, there is a gap in the proof. The author sincerely thanks Prof. M Fragoulopoulou for pointing out this. It is implicitely used in the 'proof' therein that the completion of a hermitian normed algebra is hermitian. By Gelfand theory, this is certainly true in the commutative case, but is not true in non-commutative case (see ([17], p. 18)). Thus ([15], Corollary 2.4) remains valid in commutative case; and the above proposition partially repairs the gap in the non-commutative case. Consequently ([15], Lemma 2.15, Theorem 2.14) remains valid for Frechet algebras. Is a hermitian Frechet algebra with a C^*-enveloping algebra a Q-algebra? (ii) The algebra $C(\mathbb{R})$ of continuous functions on $\mathbb{R}$ exhibits that the condition *-*sb* can not be omitted from the above proposition. It also follows from above that a *-*sb* σ-C^*-algebra is a C^*-algebra.

(4.3) In Theorem 1.2, the assumption that π is closed can not be omitted. Let $A = C^\infty(\mathbb{R})$, the Frechet *-algebra of C^∞ functions on $\mathbb{R}$, with pointwise operations and the topology of uniform convergence on compact subsets of $\mathbb{R}$ of functions as well as their derivatives. Then $E(A) = C(\mathbb{R})$, the algebra of continuous functions on $\mathbb{R}$ with the compact open topology. On the Hilbert space $H = L^2(\mathbb{R})$, the *-representation π of A with $\mathcal{D}(\pi) = C_c^\infty(\mathbb{R})$, $\pi(a)f = af$, cannot be extended to a *-representation of $C(\mathbb{R})$ with the same domain ([10], Example 4.7).

(4.4) Theorem 1.1 means that a Fechet *-algebra has a C^*-enveloping algebra if and only if it is a BG^*-algebra [24]. In the non-metrizable case, it follows from Theorem 1.2 that if A is a complete topological m-convex *-algebra with a C^*-enveloping algebra, then every *-representation of A which is continuous in the uniform topology is a bounded operator representation. However, the converse does not hold. This is exhibited by the BG^*-algebra $C[0,1]$ of continuous functions on [0,1] with the pro-C^*-topology τ of uniform convergence on all countable compact subsets of [0,1]. Thus Theorem 1.1 is false without the assumption that A is Frechet. It would be of interest to find an example of a topological algebra with a C^*-enveloping algebra which is not a BG^*-algebra.

(4.5) Yood [42] has shown that a *-algebra A admits a greatest C^*-seminorm if and only if $\sup|f(x)| < \infty$ for each x, where the sup is taken over all admissible states S; and by Lemma 2.10, this happens for a Frechet A if and only if A has a C^*-enveloping algebra. Yood's result is an algebraic version of ([5], Corollary 2.9) that states that a complete m-convex algebra has a C^*-enveloping algebra if and only if S is equicontinuous.

(4.6) (i) Let π be a *-representation of a complete locally m-convex *-algebra A with a bounded approximate identity. Let A have a C^*-enveloping algebra. Is π continuous in the uniform topology? In particular, let π be a bounded operator *-representation. Is π norm-continuous?

(ii) Let A be a pro-C^*-algebra (more generally, a complete m-convex *-algebra with a bounded approximate identity). Let f be a representable, not necessarily continuous, positive functional on A. Is the GNS representation π_f a bounded operator representation? Is every *-representation of A weakly unbounded?

These are motivated by the point of view ([5], Remark 2.11, p. 207) that a topological *-algebras with a C^*-enveloping algebra provide a hermitian analogue of a commutative Q-algebra. It is easy to see that a *-representation π of a locally convex Q-algebra is a bounded operator representation and is norm continuous.

5. Crossed product constructions

We recall the crossed product of a C^*-dynamical system (G, A, α). Let α be a strongly continuous action of a locally compact group G by *-automorphisms of a C^*-algebra A. Let $C_c(G, A)$ be the vector space of all continuous A-valued functions with compact supports. It is a *-algebra with twisted convolution

$$x * y(g) = \int_G x(h)\alpha_h(y(h^{-1}g))\mathrm{d}h$$

and the involution $x^*(g) = \Delta(g)^{-1}\alpha_g(x(g^{-1}))^*$. The Banach *-algebra $L^1(G, A)$ is the completion of $C_c(G, A)$ in the norm $||x||_1 = \int_G ||x(h)||\mathrm{d}h$; and the crossed product C^*-algebra $C^*(G, A, \alpha)$ is the completion of $L^1(G, A)$ in its Gelfand–Naimark pseudonorm $||x|| = \sup\{||\pi(x)|| : \pi \in R(L^1(G, A))\}$, which is, in fact, a norm. Thus it is the enveloping C^*-algebra of the Banach *-algebra $L^1(G, A)$. The C^*-algebra $C^*(G, A, \alpha)$

can also be realized as the enveloping C^*-algebra of non-normed topological *-algebras smaller than $L^1(G,A)$.

Let $\mathscr{K}$ be the collection of all compact, symmetric neighbourhoods of the identity in G. For $K \in \mathscr{K}$, let $C_K(G,A) = \{f \in C_c(G,A) : \operatorname{supp} f \subseteq K\}$, a Banach space with the norm $||f|| = \sup\{||f(x)|| : x \in K\}$. The inductive limit topology τ on $C_c(G,A)$ is the finest locally convex topology on $C_c(G,A)$ making each of the embeddings $C_K(G,A) \to C_c(G,A)$, for all $K \in \mathscr{K}$, continuous. Then $C_c(G,A)$ is a locally convex, non- m-convex, topological *-algebra with jointly continuous multiplication and continuous involution. From ([18], p. 203), $E(C_c(G,A)) = C^*(G,A,\alpha)$. This immediately leads to the following.

PROPOSITION 5.1

Let (G,A,α) be a C^*-dynamical system. Let B be any topological *-algebra containing $C_c(G,A)$ as a dense *-subalgebra and satisfying $C_c(G,A) \subseteq B \subseteq C^*(G,A,\alpha)$. Then $E(B) = C^*(G,A,\alpha)$.

For $1 \leq p < \infty$, let $A^p(G,A) = L^1(G,A) \cap L^p(G,A)$, a Banach *-algebra with the norm $|x|_p = ||x||_1 + ||x||_p$. The above applies to $B = \bigcap\{A^p(G,A) : 1 \leq p < \infty\}$, a locally m-convex Q-Frechet *-algebra with the topology of $|\ |_p$-convergence for each p.

Smooth elements of a Lie group action

Let A be a unital C^*-algebra and G be a Lie group acting on A. Let Δ denote the infinitesimal generators of actions of 1-parameter subgroups of G on A, viz.,

$$\Delta = \{(\mathrm{d}/\mathrm{d}t)\alpha_{u(t)}|_{t=0} : t \to u(t) \text{ is a continuous homomorphism of } \mathbb{R} \text{ into } G\}.$$

Then Δ consists of derivations and it is a finite dimensional vector space ([11], p. 40) having basis, say $\delta_1, \delta_2, \ldots, \delta_d$. Then *$C^n$-elements* $(1 \leq n < \infty)$ and *C^∞-elements* of A for the action α are defined as follows.

$$C^n(A) = \{x \in A : x \in \operatorname{Dom}(\delta_{i_1}\delta_{i_2}\ldots\delta_{i_n}) \text{ for all } n\text{-tuples } \{\delta_{i_1}, \ldots, \delta_{i_n}\} \text{ in } \Delta\}$$
$$C^\infty(A) = \bigcap\{C^n(A) : n \in \mathbb{N}\}.$$

By ([11], Proposition 2.2.1), each $C^n(A)$ and $C^\infty(A)$ are dense *-subalgebras of A; and $C^n(A)$ is a Banach *-algebra with the norm

$$||x||_n = ||x|| + \sum_{k=1}^{n} \sum_{i_1,i_2,\ldots,i_k=1}^{d} ||\delta_{i_1}\delta_{i_2}\ldots\delta_{i_k}(x)||/k!.$$

Then $C^\infty(A) = \operatorname{proj\,lim} C^n(A)$ is a Frechet *-algebra with the topology defined by the norms $\{||\ ||_n : n = 1, 2, \ldots\}$.

Lemma 5.2. *$C^\infty(A)$ has a C^*-enveloping algebra and $E(C^\infty(A)) = A$.*

Proof. It is well known that $C^n(A)$ and $C^\infty(A)$ are spectrally invariant in A. Hence $(C^n(A), ||\ ||)$ and $(C^\infty(A), ||\ ||)$ are Q-algebras in the norm $||\ ||$ from the C^*-algebra A. Since $||\ || \leq ||\ ||_n$, $(C^\infty(A), \tau)$ is also a Q-algebra. By Lemma 2.10, $(C^\infty(A), \tau)$ is an algebra with a C^*-enveloping algebra. Let $\pi : B \to B(H)$, where $B = C^n(A)$ or $C^\infty(A)$, be a bounded operator *-representation on a Hilbert space H. Then for all $x \in B$,

$$\begin{aligned}||\pi(x)||^2 &= ||\pi(x^*x)|| = r_{B(H)}(\pi(x^*x)) \leq r_{\pi(B)}(\pi(x^*x)) \\ &\leq r_B(x^*x) \leq ||x||^2.\end{aligned}$$

Hence π is $||\ \ ||$-continuous; and by the density of $C^\infty(A)$ in A, π extends uniquely to a *-representation of A on H. It follows that $E(C^\infty(A)) = C^*(C^n(A)) = A$ for all n.

An element $x \in A$ is *analytic* if $x \in C^\infty(A)$ and there exists a scalar $t > 0$ such that

$$\sum_{k=0}^{\infty}\left(\sum_{i_1,i_2,\ldots,i_k=1}^{d} ||\delta_{i_1}\delta_{i_2}\cdots\delta_{i_k}(x)||/k!\right)t^k < \infty;$$

whereas x is *entire* if $x \in C^\infty(A)$ and for all $t > 0$, it holds that

$$\sum_{k=0}^{\infty}\left(\sum_{i_1,i_2,\ldots,i_k=1}^{d} ||\delta_{i_1}\delta_{i_2}\cdots\delta_{i_k}(x)||/k!\right)t^k < \infty.$$

Let $C^\omega(A)$ (respectively $C^{e\omega}(A)$) denote the set of all analytic (respectively entire) elements of A. Then each of $C^\omega(A)$ and $C^{e\omega}(A)$ is a *-subalgebra of A and $C^{e\omega}(A) \subset C^\omega(A) \subset C^\infty(A)$. For each $t > 0$ and $x \in C^n(A)$, define

$$p_n^t(x) = ||x|| + \sum_{k=1}^{n}\left(\sum_{i_1,i_2,\ldots,i_k=1}^{d} ||\delta_{i_1}\cdots\delta_{i_k}(x)||/k!\right)t^k.$$

Then $||\ \ ||_n$ and $p_n^t(\ \ )$ are equivalent norms. Hence $P^t = (p_n^t(\ \ ))$ and $p = (||\ \ ||_n)$ define the same C^∞-topology τ on $C^\infty(A)$. Let $A_t = \{x \in C^\infty(A) : p^t(x) = \sup_k p_k^t(x) < \infty\}$, a *-subalgebra of $C^\infty(A)$, which is a Banach *-algebra with norm $p^t(\ \ )$, and which consists of elements of $C^\infty(A)$ whose numerical ranges defined with respect to P^t are bounded. For $t < s$, the inclusion $A_s \to A_t$ is norm decreasing. Thus

$$C^{e\omega}(A) = \bigcap\{A_t : t > 0\} = \bigcap_{n=1}^{\infty} A_n = \text{proj lim} A_n,$$

a Frechet m-convex, *-algebra with the topology $\tau_{e\omega}$ defined by the family of norms $\{p^t(\ \ ) : t \in \mathbb{N}\}$ (setting $p^\circ(\ \ ) = ||\ \ ||$). Further,

$$C^\omega(A) = \bigcup_{t>0} A_t = \bigcup_{n=1}^{\infty} A_{1/n} = \text{ind lim} A_{1/n}$$

with the linear inductive limit topology τ_ω. By ([21], Corollary 10.2, Lemma 10.2, p. 317) and ([32], Proposition 6.6, p. 59), $(C^\omega(A), \tau_\omega)$ is a complete m-convex *-algebra which is a Q-algebra. Thus $C^\omega(A)$ is an algebra with a C^*-enveloping algebra. Further if each A_t is dense and spectrally invariant in $C^\infty(A)$, then $C^{e\omega}(A)$ is an algebra with a C^*-enveloping algebra and $E(C^{e\omega}(A)) = E(C^\omega(A)) = A$.

The smooth crossed product

We recall the smooth Frechet algebra crossed product [29]. Let B be a Frechet *-algebra. Let (p_n) be a sequence of submultiplicative *-seminorms defining the topology of B. Let β be a strongly continuous action of $\mathbb{R}$ by continuous *-automorphisms of B. Then β is called *m-tempered* (respectively *isometric*) if for each $m \in \mathbb{N}$, there exists a polynomial

$P(X)$ such that $p_m(\beta_r(x)) \leq P(r)p_m(x)$ for all $x \in B$, $r \in \mathbb{R}$ (respectively for each $m \in \mathbb{N}$, $p_m(\beta_r(x)) = p_m(x)$ for all $x \in B$, all $r \in \mathbb{R}$). Let $S(\mathbb{R})$ be the Schwartz space consisting of the rapidly decreasing C^∞-functions on $\mathbb{R}$. It is a Frechet space with the Schwartz topology. The completed projective tensor product $S(\mathbb{R}) \otimes B = S(\mathbb{R}, B)$ consists of B-valued Schwartz functions on $\mathbb{R}$. If β is m-tempered, then $S(\mathbb{R}, B)$ becomes an m-convex Frechet algebra with twisted convolution

$$(f * g)(r) = \int_{\mathbb{R}} f(s)\beta_s(g(r-s))\mathrm{d}s.$$

This Frechet algebra is called the *smooth Schwartz crossed product* of B by the action β of $\mathbb{R}$, and is denoted by $S(\mathbb{R}, B, \beta)$. In general, $S(\mathbb{R}, B, \beta)$ need not be a *-algebra ([34], § 4). If β is isometric, then the completed projective tensor product

$$\begin{aligned} L^1(\mathbb{R}) \otimes B &= L^1(\mathbb{R}, B) \\ &= \{f : \mathbb{R} \to B \text{ measurable function} : \int_{\mathbb{R}} p_m(f(r))\mathrm{d}r < \infty \text{ for all } m \in \mathbb{N}\} \end{aligned}$$

is a Frechet *-algebra with twisted convolution and the involution $f^*(r) = \beta_r(f(-r)^*)$, denoted by $L^1(\mathbb{R}, B, \beta)$. One has $S(\mathbb{R}, B, \beta) \subset L^1(\mathbb{R}, B, \beta)$.

The following is closely related with ([29], Lemma 1.1.9).

Lemma 5.3. *Let A be a dense Frechet *-subalgebra of a Frechet *-algebra B. Assume that A and B can be expressed as inverse limits of Banach *-algebras A_n and B_n respectively, where A_n is dense in B_n for all n; the inclusions $A \to A_n$, $B \to B_n$ have dense ranges for all n; and each A_n is spectrally invariant in B_n. Then A is spectrally invariant in B and $E(A) = E(B)$.*

Proof. By ([15], Theorem 4.3), $E(A) = \text{proj lim} E(A_n)$ and $E(B_n) = \text{proj lim} E(B_n)$. Since $A_n \to B_n$ is spectrally invariant with dense range, A_n is a Q-normed algebra in the norm of B_n. Hence every C^*-seminorm on A_n is continuous in the norm of B_n; and extends uniquely to B_n. Thus A_n and B_n have the same collection of C^*-seminorms. It follows that $E(A_n) = E(B_n)$ for all n; and so $E(A) = E(B)$.

PROPOSITION 5.4

Let α be an m-tempered strongly continuous action of $\mathbb{R}$ by continuous *-automorphisms of a Frechet *-algebra B contained as a dense *-subalgebra of a C^*-algebra A such that $E(B) = A$. Then $E(C^\infty(B)) = A$.

Proof. Let $\|\ \|$ denote the C^*-norm on A. Let (p_n) be an increasing sequence of submultiplicative *-seminorms defining the topology of B. In view of the continuity of the inclusion $B \to A$, the increasing sequence $q_n(\ ) = p_n(\ ) + \|\ \|$ of norms also determines the topology of B. Let $B_n = (B, q_n)$ be the completion, which is a Banach *-algebra. Then $B = \text{proj.}\lim B_n = \bigcap B_n$. Now, for any $n \in \mathbb{N}$, $r \in \mathbb{R}$, and $x \in B$,

$$\begin{aligned} q_n(\alpha_r(x)) &= \|\alpha_r(x)\| + p_n(\alpha_r(x)) \\ &= \|x\| + \text{poly}\,(r)p_n(x) = \text{poly}'\,(r)q_n(x) \end{aligned}$$

for some polynomial poly'( ). It follows that α is m-tempered for $(q_n(\ ))$ also; and it induces an action $\alpha^{(n)}$ of $\mathbb{R}$ by continuous *-automorphisms of B_n. Let $B_{n,m}$ be the Banach

*-algebra consisting of all C^m-vectors in B_n for $\alpha^{(n)}$. By ([33], Theorem 2.2), $B_{n,m} \to B_n$ are spectrally invariant embeddings with dense ranges. Also, $C^\infty(B) = \text{proj lim}_{n,m} B_{n,m} = \text{proj lim}_n B_{n,n}$. Now Lemma 5.2 implies that $C^\infty(B)$ is spectrally invariant in B and $E(C^\infty(B)) = A$.

PROPOSITION 5.5

Let α be a strongly continuous action of $\mathbb{R}$ by *-automorphisms of a C^*-algebra A. The following hold.

(a) *The Frechet algebras $S(\mathbb{R}, A, \alpha)$ and $S(\mathbb{R}, C^\infty(A), \alpha)$ are Q-algebras.*
(b) *The embeddings $S(\mathbb{R}, C^\infty(A), \alpha) \to S(\mathbb{R}, A, \alpha) \to C^*(\mathbb{R}, A, \alpha)$ are continuous, spectrally invariant and have dense ranges.*
(c) *The Frechet algebra $S(\mathbb{R}, C^\infty(A), \alpha)$ is *-algebra and $E(S(\mathbb{R}, C^\infty(A), \alpha) = C^*(\mathbb{R}, A, \alpha)$.*

Proof. By ([34], Theorem A.2), α leaves $C^\infty(A)$ invariant. In ([34], Corollary 4.9), taking the scale σ to be the weight $w(r) = 1 + |r|$ on $G = \mathbb{R} = H$, it follows that $S(\mathbb{R}, C^\infty(A), \alpha)$, is a Frechet *-algebra. Now $\tilde{\alpha}_s(f)(r) = \alpha_s(f(r))$ defines an action $\tilde{\alpha}$ of $\mathbb{R}$ on the Frechet algebra $S(\mathbb{R}, A, \alpha)$ for which, by ([29], p. 189), $C^\infty(S(\mathbb{R}, A, \alpha)) = S(\mathbb{R}, C^\infty(A), \alpha)$ homeomorphically. Note that the embeddings

$$S(\mathbb{R}, C^\infty(A), \alpha) \to S(\mathbb{R}, A, \alpha) \to L^1(\mathbb{R}, A, \alpha) \to C^*(\mathbb{R}, A, \alpha)$$

are continuous; $S(\mathbb{R}, C^\infty(A), \alpha)$ is dense in $S(\mathbb{R}, A, \alpha)$ by ([34], Theorem A.2); and $S(\mathbb{R}, A, \alpha)$ is dense in $L^1(\mathbb{R}, A, \alpha)$; which, in turn, is dense in $C^*(\mathbb{R}, A, \alpha)$.

Now let $\{| \ |_n\}$ be an increasing sequence of submultiplicative seminorms defining the topology of $S(\mathbb{R}, A, \alpha)$. Let $(B_n, | \ |_n)$ be the Hausdorff completion of $S(\mathbb{R}, A, \alpha)$ in $| \ |_n$. Then B_n is a Banach algebra and $S(\mathbb{R}, A, \alpha) = \text{proj.} \lim B_n$. Since $||\alpha_r(x)|| = ||x||$, the action $\tilde{\alpha}$ of $\mathbb{R}$ on $S(\mathbb{R}, A, \alpha)$ extends to a strongly continuous action $\tilde{\alpha}^{(n)}$ of $\mathbb{R}$ by automorphisms of B_n. Let $C^m(B_n)$ be the Banach algebra of all C^m-vectors in B_n for the action of $\tilde{\alpha}^{(n)}$. As noted in ([29], p. 189), $C^n(B_n)$ is dense and spectrally invariant in B_n; and $S(\mathbb{R}, C^\infty(A), \alpha) = \text{proj lim}\, C^n(B_n)$. Let $x \in S(\mathbb{R}, C^\infty(A), \alpha)$, $x = (x_n)$ being a coherent sequence with $x_n \in C^n(B_n)$ for all $n \in \mathbb{N}$. Now

$$sp_{S(\mathbb{R}, C^\infty(A), \alpha)}(x) = \bigcup_n sp_{C^n(B_n)}(x_n) = \bigcup_n sp_{B_n}(x_n) = sp_{S(\mathbb{R}, A, \alpha)}(x).$$

Thus $S(\mathbb{R}, C^\infty(A), \alpha)$ is spectrally invariant in $S(\mathbb{R}, A, \alpha)$; which in turn is spectrally invariant in $C^*(\mathbb{R}, A, \alpha)$ by ([33], Corollary 7.16). Thus each of $S(\mathbb{R}, C^\infty(A), \alpha)$ and $S(\mathbb{R}, A, \alpha)$ are Q-normed algebras in the C^*-norm of $C^*(\mathbb{R}, A, \alpha)$; and hence are Q-algebras in their respective Frechet topologies. Using Lemma 2.10, $E(S(\mathbb{R}, C^\infty(A), \alpha)) = C^*(\mathbb{R}, A, \alpha)$.

Proof of Theorem 1.5. Since $C^\infty(B) = B$, the Frechet m-convex algebra $S(\mathbb{R}, B, \alpha)$ is a *-algebra by ([34], Corollary 4.9). Since B is Frechet and sits in the C^*-algebra A, B is *-semisimple. Similarly, since the inclusion $S(\mathbb{R}, B, \alpha) \to C^*(\mathbb{R}, A, \alpha)$ is continuous and one–one, $S(\mathbb{R}, B, \alpha)$ is also *-semisimple. To prove that $E(S(\mathbb{R}, B, \alpha)) = C^*(\mathbb{R}, A, \alpha)$, it is sufficient to prove that any *-representation $\sigma : S(\mathbb{R}, B, \alpha) \to B(H_\sigma)$ extends to a *-representation $(\tilde{\sigma}) : C^*(\mathbb{R}, A, \alpha) \to B(H_\sigma)$. This would imply that the C^*-norm on $S(\mathbb{R}, B, \alpha)$ induced by the C^*-algebra norm on $C^*(\mathbb{R}, A, \alpha)$ is the greatest (automatically continuous) C^*-seminorm on $S(\mathbb{R}, B, \alpha)$. This is shown below by arguments analogous to those in ([25], Proposition 7.6.4, p. 255).

Let (x_λ) be a bounded approximate identity for A contained in B and which is also a bounded approximate identity for B. For each $n \in \mathbb{N}$, let $f_n \in C_c^\infty(\mathbb{R})$ be such that $0 \le f_n \le 1, f_n(x) = 1$ for all $x \in [-n, n]$, and $\operatorname{supp} f_n \subset [-n-1, n+1]$. Then (f_n) is a bounded approximate identity for $S(\mathbb{R})$ (pointwise multiplication) contained in $C_c^\infty(\mathbb{R})$. The inverse Fourier transforms g_n of f_n constitute a bounded approximate identity for $S(\mathbb{R})$ with convolution. Thus $y_{n,\lambda} = g_n \otimes x_\lambda$ constitute a bounded approximate identity for $S(\mathbb{R}, B, \alpha)$. Given a *-representation $\sigma : S(\mathbb{R}, B, \alpha) \to B(H_\sigma)$ automatically continuous, let $\mathcal{U}(H_\sigma)$ be the group of all unitary operators on H_σ. Define $\pi : B \to B(H_\sigma)$ and $U : \mathbb{R} \to \mathcal{U}(H_\sigma)$ by

$$\begin{aligned} \pi(x) &= \lim_{(n,\lambda)} \sigma(xy_{(n,\lambda)}(\quad)), \\ U_t &= \lim_{(n,\lambda)} \sigma(\alpha_t(y(\cdot - t))). \end{aligned}$$

The limits are taken in the weak sense; and they exist. As in ([25], § 7.6, p. 256), it is verified that π is a *-representation of B; U is a unitary representation of $\mathbb{R}$; $U_t \pi(x) U_t^* = \pi(\alpha_t(x))$ for all $t \in \mathbb{R}$, all $x \in B$; and for all $y \in S(\mathbb{R}, B, \alpha)$, $\sigma(y) = \int \pi(y(t))\, U_t \mathrm{d}t$. Now, since $E(B) = A, \pi$ extends to a *-representation $\tilde{\pi} : A \to B(H_\sigma)$ so that $(\tilde{\pi}, U, H_\sigma)$ is a covariant representation of the C^*-dynamical system $(\mathbb{R}, A, \alpha)$. Then $\tilde{\sigma}(y) = \int \tilde{\pi}(y(t)) U_t \, \mathrm{d}t$ defines a non-degenerate *-representation of the Banach *-algebra $L^1(\mathbb{R}, B, \alpha)$; and hence extends uniquely to a *-representation $\tilde{\sigma}$ of $C^*(\mathbb{R}, B, \alpha)$. This $\tilde{\sigma}$ is the desired extension of σ. This shows that $E(S(\mathbb{R}, B, \alpha)) = C^*(\mathbb{R}, A, \alpha)$.

Further, suppose that the action α of $\mathbb{R}$ on B is isometric. Then by [29], $L^1(\mathbb{R}, B, \alpha)$ is a *-algebra, which is a Frechet m-convex *-algebra; and

$$S(\mathbb{R}, B, \alpha) \to L^1(\mathbb{R}, B, \alpha) \to L^1(\mathbb{R}, A, \alpha) \to C^*(\mathbb{R}, A, \alpha)$$

are continuous embeddings with dense ranges. It follows that $E(L^1(\mathbb{R}, B, \alpha) = C^*(\mathbb{R}, A, \alpha)$. This completes the proof of the theorem.

Actions on topological spaces

(a) Let M be a locally compact Hausdorff space. Let $\sigma : M \to [0, \infty)$ be a Borel function, $\sigma(m) \ge 1$ for all $m \in M$. Assume that σ is bounded on compact subsets of M. Following ([34], § 5), let

$$C^\sigma(M) = \{f \in C_0(M) : ||\sigma^d f|| < \infty \text{ for all } d \in \mathbb{N}\},$$

called the algebra of continuous functions on M vanishing at infinity σ-rapidly. It is shown in [34] that $C^\sigma(M)$ is a Frechet m-convex *-algebra with the topology defined by seminorms

$$||\sigma^d f|| = \sup\{|(\sigma(x))^d f(x)| : x \in M\}, \quad \mathrm{d} \in \mathbb{N};$$

and that $C_c(M) \to C^\sigma(M) \to C_0(M)$ are continuous embeddings with dense ranges. Thus $E(C^\sigma(M)) = C_0(M)$. In fact, $C^\sigma(M)$ is an ideal in $C_0(M)$; hence inverse closed in $C_0(M)$; and so is a Q-algebra.

(b) Let G be a Lie group acting on M. If $f \in C^\sigma(M)$, define $\alpha_g(f)(m) = f(g^{-1}m)$. By ([34], § 5), if σ is uniformly G-translationally equivalent (in the sense that for every compact $K \subset G$, there exists $l \in \mathbb{N}$ and $C > 0$ such that $\sigma(gm) \leq C\sigma(m)^l$ for all $g \in G$, $m \in M$), then $g \rightarrow \alpha_g$ defines a strongly continuous action of G by continuous *-automorphisms of $C^\sigma(M)$. Then the space $C^\infty(C^\sigma(M))$ consisting of C^∞-vectors for the action α of G on $C^\sigma(M)$ is an m-convex Frechet *-algebra with a C^*-enveloping algebra and $E(C^\infty(C^\sigma(M)) = C_0(M)$.

(c) In particular, let $G = \mathbb{R}$, M be a compact C^∞-manifold, and let the action of $\mathbb{R}$ on M be smooth. Then the induced action α on $C(M)$ is smooth, so that $\alpha_r(C^\infty(M)) \subseteq C^\infty(M)$ for all $r \in \mathbb{R}$. It follows from Theorem 5.1 that $E(S(\mathbb{R}, C^\infty(M), \alpha) = C^*(\mathbb{R}, C(M), \alpha)$ the covariance C^*-algebra.

6. The Pedersen ideal of a C^*-algebra

Let A be a non-unital C^*-algebra. Let K_A be its Pedersen ideal. It is a hereditary, minimal dense *-ideal of A. For $a \in A$, let $L_a = (Aa)^-$, $R_a = (aA)^-$, I_a be the closed *-ideal of A generated by aa^*. Since $a \in L_a \bigcap R_a$, $aa^* \in I_a$. Let $K_A^+ = K_A \bigcap A^+$ be the positive part of K_A endowed with the order relation induced from that of A^+. Let $K_A^{nc} = \bigcup\{I_a : a \in K_A^+\}$.

Lemma 6.1. *K_A^{nc} is a dense *-ideal of A containing K_A; and $A = C^*$-ind lim $\{I_a : a \in K_A^+\}$.*

Proof. Let $a \in K_A^+$. Then $a^2 = aa^* \in I_a$; and I_a being a C^*-algebra, $a = (a^2)^{1/2} \in I_a$. Thus $K_A^+ \subseteq K_A^{nc}$. Observe that for any $x = x^* \in K_A$, $x \in I_x$. Indeed, $x^2 \in K_a^+$; hence $x^2 \in I_x$ and $|x| \in I_x$. But than taking the Jordan decomposition $x = x^+ - x^-$ in A, $(x^+)^2 = (x^+)^2 + x^+x^- = x^+|x| \in I_x$; so that $x^+ \in I_x$, $x^- \in I_x$, and $x \in I_x$. In particular, $x^2 \in I_{x^2}$ and $|x| \in I_{x^2}$. By repeating this argument, $x \in I_{x^2} \subset K_A^{nc}$ for any $x = x^* \in K_A$. It follows that $K_A \subset K_A^{nc}$. Now, by ([28], Lemma 1), $0 \leq a \leq b$ in A implies $L_a \subseteq L_b$, $R_a \subseteq R_b$ and $I_a \subseteq I_b$; and $K_A = \bigcup\{L_a : a \in K_A^+\} = \bigcup\{R_a : a \in K_A^+\}$. The family $\{I_a : a \in K_A^+\}$ forms an inductive system of C^*-algebras; and C^*-ind $\lim\{I_a : a \in K_A^+\} = (\bigcup\{I_a : a \in K_A^+\})^- = A$, $(\quad)^-$ denoting the norm closure. This proves the lemma.

Let t_1 (respectively t_2) be the finest locally convex linear topology (respectively finest locally m-convex topology) on K_A^{nc} making continuous the embeddings $I_a \rightarrow K_A^{nc}$, where $a \in K_A^+$. Then (K_A^{nc}, t_1) (respectively (K_A^{nc}, t_2)) is the *linear topological inductive limit* (respectively *topological algebraic inductive limit*) of $\{I_a : a \in K_A^+\}$ ([21], ch. IV).

Proof of Theorem 1.6. In the present set up, ([21], p. 115, 118, 125) implies that $t_1 = t_2$, equal to τ say, and (K_A^{nc}, τ) is a complete m-barrelled locally m-convex *-algebra; and the $\|\quad\|$-topology on K_A^{nc} is coarser than τ. Since K_A^{nc} is an ideal, it is inverse closed in its $\|\quad\|$-completion A, and hence $(K_A^{nc}, \|\quad\|)$ and $(K_A, \|\quad\|)$ are Q-algebras. This implies that any *-homomorphism from K_A^{nc} into $B(H)$ for a Hilbert space H is $\|\quad\|$-continuous and extends uniquely to A. Thus $\|\quad\|$ is the greatest C^*-seminorm on K_A^{nc}. To show that $\|\quad\|$ is the greatest τ-continuous C^*-seminorm on K_A^{nc} so that $E(K_A^{nc}) = A$, it is sufficient to show that (K_A^{nc}, τ) is a Q-algebra. To that end, in view of ([23], Lemma E.2), we show that 0 is a τ-interior point of the set $(K_A^{nc})_{-1}$ of quasiregular elements of K_A^{nc}. Note that, by ([21], p. 114), basic τ-neighbourhoods of 0 in K_A^{nc} are precisely of the form $V = |co|\{\bigcup(U_a : a \in K_A^+)\}$, where $|co|$ denotes the absolutely convex hull and U_a denotes a convex balanced neighbourhood of 0 in $(I_a, \|\quad\|)$. For any $a \in K_A^+$, $(I_a, \|\quad\|)$ is a Q-algebra, and being an ideal in A, $(I_a)_{-1} = A_{-1} \bigcap I_a$. Hence, for the zero neighbourhood $U_a = \{x \in I_a : \|x\| \leq 1\}$ in $(I_a, \|\quad\|)$,

$$U_a \subseteq (I_a)_{-1} = (K_A^{nc})_{-1} \bigcap I_a \subset (K_A^{nc})_{-1}; \text{ and}$$

$$|co|\left\{\bigcup(U_a : a \in K_A^+)\right\} = \left\{x \in K_A^{nc} : ||x|| \leq 1\right\} = U \text{ (say)}$$

is a zero neighbourhood in (K_A^{nc}, τ) contained in $(K_A^{nc})_{-1}$. It follows that (K_A^{nc}, τ) and (K_A, τ) are Q-algebras. Now, as in the proof of ([28], Theorem 4), $K_A^{nc} = \cup\{I_{e_\lambda}\}$, (e_λ) being a bounded approximate identity for A contained in K_A. Thus if A has countable bounded approximate identity, then K_A^{nc} is an LFQ-algebra; and τ is the finest (unique) locally convex topology on K_A^{nc} such that for each λ, $\tau|_{I_{e_\lambda}}$ is the norm topology.

7. The groupoid C^*-algebra

We follow the terminology and notations of [31]. Let G be a locally compact groupoid, i.e., a locally compact space G with a specified subset $G^2 \subseteq G \times G$ so that two continuous maps $G \to G$, $x \to x^{-1}$, and $G^2 \to G$, $(x, y) \to xy$ are defined satisfying $(xy)z = x(yz)$, $x^{-1}(xy) = y$ and $(zx)x^{-1} = z$. The unit space of G is $G^o = \{xx^{-1} : x \in G\} = \{x^{-1}x : x \in G\}$. Let $r(x) = xx^{-1}$ and $d(x) = x^{-1}x$. Assume that there exists a left Haar system $\{\lambda^u : u \in G^o\}$ on G, i.e., a family of measures λ^u on G such that $\operatorname{supp} \lambda^u = r^{-1}(u)$; for each $f \subseteq C_c(G)$, $u \to \int f \mathrm{d}\lambda^u$ is continuous; and for all $x \in G$ and $f \in C_c(G)$, $\int f(xy)\mathrm{d}\lambda^{d(x)}(y) = \int f(y)\mathrm{d}\lambda^{r(x)}(y)$. Let σ be a continuous 2-cocycle in $Z^2(G, T)$. Let t denote the usual inductive limit topology on $C_c(G)$. Then $(C_c(G), t)$ is a topological *-algebra with jointly continuous multiplication

$$f * g(x) = \int f(xy)g(y^{-1})\sigma(xy, y^{-1})\mathrm{d}\lambda^{d(x)}(y)$$

and the involution $f^*(x) = (f(x^{-1})\sigma(x, x^{-1}))^-$ ([31], Proposition II.1.1, p. 48). The I-norm on $C_c(G, \sigma)$ is $||f||_I = \max(||f||_{I,r}, ||f||_{I,l})$, where

$$||f||_{I,r} = \sup\left\{\int |f|\mathrm{d}\lambda^u : u \in G^o\right\}, \quad ||f||_{I,l} = \sup\left\{\int |f|\mathrm{d}\lambda_u : u \in G^o\right\},$$

$\lambda_u = (\lambda^u)^{-1}$ being the image of λ^u by the inverse map $x \to x^{-1}$ ([31], p. 50). Then $||\ \ ||_I$ is a submultiplicative *-norm on $C_c(G, \sigma)$. The L^1-algebra of (G, σ) is the completion $A = (C_c(G, \sigma), ||\ \ ||_I)$, a Banach *-algebra. For f in $C_c(G, \sigma)$, define $||f|| = \sup\{||\pi(f)||\}$, π running over all weakly continuous, non-degenerate *-representations $\pi : (C_c(G, \sigma), t) \to B(H_\pi)$ satisfying $||\pi(f)|| \leq ||f||_I$ for all f. Then $||\ \ ||$ defines a C^*-norm on $C_c(G, \sigma)$; and the groupoid C^*-algebra of (G, σ) is $C^*(G, \sigma) = (C_c(G, \sigma), ||\ \ ||)^-$, the completion. The following can be proved using cyclic decomposition and ([31], Corollary II.1.22, p. 72).

PROPOSITION 7.1

Let G be second countable having sufficiently many non-singular G-Borel sets. Then $E(C_c(G, \sigma)) = C^*(G, \sigma)$.

8. The universal *-algebra on generators with relations

Let G be any set. Let $F(G)$ be the free associative *-algebra on *generators* G, viz., the *-algebra of all polynomials in non-commuting variables $G \coprod G^*$ where $G^* = \{x^* : x \in G\}$. Let R be a collection of statements about elements of G, called *relations* on G, assumed throughout to be such that they make sense for elements of a locally m-convex *-algebra. A *Banach (respectively C^*-) representation* of (G,R) is a function ρ from G to a Banach *-algebra (respectively a C^*-algebra) $\rho: G \to A$ such that $\{\rho(g) : g \in G\}$ satisfies the relations R in A. Let $\mathrm{Rep}_B(G,R)$ (respectively $\mathrm{Rep}(G,R)$) be the set of all Banach representations (respectively C^*-representations) of (G,R). Motivated by ([27], Definition 1.3.4), it is assumed that R satisfies the following.

(i) The function $\rho: G \to \{0\}$ is a Banach representation of (G,R).
(ii) Let $\rho: G \to A$ be a representation of (G,R) in a Banach *-algebra A. Let B be a closed *-subalgebra of A containing $\rho(G)$. Then ρ is a representation of (G,R) in B.
(iii) Let ρ be a representation of (G,R) in a complete locally m-convex *-algebra A. Let $\phi: A \to B$ be a continuous *-homomorphism into a Banach *-algebra B. Then $\phi \circ \rho$ is a representation of (G,R) in B.
(iv) Let A be a complete locally m-convex *-algebra expressed as an inverse limit of Banach *-algebras viz. $A = \mathrm{proj.lim} A_p$. Let $\pi_p : A \to A_p$ be the natural maps. Let $\rho: G \to A$ be a function such that for all p, $\pi_p \circ \rho$ is a representation of (G,R). Then ρ is a representation of (G,R).

DEFINITION 8.1

(a) (Blackadar) (G,R) is *C^*-bounded* if for each g in G, there exists a scalar $M(g)$ such that $||\rho(g)|| \leq M(g)$ for all $\rho \in \mathrm{Rep}\,(G,R)$.
(b) (Blackadar) (G,R) is *C^*-admissible* if it is C^*-bounded and the following holds.
(bC^*) If (ρ_α) is a family of representations $\rho_\alpha : G \to B(H_\alpha)$ of (G,R) on Hilbert spaces H_α, then $\oplus\rho_\alpha : G \to B(\oplus H_\alpha)$ is a representation of (G,R).
(c) (G,R) is *weakly Banach admissible* if given finitely many representations $\rho_i : G \to A_i$ $(1 \leq i \leq n)$ of G into Banach *-algebras, the map $g \to \rho_1(g) \oplus \rho_2(g) \oplus \ldots \oplus \rho_n(g)$ is a representation of (G,R) in $\oplus A_i$. (G,R) is *weakly C^*-admissible* [27] if this holds with Banach algebras replaced by C^*-algebras.

The class of relations making sense for elements of a Banach *-algebra is smaller than the class of relations making sense for elements of a C^*-algebra. The usual algebraic relations involving the four elementary arithmetic operations on elements of G and G^* do make sense for Banach *-algebras; but relations like $x^+ \geq x^-$ for $x = x^*$ in G, or like $|x| \geq |y|$ for elements x, y in G, which make sense for C^*-algebras, fail to make sense for Banach *-algebras. We refer to [27] for relations satisfying (i)–(iv) except (ii). The relation (suggested by the referee). "The elements a, b and c generate A" fails to satisfy Definition 8.1(c). Our definition of weakly Banach admissible relations is very much ad hoc aimed at exploring a method of constructing non-abelian locally m-convex *-algebras.

Lemma 8.2. (a) *Let (G,R) be weakly Banach admissible. Then there exists a complete m-convex *-algebra $A(G,R)$ and a representation $\rho : G \to A(G,R)$ such that given any representation $\sigma : G \to B$ into a complete m-convex *-algebra B, there exists a continuous *-homomorphism $\phi : A(G,R) \to B$ satisfying $\phi \circ \rho = \sigma$.*
(b) ([27], Proposition 1.3.6). *Let (G,R) be weakly C^*-admissible. Then there exists a pro-C^*-algebra $C^*(G,R)$ and a representation $\rho_\infty : G \to C^*(G,R)$ such that given any representation $\sigma : G \to B$ of G into a pro-C^*-algebra B, there exists a continuous *-homomorphism $\phi : C^*(G,R) \to B$ such that $\phi \circ \rho_\infty = \sigma$.*

Proof. (a) Let $K = K(F(G))$ be the set of all submultiplicative *-seminorms p on $F(G)$ of the form $p(x) = ||\sigma(x)||$, σ running through all Banach representations of G. For $p \in K$, let $N_p = \{x \in F(G) : p(x) = 0\}$ and $N_a = \cap\{N_p : p \in K\}$ a *-ideal of $F(G)$. Let $B = F(G)/N_a$. Take $\tilde{p}(x + N_a) = p(x)$. Let t be the Hausdorff topology defined by $\{\tilde{p} : p \in K\}$. Let $A(G,R)$ be the completion of (B,t). Let $\rho : G \to A(G,R)$ be $\rho(g) = g + N_a$.

Claim 1. ρ is a representation of G in $A(G,R)$.

Let q be any t-continuous submultiplicative *-seminorm on $A(G,R)$. Let A_q be the Banach *-algebra obtained by the Hausdorff completion of $(A(G,R),q)$. By (iv) above, it is sufficient to prove that $\pi_q \circ \rho : G \to A_q$ is a representation of (G,R). Since q is t-continuous, there exists $p_1, p_2, \ldots, p_k$ in K such that $q(x) \leq c \max p_i(x)$ for all $x \in F(G)$; and each p_i is of form $p_i(x) = ||\sigma_i(x)||$, $\sigma_i : G \to A(i)$ being a representation into some Banach algebra $A(i)$. By (c) of Definition 8.1, there exists a Banach *-algebra B and a representation $\sigma : G \to B$ such that $q(x) \leq ||\sigma(x)||$ for all $x \in F(G)$. In view of (ii), we assume that B is generated by $\sigma(G)$. Let $\phi : B \to A_q$ be $\phi(\sigma(x)) = (x + N_a) + \ker q = \pi_q(\rho(x))$. Then ϕ is well defined, continuous and $\phi \circ \sigma = \pi_q \circ \rho$. By the assumption (iii) above, $\phi \circ \sigma$ is a representation of G.

Claim 2. Given any representation $\sigma : G \to C$ into a complete m-convex *-algebra C, there exists a unique continuous *-homomorphism $\phi : A(G,R) \to C$ such that $\phi \circ \rho = \sigma$.

Let $C = \text{proj.}\lim C_\alpha$, an inverse limit of Banach *-algebras C_α, $\pi_\alpha : C \to C_\alpha$ being the projection maps. By (iii) of above, $\pi \circ \sigma$ is a Banach representation of (G,R). By the construction of $A(G,R)$, there exist continuous *-homomorphisms $\phi_\alpha : A(G,R) \to C_\alpha$ such that $\phi_\alpha \circ \rho = \pi_\alpha \circ \sigma$. Hence by the definition of an inverse limit, there exists a continuous *-homomorphism $\phi : A(G,R) \to C$ such that $\phi \circ \rho = \sigma$.
(b) We only outline the (needed) construction of $C^*(G,R)$ from [27]. Let S be the set of all C^*-seminorms on $F(G)$ of form $q(x) = ||\sigma(x)||$, σ running over all representations of G into C^*-algebras. Let $N_q = \{x \in F(G) : q(x) = 0\}$ and $N = \cap\{N_q : q \in S\}$. Let τ be the pro-C^*-topology on $F(G)/N$ defined by $\tilde{q}(x + N) = q(x)$, $q \in S$. Then $C^*(G,R)$ is the completion of $(F(G)/N, \tau)$. The map $\rho_\infty : G \to C^*(G,R)$ where $\rho_\infty(x) = x + N$ is the canonical representation.

The following brings out the essential point in arguments in claim 1 above.

Lemma 8.3. *There exists a natural one-to-one correspondence between* $\text{Rep}_B(G,R)$ (*respectively* $\text{Rep}\,(G,R)$) *and t-continuous Banach *-representations (respectively C^*-algebra representations) of* $A(G,R)$.

Lemma 8.4. $\text{srad}\,(A(G,R)) \bigcap (F(G)/N_a) = \text{srad}\,(F(G)/N_a) = \{x + N_a : x \in N\}$.

Proof. Let $C = F(G)/N_a$. Let $x + N_a \in C \bigcap \text{srad} A$. Then $\pi(x + N_a) = 0$ for all continuous *-homomorphisms $\pi : A \to B(H_\pi)$. By Lemma 8.3, $p(x) = 0$ for all $p \in S$. Hence $x \in N$, and $x + N_a \in \text{srad}\,(F(G)/N_a)$. Conversely, let $x \in N$. Then $q(x) = 0$ for all $q \in S$. Again by Lemma 8.3, $||\pi(x + N_a)|| = 0$ for all $\pi \in R(A)$, hence $x + N_a \in \text{srad} A$.

Proof of Theorem 1.7. (1) Let $A = A(G,R)$. Let $\phi : (F(G)/N_a, t) \to (F(G)/N_a, \tau)$ be $\phi(X + N_a) = x + N$. Then ϕ is a well defined, continuous *-homomorphism; hence extends as a continuous surjective *-homomorphism $\phi : A \to C^*(G,R)$. The universal property of $C^*(G,R)$, Lemma 8.3 and weak Banach admissibility of R imply the following whose proof we omit.

Assertion 1. Given any continuous *-homomorphism $\pi : A(G,R) \to B$ to a pro-C^*-algebra B, there exists a continuous *-homomorphism $\tilde{\pi} : C^*(G,R) \to B$ such that $\pi = \tilde{\pi} \circ \phi$.

$$\begin{tikzcd}
A(G,R) \arrow[r, "\pi"] \arrow[dr, "\phi"] & B \\
G \arrow[u, "\rho"] \arrow[r, "\rho_\infty"'] & C^*(G,R) \arrow[u, "\tilde{\pi}"']
\end{tikzcd}$$

By applying the above to the maps ϕ and $j : A \to E(A), j(x) = x + \operatorname{srad}(A)$, it follows that there exist continuous *-homomorphisms $\tilde{\phi} : E(A) \to C^*(G,R)$ and $\tilde{j} : C^*(G,R) \to E(A)$ such that the following diagrams commute.

$$\begin{tikzcd}
 & A \arrow[dl, "j"'] \arrow[dr, "\phi"] & \\
E(A) \arrow[rr, shift left, "\tilde{\phi}"] & & C^*(G,R) \arrow[ll, shift left, "\tilde{j}"]
\end{tikzcd}$$

Assertion 2. The maps $\tilde{\phi}$ and $\tilde{j}$ are inverse of each other.

Indeed, $\tilde{j}$ is one–one on $F(G)/N$. For given $x \in F(G)$,

$$0 = \tilde{j}(x+N) = \tilde{j} \circ \phi(x+N_a) = j(x+N_a)$$

which implies $(x+N_a) + \operatorname{srad}(A) = 0$ and $(x+N_a) \in \operatorname{srad}(A)$. Hence $x \in N$ by Lemma 8.4, so that $x + N = 0$. Similarly $\tilde{\phi}$ is one–one on $F(G)/N$. Also,

$$\begin{aligned}(\tilde{\phi} \circ \tilde{j})(x+N) &= \tilde{\phi} \circ \tilde{j} \circ \phi(x+N_a) \\ &= \tilde{\phi} \circ j(x+N_a) = \phi(x+N_a) = x+N,\end{aligned}$$

which implies that $\tilde{\phi} = \tilde{j}^{-1}$ on $F(G)/N_a$; and $\tilde{j} = \tilde{\phi}^{-1}$ on $F(G)/N_a + \operatorname{srad} A$. By continuity and density, $\tilde{\phi}$ establishes a homeomorphic *-isomorphism $\tilde{\phi} : E(A) \to C^*(G,R)$ with $\tilde{\phi}^{-1} = \tilde{j}$.

(2) Let (G,R) be C^*-admissible. Then $\sup\{\|\sigma(x)\| : \sigma \in \operatorname{Rep}(G,R)\} < \infty$; and $\pi = \oplus\{\sigma : \sigma \in \operatorname{Rep}(G,R)\} \in \operatorname{Rep}(G,R)$. Thus $q(x) = \|\pi(x)\|$ defines the greatest member of $S(F(G))$, q is a C^*-norm, and it is the greatest t-continuous C^*-seminorm on $F(G)/N$. Thus the topology τ on $C^*(G,R)$ is determined by q. Conversely suppose that $C^*(G,R)$ is a C^*-algebra so that $\|z\|_\infty = \sup\{q(z) : q$ is a continuous C^*-seminorm on $C^*(G,R)\} < \infty$ for all $z \in C^*(G,R)$, and τ is determined by the C^*-norm $\| \ \|_\infty$. Let $p_\infty(x) = \|x+N\|_\infty = \sup\{q(x) : q \in S\}$ for all $x \in F(G)$. Then $p_\infty \in S$ and $\ker p_\infty = N$. There exists a C^*-representation $\sigma : G \to C$ such that $p_\infty(g) = \|\sigma(g)\|$ for all $g \in G$; and this defines a continuous C^*-representation $\sigma : C^*(G,R) \to C$. It is clear that R is C^*-bounded. We verify (bC^*) of Definition 8.1. Let $\{\rho_\alpha\} \subseteq \operatorname{Rep}(G,R)$ with $\rho_\alpha : G \to B(H_\alpha)$

for some Hilbert space H_α. Let $H = \oplus H_\alpha$. For $x \in F(G)$, let $\lambda(x) = \oplus \rho_\alpha(x)$. By the C^*-boundedness of (G,R), $\lambda(x) \in B(H)$. This defines a *-homomorphism $\lambda : F(G) \rightarrow B(H)$ satisfying $||\lambda(x)|| = \sup ||\rho_\alpha(x)|| \leq p_\infty(x)$ for all $x \in F(G)$. Since $\ker p_\infty = N$, λ factors to a *-representation $\tilde{\lambda} : F(G)/N \rightarrow B(H)$ satisfying $||\tilde{\lambda}(z)|| \leq ||z||_\infty$. As $||\ ||_\infty$ is τ-continuous, so is $\tilde{\lambda}$. By lemma 8.3, $\{\lambda(g) : g \in G\}$ satisfies the relations R in $B(H)$. Thus (G,R) is C^*-admissible.

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On the equisummability of Hermite and Fourier expansions

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Abstract. We prove an equisummability result for the Fourier expansions and Hermite expansions as well as special Hermite expansions. We also prove the uniform boundedness of the Bochner-Riesz means associated to the Hermite expansions for polyradial functions.

Keywords. Hermite functions; special Hermite expansions; equisummability.

1. Introduction

This paper is concerned with a comparative study of the Bochner-Riesz means associated to the Hermite and Fourier expansions. Recall that the Bochner-Riesz means associated to the Fourier transform on $\mathbb{R}^n$ are defined by

$$S_t^\delta f(x) = (2\pi)^{-n/2} \int_{|y|\leq t} e^{ix\cdot y}\left(1 - \frac{|y|^2}{t^2}\right)^\delta \hat{f}(y)dy,$$

where

$$\hat{f}(y) = (2\pi)^{-n/2} \int e^{-ix.y} f(x)dx$$

is the Fourier transform on $\mathbb{R}^n$. Let $\Phi_\alpha, \alpha \in N^n$ be the n-dimensional Hermite functions which are eigenfunctions of the Hermite operator $H = -\Delta + |x|^2$ with the eigenvalue $(2|\alpha| + n)$ where $|\alpha| = \alpha_1 + \cdots + \alpha_n$. Let P_k be the orthogonal projection of $L^2(\mathbb{R}^n)$ onto the kth eigenspace spanned by $\Phi_\alpha, |\alpha| = k$. More precisely,

$$P_k f(x) = \sum_{|\alpha|=k} \left(\int f(y)\Phi_\alpha(y)dy\right) \Phi_\alpha(x).$$

Then the Bochner-Riesz means associated to the Hermite expansions are defined by

$$S_R^\delta f(x) = \sum \left(1 - \frac{2k+n}{R}\right)_+^\delta P_k f(x).$$

For the properties of Hermite functions and related results, see [6].

In our study of the Bochner-Riesz means associated to Hermite and special Hermite expansions we make use of a transplantation theorem of Kenig-Stanton-Tomas [2]. Let us briefly recall their result. Let P be a differential operator acting on $C_0^\infty(\mathbb{R}^n) \subset L^2(\mathbb{R}^n)$ which is self adjoint. Let

$$Pf = \int \lambda \mathrm{d}E_\lambda$$

be the spectral resolution of P. Let m be a bounded function on $\mathbb{R}$ and define

$$m_R(P) = \int m\left(\frac{\lambda}{R}\right)\mathrm{d}E_\lambda.$$

Let K be a subset of $\mathbb{R}^n$ with positive measure and define the projection operator Q_k on $L^2(\mathbb{R}^n)$ by

$$Q_k f(x) = \chi_K(x) f(x),$$

where $\chi_K(x)$ is the characteristic function of K. Let $p(x,\xi)$ be the principal symbol of P. Since P is symmetric p is real valued. Then we have the following theorem.

Theorem 1.1. *Assume* $1 \le p \le \infty$ *and that there is a set of positive measure* K_0 *for which the operators* $Q_{K_0} m_R(P) Q_{K_0}$ *are uniformly bounded on* $L^p(\mathbb{R}^n)$. *If* x_0 *in* K_0 *is any point of density, then* $m(p(x_0,\xi))$ *is a Fourier multiplier of* $L^p(\mathbb{R}^n)$.

Let B be any compact set in $\mathbb{R}^n$ containing origin as a point of density and let χ_B be the operator

$$\chi_B f(x) = \chi_B(x) f(x).$$

Then from Theorem 1.1 it follows that the uniform boundedness of $\chi_B S_R^\delta \chi_B$ on $L^p(\mathbb{R}^n)$ implies the uniform boundedness of S_t^δ on $L^p(\mathbb{R}^n)$. Thus once we have the local summability theorem for Hermite expansions then a global result is true for the Fourier transform. At this point a natural question arises, to what extend the converse is true? In this paper we answer this question in the affirmative in dimensions one and two and partially in higher dimensions. We also study the equisummability of the special Hermite expansions, namely the eigenfunction expansion associated to the operator

$$L = -\Delta + \frac{1}{4}|z|^2 - i\sum\left(x_j\frac{\partial}{\partial y_j} - y_j\frac{\partial}{\partial x_j}\right)$$

on $\mathbb{C}^n$. In this case we show that the local uniform boundedness of the Bochner-Riesz means for the special Hermite operator is equivalent to the uniform boundedness of S_t^δ on $\mathbb{R}^{2n}$. Using a recent result of Stempak and Zienkiewicz [4], on the restriction theorem we study the Bochner-Riesz means associated to the Hermite expansions on $\mathbb{R}^{2n}$ for functions having some homogeneity. We also prove a weighted version for the Hermite expansions which slightly improves the local estimates proved in [5]. Eigenfunction expansions associated to special Hermite operator L has been studied by Thangavelu [6].

2. Hermite expansions on $\mathbb{R}^n$

The Hermite functions h_k on $\mathbb{R}$ are defined by

$$h_k(x) = (2^k k!\sqrt{\pi})^{-\frac{1}{2}}(-1)^k \frac{\mathrm{d}^k}{\mathrm{d}x^k}(\mathrm{e}^{-x^2})\mathrm{e}^{\frac{1}{2}x^2}.$$

In the higher dimensions the Hermite functions are defined by taking tensor products:

$$\Phi_\alpha(x) = h_{\alpha_1}(x_1)\dots h_{\alpha_n}(x_n).$$

Given $f \in L^p(\mathbb{R})$ consider the Hermite expansion

$$f(x) = \sum_{k=0}^{\infty}(f, h_k)h_k(x),$$

where $(f, h_k) = \int f(x)h_k(x)\mathrm{d}x$.

Let $S_N f(x) = \sum_{k=0}^{N}(f, h_k)h_k(x)$ be the partial sums associated to the above series. In 1965, Askey-Wainger [1] proved the following celebrated theorem.

Theorem 2.1. *$S_N f \to f$ in the L^p norm iff $\frac{4}{3} < p < 4$.*

Let S_t be the partial sum operator associated to the Fourier transform on $\mathbb{R}$. Then it is well known that $S_t f \to f$ in L^p norm for all $1 < p < \infty$. In this section we show that on a subclass of $L^p(\mathbb{R})$ the same is true for the Hermite expansions.

In the higher dimensions it is convenient to work with Cesaro means rather than Riesz means. These are defined by

$$\sigma_N^\delta f(x) = \frac{1}{A_N^\delta}\sum_{k=0}^{N} A_{N-k}^\delta P_k f(x),$$

where A_k^δ are the binomial coefficients defined by $A_k^\delta = \frac{\Gamma(k+\delta+1)}{\Gamma(k+1)\Gamma(\delta+1)}$. It is well known that σ_N^δ are uniformly bounded on $L^p(\mathbb{R}^n)$ iff S_R^δ are uniformly bounded. We have the following equisummability result. Let E stand for the operator $Ef(x) = \mathrm{e}^{-\frac{1}{2}|x|^2} f(x)$.

Theorem 2.2. *$E\sigma_N^\delta E$ are uniformly bounded on $L^p(R^n)$ iff S_t^δ are uniformly bounded, provided $\delta \geq \max\{0, \frac{n}{2} - 1\}$.*

As a corollary we have the following.

COROLLARY 2.3

Let $1 < p < \infty$. Then for the partial sum operators associated to the one dimensional Hermite expansion we have the uniform estimate

$$\int |S_N f(x)|^p \mathrm{e}^{-\frac{p}{2}x^2}\mathrm{d}x \leq C \int |f(y)|^p \mathrm{e}^{\frac{p}{2}y^2}\mathrm{d}y.$$

Thus for $f \in L^p(\mathrm{e}^{\frac{p}{2}y^2}\mathrm{d}y)$, $1 < p < \infty$ the partial sums converge to f in $L^p(\mathrm{e}^{-\frac{p}{2}x^2}\mathrm{d}x)$.

For a general weighted norm inequality for Hermite expansions, see Muckenhoupt's paper [3].

The celebrated theorem of Carleson-Sjolin for the Fourier expansion on $\mathbb{R}^2$ says that if $\delta > 2(\frac{1}{p} - \frac{1}{2}) - \frac{1}{2}$, $1 \leq p < \frac{4}{3}$ then S_t^δ are uniformly bounded on $L^p(\mathbb{R}^2)$. As a corollary to this we obtain the following result for the Cesaro means σ_N^δ on $\mathbb{R}^2$.

COROLLARY 2.4

Let $n = 2$, $1 \leq p < \frac{4}{3}$ and $\delta > 2(\frac{1}{p} - \frac{1}{2}) - \frac{1}{2}$. Then for $f \in L^p(\mathbb{R}^2)$

$$\int |\sigma_N^\delta f(x)|^p \mathrm{e}^{-\frac{p}{2}|x|^2}\mathrm{d}x \leq C \int |f(y)|^p \mathrm{e}^{\frac{p}{2}|y|^2}\mathrm{d}y.$$

It is an interesting and more difficult problem to establish the above without the exponential factors.

We now proceed to prove Theorem 2.2. It is a trivial matter to see that uniform boundedness of $E\sigma_N^\delta E$ implies the same for $\chi_B\sigma_N^\delta\chi_B$ for any compact subset B of $\mathbb{R}^n$. In fact, if $E\sigma_N^\delta E$ are uniformly bounded then

$$\begin{aligned}
&\int |\chi_B\sigma_N^\delta\chi_B f|^p\mathrm{d}x\\
&= \int_B \mathrm{e}^{-\frac{p}{2}|x|^2}\mathrm{e}^{\frac{p}{2}|x|^2}|\sigma_N^\delta(\mathrm{e}^{-\frac{1}{2}|y|^2}(\chi_B f(y)\mathrm{e}^{\frac{1}{2}|y|^2}))|^p\mathrm{d}x\\
&\leq C\int |E\sigma_N^\delta E(\chi_B f(y)\mathrm{e}^{\frac{1}{2}|y|^2})|^p\mathrm{d}x\\
&\leq C\int |f(x)|^p\mathrm{d}x
\end{aligned}$$

which proves the one way implication, by the transplantation theorem [2]. To prove the converse we proceed as follows. Let

$$\Phi_k(x,y) = \sum_{|\alpha|=k}\Phi_\alpha(x)\Phi_\alpha(y)$$

be the kernel of the projection operator P_k. Then the kernel $\sigma_N^\delta(x,y)$ of the Cesaro means is given by

$$\sigma_N^\delta(x,y) = \frac{1}{A_N^\delta}\sum_{k=0}^{N}A_{N-k}^\delta\Phi_k(x,y).$$

We first obtain a usable expression for this kernel in terms of certain Laguerre functions. Let $L_k^\alpha(t)$ be the Laguerre polynomials of the type $\alpha > -1$ defined by

$$\mathrm{e}^{-t}t^\alpha L_k^\alpha(t) = (-1)^k\frac{1}{k!}\frac{\mathrm{d}^k}{\mathrm{d}t^k}(\mathrm{e}^{-t}t^{k+\alpha}),\quad t>0.$$

We have the following expression.

PROPOSITION 2.5

$$\sigma_N^\delta(x,y) = \frac{1}{A_N^\delta}\sum_{k=0}^{N}(-1)^k L_{N-k}^{\delta+\frac{n}{2}}\left(\frac{1}{2}|x-y|^2\right)\mathrm{e}^{-\frac{1}{4}|x-y|^2}L_k^{\frac{n}{2}-1}\left(\frac{1}{2}|x+y|^2\right)\mathrm{e}^{-\frac{1}{4}|x+y|^2}.$$

Proof. The generating function identity for the projection kernels $\Phi_k(x,y)$ reads

$$\sum_{k=0}^{\infty}r^k\Phi_k(x,y) = \pi^{-\frac{n}{2}}(1-r^2)^{-\frac{n}{2}}\,\mathrm{e}^{-\frac{1}{2}\frac{1+r^2}{1-r^2}(|x|^2+|y|^2)+\frac{2rx.y}{1-r^2}}.$$

Since

$$(1-r)^{-\delta-1} = \sum_{k=0}^{\infty}A_k^\delta r^k$$

the generating function for $\sigma_k^\delta(x,y)$ is given by

$$\sum_{k=0}^{\infty} r^k A_k^\delta \sigma_k^\delta(x,y) = (1-r)^{-\delta-\frac{n}{2}-1}(1+r)^{-\frac{n}{2}}\, \mathrm{e}^{-\frac{1}{2}\frac{1+r^2}{1-r^2}(|x|^2+|y|^2)+\frac{2rx.y}{1-r^2}}.$$

The right hand side of the above expression can be written as

$$(1-r)^{-\delta-\frac{n}{2}-1}\mathrm{e}^{-\frac{1}{4}\frac{1+r}{1-r}|x-y|^2}(1+r)^{-\frac{n}{2}}\, \mathrm{e}^{-\frac{1}{4}\frac{1-r}{1+r}|x+y|^2}.$$

Now the generating function for the Laguerre polynomials L_k^α is

$$\sum_{k=0}^{\infty} r^k L_k^\alpha\left(\frac{1}{2}t^2\right)\mathrm{e}^{-\frac{1}{4}t^2} = (1-r)^{-\alpha-1}\mathrm{e}^{-\frac{1}{4}\frac{1+r}{1-r}t^2}.$$

Therefore, we have

$$\sum_{k=0}^{\infty} r^k A_k^\delta \sigma_k^\delta(x,y) = \left(\sum_{j=0}^{\infty} r^j L_j^{\delta+\frac{n}{2}}\left(\frac{1}{2}|x-y|^2\right)\mathrm{e}^{-\frac{1}{4}|x-y|^2}\right)$$
$$\left(\sum_{i=0}^{\infty} (-r)^i L_i^{\frac{n}{2}-1}\left(\frac{1}{2}|x+y|^2\right)\mathrm{e}^{-\frac{1}{4}|x+y|^2}\right).$$

Equating the coefficients of r^k on both sides we obtain the proposition.

The Laguerre functions L_k^α are expressible in terms of Bessel functions J_α. More precisely, we have the formula

$$\mathrm{e}^{-x}x^{\frac{\alpha}{2}}L_k^\alpha(x) = \frac{1}{\Gamma(k+1)}\int_0^\infty \mathrm{e}^{-t}t^{k+\frac{\alpha}{2}}J_\alpha(2\sqrt{tx})\mathrm{d}t.$$

Using this, the kernel $\mathrm{e}^{-\frac{1}{2}|x|^2}\sigma_N^\delta(x,y)\,\mathrm{e}^{-\frac{1}{2}|y|^2}$ of the operator $E\sigma_N^\delta E$ is given by.

$$\mathrm{e}^{-\frac{1}{2}|x|^2}\sigma_N^\delta(x,y)\mathrm{e}^{-\frac{1}{2}|y|^2} =$$
$$\frac{C}{A_N^\delta}\int_0^\infty\int_0^\infty \mathrm{e}^{-t}\mathrm{e}^{-s}\frac{(t-s)^N}{N!}t^\delta s^{\frac{n}{2}-1}t^{\frac{n}{2}}\frac{J_{\delta+\frac{n}{2}}(\sqrt{2t}|x-y|)}{(\sqrt{2t}|x-y|)^{\delta+\frac{n}{2}}}\frac{J_{\frac{n}{2}-1}(\sqrt{2s}|x+y|)}{(\sqrt{2s}|x+y|)^{\frac{n}{2}-1}}\mathrm{d}t\mathrm{d}s,$$

where C depends only on δ. Now the kernel of the Bochner-Riesz means S_t^δ on $\mathbb{R}^n$ is given by

$$S_t^\delta(x,y) = t^n\frac{J_{\delta+\frac{n}{2}}(t|x-y|)}{(t|x-y|)^{\delta+\frac{n}{2}}}.$$

When $n=1$,

$$\frac{J_{-\frac{1}{2}}(t)}{t^{-\frac{1}{2}}} = \left(\frac{2}{\pi}\right)^{1/2}\cos t$$

and hence

$$E\sigma_N^\delta Ef(x) = C\frac{1}{A_N^\delta}\int_0^\infty\int_0^\infty \mathrm{e}^{-t}\mathrm{e}^{-s}\frac{(t-s)^N}{N!}t^\delta s^{-\frac{1}{2}}T_t^\delta f(x)\mathrm{d}t\mathrm{d}s,$$

where

$$T_t^\delta f(x) = \int_{\mathbb{R}} S^\delta_{\sqrt{2t}}(x,y)\cos(\sqrt{2s}|x+y|)f(y)\mathrm{d}y$$

and C an absolute constant.

By Minkowski's integral inequality we get

$$\begin{aligned}||E\sigma_N^\delta Ef||_p &\leq C\frac{1}{N^\delta}\int_0^\infty\int_0^\infty \mathrm{e}^{-t}\mathrm{e}^{-s}\frac{|t-s|^N}{N!}t^\delta s^{-\frac{1}{2}}||T_t^\delta f||_p\mathrm{d}t\mathrm{d}s\\ &\leq C||f||_p\end{aligned}$$

since

$$\begin{aligned}&\int_0^\infty\int_0^\infty \mathrm{e}^{-t}\mathrm{e}^{-s}|t-s|^N t^\delta s^{-\frac{1}{2}}\mathrm{d}t\,\mathrm{d}s\\ &\quad\leq \int_0^\infty \mathrm{e}^{-t}t^\delta\left(\int_0^t \mathrm{e}^{-s}t^N s^{-\frac{1}{2}}\mathrm{d}s + \int_t^\infty \mathrm{e}^{-s}s^N s^{-\frac{1}{2}}\,\mathrm{d}s\right)\mathrm{d}t\\ &\quad\leq C\int_0^\infty \mathrm{e}^{-t}t^\delta t^N\mathrm{d}t + \Gamma\left(N+\frac{1}{2}\right)\int_0^\infty \mathrm{e}^{-t}t^\delta\,\mathrm{d}t\\ &\quad\leq CN!N^\delta\end{aligned}$$

which proves the theorem in one dimension.

When $n \geq 2$ we have the Bessel functions $J_{\frac{n}{2}-1}$ inside the integral. If $\mathrm{d}\mu$ is the surfa measure on the unit circle $|x| = 1$ in $\mathbb{R}^n$ then we have

$$C\frac{J_{\frac{n}{2}-1}(|x|)}{|x|^{\frac{n}{2}-1}} = \int_{|y|=1}\mathrm{e}^{ix.y}\mathrm{d}\mu(y),$$

where C is an absolute constant. If we use this in the above we get $E\sigma_N^\delta Ef(x)$ equals

$$\frac{C}{A_N^\delta}\int_{|\xi|=1}\int_0^\infty\int_0^\infty \mathrm{e}^{-t}\mathrm{e}^{-s}\frac{(t-s)^N}{N!}t^\delta s^{\frac{n}{2}-1}S^\delta_{\sqrt{2t}}(f(y)\mathrm{e}^{i\sqrt{2s}y.\xi})(x)\mathrm{e}^{i\sqrt{2s}x.\xi}\mathrm{d}t\mathrm{d}s\mathrm{d}\mu(\xi$$

As before, using Minkowski's inequality we get

$$||E\sigma_N^\delta Ef||_p \leq C||f||_p$$

since

$$\begin{aligned}&\int_0^\infty\int_0^\infty \mathrm{e}^{-t}\mathrm{e}^{-s}|t-s|^N t^\delta s^{\frac{n}{2}-1}\mathrm{d}t\mathrm{d}s\\ &\quad\leq \int_0^\infty \mathrm{e}^{-t}t^\delta\left(\int_0^t \mathrm{e}^{-s}t^N s^{\frac{n}{2}-1}\mathrm{d}s + \int_t^\infty \mathrm{e}^{-s}s^{N+\frac{n}{2}-1}\mathrm{d}s\right)\mathrm{d}t\\ &\quad\leq C\int_0^\infty \mathrm{e}^{-t}t^\delta t^N\mathrm{d}t + \Gamma\left(N+\frac{n}{2}\right)\int_0^\infty \mathrm{e}^{-t}t^\delta\mathrm{d}t\\ &\quad\leq C\Gamma(N+\delta+1)\end{aligned}$$

provided $\delta \geq \frac{n}{2} - 1$. This completes the proof.

3. Special Hermite expansions

Let $\Phi_{\alpha\beta}$, $\alpha, \beta \in N^n$, be the special Hermite functions on $\mathbb{C}^n$ which form an orthonormal basis for $L^2(\mathbb{C}^n)$. The special Hermite expansion of a function f in $L^p(\mathbb{C}^n)$ is given by

$$f = \sum\sum(f, \Phi_{\alpha\beta})\Phi_{\alpha\beta}.$$

The functions $\Phi_{\alpha\beta}$ are the eigenfunctions of the operator L with eigenvalues $(2|\beta| + n)$. Let

$$Q_k f = \sum_{|\alpha|=k}\sum_{\beta}(f, \Phi_{\alpha\beta})\Phi_{\alpha\beta}$$

be the projection onto the kth eigenspace. Then we have

$$Q_k f(z) = (2\pi)^{-n} f \times \varphi_k(z),$$

where $\varphi_k(z) = L_k^{n-1}(\frac{1}{2}|z|^2)e^{-\frac{1}{4}|z|^2}$ are the Laguerre functions and $f \times g$ is the twisted convolution

$$f \times g(z) = \int_{\mathbb{C}^n} f(z-w)g(w)e^{\frac{i}{2}\operatorname{Im} z.\bar{w}}dw.$$

The special Hermite expansion then takes the compact form

$$f = (2\pi)^{-n}\sum_{k=0}^{\infty} f \times \varphi_k.$$

The Cesaro means are then defined by

$$\sigma_N^\delta f(z) = \frac{(2\pi)^{-n}}{A_N^\delta}\sum_{k=0}^{N} A_{N-k}^\delta f \times \varphi_k(z).$$

In this section we prove the following theorem.

Let S_t^δ be the Bochner-Riesz means for the Fourier transform on $\mathbb{R}^{2n} = \mathbb{C}^n$.

Theorem 3.1. *Let B be any compact subset of $\mathbb{C}^n$ containing the origin. Then $\chi_B\sigma_N^\delta\chi_B$ are uniformly bounded on L^p, $1 \le p \le \infty$ if and only if S_t^δ are uniformly bounded on the same L^p.*

Proof. The kernel $\sigma_N^\delta(z)$ of σ_N^δ is given by

$$\sigma_N^\delta(z) = \frac{(2\pi)^{-n}}{A_N^\delta}\sum_{k=0}^{N} A_{N-k}^\delta \varphi_k(z).$$

Using the formula

$$\sum_{k=0}^{N} A_{N-k}^\delta L_k^\alpha(t) = L_N^{\alpha+\delta+1}(t)$$

we have

$$\sigma_N^\delta(z) = \frac{(2\pi)^{-n}}{A_N^\delta} L_N^{\delta+n}\left(\frac{1}{2}|z|^2\right)e^{-\frac{1}{4}|z|^2}.$$

As in the previous section we can express the Laguerre function in terms of the Bessel functions, thus getting

$$\sigma_N^\delta(z) = \frac{(2\pi)^{-n}}{A_N^\delta}\frac{1}{\Gamma(N+1)}\mathrm{e}^{\frac{1}{4}|z|^2}\int_0^\infty \mathrm{e}^{-t}t^{\delta+N+n}\frac{J_{\delta+n}(\sqrt{2t}|z|)}{(\sqrt{2t}|z|)^{\delta+n}}\mathrm{d}t.$$

Now, $\sigma_N^\delta f = f \times \sigma_N^\delta$ so that

$$\sigma_N^\delta f(z) = \int \sigma_N^\delta(z, w) f(w)\mathrm{d}w,$$

where

$$\sigma_N^\delta(z, w) = \mathrm{e}^{\frac{i}{2}\mathrm{Im}\, z.\overline{w}}\sigma_N^\delta(z - w).$$

Writing $|z - w|^2 = |z|^2 + |w|^2 + 2\,\mathrm{Re}\, z \cdot \overline{w}$ we have

$$\begin{aligned}\sigma_N^\delta(z, w) &= \mathrm{e}^{\frac{1}{4}|z|^2}\mathrm{e}^{-\frac{1}{2}z.\overline{w}}\frac{1}{A_N^\delta\Gamma(N+1)}\int_0^\infty \mathrm{e}^{-t}t^{\delta+n+N}\frac{J_{\delta+n}(\sqrt{2t}|z-w|)}{(\sqrt{2t}|z-w|)^{\delta+n}}\mathrm{e}^{\frac{1}{4}|w|^2}\mathrm{d}t\\ &= \left(\sum_\alpha\left(-\frac{1}{2}\right)^{|\alpha|}\frac{(z.\overline{w})^\alpha}{\alpha!}\right)\mathrm{e}^{\frac{1}{4}|z|^2}\frac{1}{A_N^\delta\Gamma(N+1)}\;t\\ &\quad\times\int_0^\infty \mathrm{e}^{-t}t^{\delta+n+N}\frac{J_{\delta+n}(\sqrt{2t}|z-w|)}{(\sqrt{2t}|z-w|)^{\delta+n}}\mathrm{e}^{\frac{1}{4}|w|^2}\mathrm{d}t,\end{aligned}$$

where $(z \cdot \overline{w})^\alpha = (z_1\overline{w}_1)^{\alpha_1}\cdots(z_n\overline{w}_n)^{\alpha_n}$. Therefore,

$$\chi_B\sigma_N^\delta\chi_B f(z) = \frac{1}{A_N^\delta\Gamma(N+1)}\sum_\alpha\left(-\frac{1}{2}\right)^{|\alpha|}\frac{1}{\alpha!}\int_0^\infty \mathrm{e}^{-t}t^{\delta+N}T_{\alpha,\delta}^t f(z)\mathrm{d}t,$$

where

$$T_{\alpha,\delta}^t f(z) = \chi_B(z)z^\alpha\mathrm{e}^{\frac{1}{4}|z|^2}\int_{\mathbb{C}^n} t^n\frac{J_{\delta+n}(\sqrt{2t}|z-w|)}{(\sqrt{2t}|z-w|)^{\delta+n}}\chi_B(w)\overline{w}^\alpha\mathrm{e}^{\frac{1}{4}|w|^2}f(w)\mathrm{d}w.$$

If we assume that S_t^δ are uniformly bounded we get

$$||T_{\alpha,\delta}^t f||_p \leq CR^{2|\alpha|}||f||_p,$$

when B is contained in the ball $\{z : |z| \leq R\}$. Using this in the above equation we get

$$||\chi_B\sigma_N^\delta\chi_B f||_p \leq C_B||f||_p.$$

The converse is the transplantation theorem of Kenig-Stanton-Tomas.

In [5], Thangavelu has established the following local estimates for the Cesa means.

Theorem 3.2. *Let* $\frac{2(2n+1)}{2n-1} \leq p \leq \infty$ *and* $\delta > \delta(p) = 2n(\frac{1}{p} - \frac{1}{2}) - \frac{1}{2}$ *then for any compa subset* B *of* $\mathbb{C}^n$

$$\int_B |\sigma_N^\delta f(z)|^p\mathrm{d}z \leq C_B\int |f(z)|^p\mathrm{d}z.$$

Recently Stempak and Zienkiewicz have proved the global estimate

$$\int_{\mathbb{C}^n} |\sigma_N^\delta f(z)|^p \mathrm{d}z \leq C \int_{\mathbb{C}^n} |f(z)|^p \mathrm{d}z$$

for the above range. The key point is the restriction theorem namely, the estimate

$$||f \times \varphi_k||_2 \leq C k^{n(\frac{1}{p}-\frac{1}{2})-\frac{1}{2}} ||f||_p$$

which they established in the range $1 \leq p \leq \frac{2(2n+1)}{2n+3}$. In the next section we use this restriction theorem in order to prove a positive result for the Hermite expansions on $\mathbb{R}^{2n}$.

4. Hermite expansions on $\mathbb{R}^{2n}$

In this section we consider the operator $-\Delta + \frac{1}{4}|z|^2$ rather than the operator $-\Delta + |z|^2$. If $\Phi_\mu(x, y), \mu \in N^{2n}$ are the eigenfunctions of the operator $-\Delta + |z|^2$ then $\Psi_\mu(z) = \Phi_\mu(\frac{x}{\sqrt{2}}, \frac{y}{\sqrt{2}})$ are the eigenfunctions of $-\Delta + \frac{1}{4}|z|^2$ with eigenvalues $(|\mu| + n)$. The operator $-\Delta + \frac{1}{4}|z|^2$ has another family of eigenfunctions namely the special Hermite functions. In fact, $\Phi_{\alpha\beta}$ are eigenfunctions of the operator $-\Delta + \frac{1}{4}|z|^2$ with eigenvalue $(|\alpha| + |\beta| + n)$; here $\alpha, \beta \in N^n$.

In this section we study the expansion in terms of Ψ_μ for functions having some homogeneity. The torus $T(n) = \{(\mathrm{e}^{i\theta_1}, \mathrm{e}^{i\theta_2}, \ldots, \mathrm{e}^{i\theta_n}) : \theta \in \mathbb{R}^n\}$ acts on functions on $\mathbb{C}^n$ by $\tau_\theta f(z) = f(\mathrm{e}^{i\theta} z)$ where $\mathrm{e}^{i\theta} z = (\mathrm{e}^{i\theta_1} z_1, \mathrm{e}^{i\theta_2} z_2, \ldots, \mathrm{e}^{i\theta_n} z_n)$. We say that a function is m-homogeneous if $\tau_\theta f(z) = \mathrm{e}^{im.\theta} f(z)$, here $m \in Z^n$ and $m.\theta = m_1.\theta_1 + \cdots + m_n \cdot \theta_n$. It is a fact that $\Phi_{\alpha\beta}$ is $(\beta - \alpha)$ homogeneous. 0-homogeneous functions are also called polyradial.

The operator $-\Delta + \frac{1}{4}|z|^2$ commutes with τ_θ for all θ, therefore $P_k \tau_\theta f = \tau_\theta P_k f$ which shows that $P_k f$ is m-homogeneous if f is . In particular, $P_k f$ is polyradial if f is. Therefore, for such functions $L(P_k f) = (-\Delta + \frac{1}{4}|z|^2) P_k f = (k + n) P_k f$. This shows that $P_k f$ is an eigenfunction of L with eigenvalue $k + n$. But the spectrum of L is $\{2k + n : k = 0, 1, \ldots\}$ which forces $P_k f = 0$ when k is odd.

PROPOSITION 4.1

Let f be polyradial on $\mathbb{C}^n$. *Then* $P_{2k+1} f = 0$ *and* $P_{2k} f = f \times \varphi_k$.

Proof. We show that when f is polyradial the operators $P_{2k} f$ and $f \times \varphi_k$ have the same kernel. Let

$$\Psi_k(z, w) = \sum_{|\mu|=k} \Psi_\mu(z) \Psi_\mu(w)$$

be the kernel of P_k. Then by Mehler's formula

$$\sum_{k=0}^{\infty} t^k \Psi_k(z, w) = \pi^{-n} (1 - t^2)^{-n} \mathrm{e}^{-\frac{1}{4}\frac{1+t^2}{1-t^2}(|z|^2+|w|^2)+\frac{t}{1-t^2}\mathrm{Re}(z.\overline{w})}$$

so that

$$\sum_{k=0}^{\infty} t^k P_k f(z) = \pi^{-n} (1 - t^2)^{-n} \int \mathrm{e}^{-\frac{1}{4}\frac{1+t^2}{1-t^2}(|z|^2+|w|^2)+\frac{t}{1-t^2}\mathrm{Re}(z.\overline{w})} f(w) \mathrm{d}w.$$

Let $w_j = u_j + iv_j = r_j\mathrm{e}^{i\theta_j}$. When f is polyradial $f(w) = f_0(r_1, r_2, \ldots, r_n)$ and so we have

$$\sum_{k=0}^{\infty} t^k P_k f(z) = \int_0^{\infty} \cdots \int_0^{\infty} \Psi(s,r) f_0(r_1, \ldots, r_n) r_1 r_2, \ldots, r_n \mathrm{d}r_1 \mathrm{d}r_2 \ldots \mathrm{d}r_n,$$

where $s = (s_1, s_2, \ldots, s_n), s_j = |z_j|$ and Ψ is given by

$$\Psi(s,r) = (1-t^2)^{-n} \int_{[0,2\pi]^n} \mathrm{e}^{-\frac{1}{4}\frac{1+t^2}{1-t^2}(r^2+s^2)} \mathrm{e}^{\frac{t}{1-t^2}\mathrm{Re}\, z.\overline{w}} \mathrm{d}\theta_1 \mathrm{d}\theta_2 \cdots \mathrm{d}\theta_n.$$

Now Re $z_j \cdot \overline{w}_j = r_j s_j \cos(\theta_j - \varphi_j)$ where $z_j = s_j \mathrm{e}^{i\varphi_j}, w_j = r_j \mathrm{e}^{i\theta_j}$. Consider the integral

$$\int_0^{2\pi} \mathrm{e}^{\frac{t}{1-t^2} r_j s_j \cos(\theta_j - \varphi_j)} \mathrm{d}\theta_j$$

which equals, if we recall the definition of the Bessel functions, $J_0(\frac{it}{1-t^2} r_j s_j)$. Thus we have proved

$$\Psi(s,r) = (1-t^2)^{-n} \mathrm{e}^{-\frac{1}{4}\frac{1+t^2}{1-t^2}(r^2+s^2)} \Pi_{j=1}^{n} J_0\left(\frac{it}{1-t^2} r_j s_j\right).$$

On the other hand when f is polyradial $f \times \varphi_k$ reduces to the finite sum

$$\begin{aligned} f \times \varphi_k &= \sum_{|\alpha|=k} (f, \Phi_{\alpha\alpha}) \Phi_{\alpha\alpha}(z) \\ &= \sum_{|\alpha|=k} \left(\int_0^{\infty} \cdots \int_0^{\infty} f_0(r_1, \ldots, r_n) \Phi_{\alpha\alpha}(r_1, \ldots, r_n) r_1, \ldots, r_n \mathrm{d}r_1, \ldots, \mathrm{d}r_n \right) \\ &\quad \times \Phi_{\alpha\alpha}(s_1, \ldots, s_n) \end{aligned}$$

where we have written

$$\Phi_{\alpha\alpha}(z) = \Phi_{\alpha\alpha}(r_1, \ldots, r_n)$$

as it is polyradial. Then $f \times \varphi_k$ is given by the integral operator

$$\begin{aligned} f \times \varphi_k(z) = \int_0^{\infty} \cdots \int_0^{\infty} & \left(\sum_{|\alpha|=k} \Phi_{\alpha,\alpha}(r_1, \ldots, r_n) \Phi_{\alpha,\alpha}(s_1, \ldots, s_n) \right) \\ & f_0(r_1, \ldots, r_n) r_1, \ldots, r_n \mathrm{d}r_1, \ldots, \mathrm{d}r_n. \end{aligned}$$

We have the formula (see [6])

$$\Phi_{\mu\mu}(z) = (2\pi)^{-\frac{n}{2}} \Pi_{j=1}^{n} L_{\mu_j}\left(\frac{1}{2}|z_j|^2\right) \mathrm{e}^{-\frac{1}{4}|z_j|^2}.$$

Recalling the generating function identity for the Laguerre polynomials of type 0,

$$\sum_{k=0}^{\infty} L_k(x) L_k(y) w^k = (1-w)^{-1} \mathrm{e}^{-\frac{w}{1-w}(x+y)} J_0\left(\frac{2(-xyw)^{\frac{1}{2}}}{1-w}\right)$$

we get, if $S_k(r,s)$ is the kernel for $f \times \varphi_k$

$$\sum_{k=0}^{\infty} t^k S_k(r,s) = (1-t)^{-n} \mathrm{e}^{-\frac{1}{4}\frac{1+t}{1-t}(r^2+s^2)} \Pi_{j=1}^{n} J_0\left(\frac{i\sqrt{t}}{1-t} r_j s_j\right).$$

Comparing the two generating functions we see that

$$\sum_{k=0}^{\infty} t^{2k} S_k(r,s) = \sum_{k=0}^{\infty} t^k \Psi_k(r,s)$$

from which follows $\Psi_{2k}(r,s) = S_k(r,s)$ and this proves the proposition.

Consider now the Bochner-Riesz means associated to the expansions in terms of $\Psi_\mu(z)$ defined by

$$S_R^\delta f(z) = \sum_{\mu} \left(1 - \frac{(|\mu|+n)}{R}\right)_+^\delta (f, \Psi_\mu) \Psi_\mu(z).$$

For these means we have the following result.

Theorem 4.2. *Let* $1 \leq p \leq 2(\frac{2n+1}{2n+3})$, $\delta > \delta(p) = 2n(\frac{1}{p} - \frac{1}{2}) - \frac{1}{2}$ *and let* $f \in L^p(\mathbb{C}^n)$ *be polyradial. Then*

$$||S_R^\delta f||_p \leq C||f||_p,$$

where C is independent of f and R.

The key ingredient in proving the above theorem is the $L^p - L^2$ estimates

$$||P_k f||_2 \leq C k^{n(\frac{1}{p}-\frac{1}{2})-\frac{1}{2}} ||f||_p$$

which now follows from the corresponding estimates for $f \times \varphi_k$. We omit the details.

We conclude this section with the following remarks. As we have observed, $P_k f$ is m-homogeneous whenever f is and so $P_k f$ can be obtained in terms of $f \times \varphi_k$ when f is m-homogeneous. So an analogue of the above theorem is true for all m-homogeneous functions. More generally, let us call a function f of type N if it has the Fourier expansion

$$f(z) = \sum_{|m| \leq N} f_m(z),$$

where

$$f_m(z) = \int f(\mathrm{e}^{i\theta} z) \mathrm{e}^{-im.\theta} \mathrm{d}\theta_1 \cdots \mathrm{d}\theta_n.$$

Note that f_m is m-homogeneous. We can show that when f is of type N then

$$||S_R^\delta f||_p \leq C_N ||f||_p,$$

under the conditions of the above theorem on p and δ where now C_N depends on N. We leave the details to the interested reader. It is an interesting problem to see if the theorem is true for all functions.

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Periodic and boundary value problems for second order differential equations

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Abstract. In this paper we study second order scalar differential equations with Sturm–Liouville and periodic boundary conditions. The vector field $f(t, x, y)$ is Caratheodory and in some instances the continuity condition on x or y is replaced by a monotonicity type hypothesis. Using the method of upper and lower solutions as well as truncation and penalization techniques, we show the existence of solutions and extremal solutions in the order interval determined by the upper and lower solutions. Also we establish some properties of the solutions and of the set they form.

Keywords. Upper solution; lower solution; order interval; truncation map; penalty function; Caratheodory function; Sobolev space; compact embedding; Dunford–Pettis theorem; Arzela–Ascoli theorem; extremal solution; periodic problem; Sturm–Liouville boundary conditions.

1. Introduction

The method of upper and lower solutions offers a powerful tool to establish the existence of multiple solutions for initial and boundary value problems of the first and second order. This method generates solutions of the problem, located in an order interval with the upper and lower solutions serving as bounds. In fact the method is often coupled with a monotone iterative technique which provides a constructive way (amenable to numerical treatment) to generate the extremal solutions within the order interval determined by the upper and lower solutions.

In this paper we employ this technique to study scalar nonlinear periodic and boundary value problems. The overwhelming majority of the works in this direction, assume that the vector field is continuous in all variables and they look for solutions in the space $C^2(0, b)$. We refer to the books by Bernfeld–Lakshmikantham [2] and Gaines–Mawhin [6] and the references therein. The corresponding theory for discontinuous (at least in the time variable t) nonlinear differential equations is lagging behind. It is the aim of this paper to contribute in the development of the theory in this direction. Dealing with discontinuous problems, leads to Caratheodory or monotonicity conditions and to Sobolev spaces of functions of one variable. It is within such a framework that we will conduct our investigation in this paper. We should mention that an analogous study for first order problems can be found in Nkashama [18].

2. Sturm–Liouville problems

Let $T=[0,b]$. We start by considering the following second order boundary value problem:

$$\left\{\begin{array}{l} -x''(t)=f(t,x(t),x'(t)) \text{ a.e. on } T \\ (B_0x)(0)=\nu_0,\ (B_1x)(b)=\nu_1 \end{array}\right\}. \tag{1}$$

Here $(B_0x)(0)=a_0x(0)-c_0x'(0)$ and $(B_1x)(b)=a_1x(b)+c_1x'(b)$, with $a_0,c_0,a_1,$ $c_1\geq 0$ and $a_0(a_1b+c_1)+c_0a_1\neq 0$. Note that if $c_0=c_1=\nu_0=\nu_1=0$, then we have the Dirichlet (or Picard in the terminology of Gaines–Mawhin [6]) problem. The vector field $f(t,x,y)$ is not continuous, but only a Caratheodory function; i.e. it is measurable in $t\in T$ and continuous in $(x,y)\in\mathbb{R}\times\mathbb{R}$ (later the continuity in y will be replaced by a monotonicity condition). Hence $x''(\cdot)$ is not continuous, but only an $L^1(T)$-function. Recently Nieto–Cabada [17] considered a special case of (1) with f independent of y. Also there is the work of Omari [19] where f is continuous.

We will be using the Sobolev spaces $W^{1,1}(T)$ and $W^{2,1}(T)$. It is well known (see for example Brezis [3], p. 125), that $W^{1,1}(T)$ is the space of absolutely continuous functions and $W^{2,1}(T)$ is the space of absolutely continuous function whose derivative is absolutely continuous too.

DEFINITION

A function $\psi\in W^{2,1}(T)$ is said to be a 'lower solution' for problem (1) if

$$\left\{\begin{array}{l} -\psi''(t)\leq f(t,\psi(t),\psi'(t)) \text{ a.e. on } T \\ (B_0\psi)(0)\leq\nu_0,\ (B_1\psi)(b)\leq\nu_1 \end{array}\right\}. \tag{2}$$

A function $\phi\in W^{2,1}(T)$ is said to be an 'upper solution' for problem (1) if the inequalities in (2) are reversed.

For the first existence theorem we will need the following hypotheses:

$\mathrm{H(f)}_1$: $f:T\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is a function such that

(i) for every $x,y\in\mathbb{R}$, $t\to f(t,x,y)$ is measurable;
(ii) for every $t\in T$, $(x,y)\to f(t,x,y)$ is continuous;
(iii) for every $r>0$ there exists $\gamma_r\in L^1(T)$ such that $|f(t,x,y)|\leq\gamma_r(t)$ a.e. on T for all $x,y\in\mathbb{R}$ with $|x|,|y|\leq r$.

H_0: There exists an upper solution ϕ and a lower solution ψ such that $\psi(t)\leq\phi(t)$ for every $t\in T$ and there exists $h\in C(\mathbb{R}_+,(0,\infty))$ such that $|f(t,x,y)|\leq h(|y|)$ for all $t\in T$ and all $x,y\in\mathbb{R}$ with $\psi(t)\leq x\leq\phi(t)$ and $\int_\lambda^\infty\frac{r\,dr}{h(r)}>\max_{t\in T}\phi(t)-\min_{t\in T}\psi(t)$, with $\lambda=\frac{\max[|\psi(0)-\phi(b)|,|\psi(b)-\phi(0)|]}{b}$.

Remark. The second part of hypothesis H_0 (the growth condition on f), is known as the 'Nagumo growth condition' and guarantees an *a priori* L^∞-bound for $x'(\cdot)$. More precisely, if H_0 holds, then there exists $N_1>0$ (depending only on ϕ,ψ,h) such that for every $x\in W^{2,1}(T)$ solution of $-x''(t)=f(t,x(t),x'(t))$ a.e. on T with $\psi(t)\leq x(t)\leq\phi(t)$ for all $t\in T$, we have $|x'(t)|\leq N_1$ for all $t\in T$ (the proof of this, is the same (with minor modifications) with that of Lemma 1.4.1, p. 26 of Bernfeld–Lakshmikantham [2]).

We introduce the order interval $K = [\psi, \phi] = \{x \in W^{1,2}(T) : \psi(t) \leq x(t) \leq \phi(t)$ for all $t \in T\}$ and we want to know if there exists a solution of (1) within the order interval K. Also we are interested on the existence of the least and the greatest solutions of (1) within K ('extremal solutions'). The next two theorems solve these problems. In theorem 1 we prove the existence of a solution in K and in theorem 2 we prove the existence of extremal solutions within K. Although the hypotheses in both theorems are the same, we decided to present them separately for reasons of clarity, since otherwise the proof would have been too long.

Theorem 1. *If hypotheses* $\mathrm{H(f)}_1$ *and* H_0 *hold, then problem* (1) *has a solution* $x \in W^{2,1}(T)$ *within the order interval* $K = [\psi, \phi]$.

Proof. As we already mentioned in a previous remark, the Nagumo growth condition (see H_0) implies the existence of $N_1 > 0$ (depending only on ψ, ϕ, h) such that $|x'(t)| \leq N_1$ for all $f \in T$, for every $x \in W^{2,1}(T)$ solution of (1) belonging in K. Set $N = 1 + \max\{N_1, ||\psi'||_\infty, ||\phi'||_\infty\}$. Also define the truncation operator τ: $W^{1,1}(T) \to W^{1,1}(T)$ by

$$\tau(x)(t) = \begin{cases} \phi(t) & \text{if } \phi(t) \leq x(t) \\ x(t) & \text{if } \psi(t) \leq x(t) \leq \phi(t) \\ \psi(t) & \text{if } x(t) \leq \psi(t) \end{cases}.$$

The fact that $\tau(x) \in W^{1,1}(T)$ can be found in Gilbarg–Trudinger [8] (p. 145) and we know that

$$\tau(x)'(t) = \begin{cases} \phi'(t) & \text{if } \phi(t) \leq x(t) \\ x'(t) & \text{if } \psi(t) \leq x(t) \leq \phi(t) \\ \psi'(t) & \text{if } x(t) \leq \psi(t) \end{cases}.$$

Also we define the truncation at N function $q_N \in C(\mathbb{R})$ by

$$q_N(x) = \begin{cases} N & \text{if } N \leq x \\ x & \text{if } -N \leq x \leq N \\ -N & \text{if } x \leq -N \end{cases}$$

and the penalty function $u : T \times \mathbb{R} \to \mathbb{R}$ by

$$u(t,x) = \begin{cases} x - \phi(t) & \text{if } \phi(t) \leq x \\ 0 & \text{if } \psi(t) \leq x \leq \phi(t) \\ x - \psi(t) & \text{if } x \leq \psi(t) \end{cases}.$$

Then we consider the following Sturm–Liouville problem

$$\left\{ \begin{array}{l} -x''(t) = f(t, \tau(x)(t), q_N(\tau(x)'(t))) - u(t, x(t)) \text{ a.e. on } T \\ (B_0x)(0) = \nu_0, (B_1x)(b) = \nu_1 \end{array} \right\}. \tag{3}$$

Denote by S the solution set of (3).

Claim # 1. $S \subseteq K = [\psi, \phi]$. Let $x \in S$. Then we have

$$-x''(t) = f(t, \tau(x)(t), q_N(\tau(x)'(t))) - u(t, x(t)) \text{ a.e. on } T. \tag{4}$$

Also since $\psi \in W^{2,1}(T)$ is a lower solution of (1), we have

$$\psi''(t) \geq -f(t, \psi(t), \psi'(t)) \qquad \text{a.e. on } T. \tag{5}$$

Adding (4) and (5), we obtain

$$\begin{aligned} \psi''(t) - x''(t) \geq\ & f(t, \tau(x)(t), q_N(\tau(x)'(t))) \\ & - f(t, \psi(t), \psi'(t)) - u(t, x(t)) \qquad \text{a.e. on } T. \end{aligned}$$

Multiplying with $(\psi - x)_+(t)$ and integrating over $T = [0, b]$, we have

$$\begin{aligned} & \int_0^b (\psi''(t) - x''(t))(\psi - x)_+(t)\mathrm{d}t \\ & \quad \geq \int_0^b [f(t, \tau(x)(t), q_N(\tau(x)'(t))) - f(t, \psi(t), \psi'(t))] \\ & \quad (\psi - x)_+(t)\mathrm{d}t - \int_0^b u(t, x(t))(\psi - x)_+(t)\mathrm{d}t. \end{aligned} \tag{6}$$

From the integration by parts formula (Green's identity), we have

$$\begin{aligned} & \int_0^b (\psi''(t) - x''(t))(\psi - x)_+(t)\mathrm{d}t \\ & \quad = (\psi' - x')(b)(\psi - x)_+(b) - (\psi' - x')(0)(\psi - x)_+(0) \\ & \quad - \int_0^b (\psi' - x')(t)(\psi - x)'_+(t)\mathrm{d}t. \end{aligned} \tag{7}$$

Using the boundary conditions for x and ψ at $t = 0$, we have

$$\begin{aligned} & a_0\psi(0) - c_0\psi'(0) \leq \nu_0 = a_0 x(0) - c_0 x'(0) \\ & \quad \Rightarrow -c_0(\psi'(0) - x'(0)) \leq -a_0(\psi(0) - x(0)). \end{aligned}$$

If $c_0 = 0$, then $\psi(0) \leq x(0)$ and so $(\psi - x)_+(0) = 0$. Therefore $-(\psi' - x')(0)$ $(\psi - x)_+(0) = 0$.

If $c_0 > 0$, then $-(\psi'(0) - x'(0)) \leq -\frac{a_0}{c_0}(\psi(0) - x(0)) \Rightarrow -(\psi'(0) - x'(0))(\psi - x)_+$ $(0) \leq -\frac{a_0}{c_0}(\psi(0) - x(0))(\psi - x)_+(0)$. Thus if $(\psi(0) - x(0)) \geq 0$, we have $-(\psi' - x')(0)$ $(\psi - x)_+(0) \leq 0$ and if $(\psi(0) - x(0)) < 0$, we have $(\psi - x)_+(0) = 0$ and so $-(\psi' - x')(0)(\psi - x)_+(0) = 0$. Therefore we always have

$$-(\psi' - x')(0)(\psi - x)_+(0) \leq 0. \tag{8}$$

From the boundary condition at $t = b$, we have

$$\begin{aligned} & a_1\psi(b) + c_1\psi'(b) \leq \nu_1 = a_1 x(b) + c_1 x'(b) \\ & \quad \Rightarrow c_1(\psi'(b) - x'(b)) \leq -a_1(\psi(b) - x(b)). \end{aligned}$$

Then arguing as above, we infer that

$$(\psi' - x')(b)(\psi - x)_+(b) \leq 0. \tag{9}$$

Finally recall that

$$(\psi - x)'(t) = \begin{cases} (\psi - x)'(t) & \text{if} \quad x(t) \leq \psi(t) \\ 0 & \text{if} \quad x(t) \geq \psi(t) \end{cases}$$

(see Gilbarg–Trudinger [8], p. 145). Hence it follows that

$$\int_0^b (\psi' - x')(t)(\psi - x)'_+(t)\mathrm{d}t = \int_{\{x \leq \psi\}} [(\psi - x)'(t)]^2 \mathrm{d}t \geq 0. \tag{10}$$

Using (8), (9), (10) in (7), we deduce that

$$\int_0^b (\psi'' - x'')(t)(\psi - x)_+(t)\mathrm{d}t \leq 0. \tag{11}$$

Also note that

$$\begin{aligned} &\int_0^b [f(t, \tau(x)(t), q_N(\tau(x)'(t))) - f(t, \psi(t), \psi'(t))](\psi - x)_+(t)\mathrm{d}t \\ &\quad = \int_{\{x \leq \psi\}} [f(t, \tau(x)(t), q_N(\tau(x)'(t))) - f(t, \psi(t), \psi'(t))](\psi - x)(t)\mathrm{d}t \\ &\quad = \int_{\{x \leq \psi\}} [f(t, \psi(t), \psi'(t)) - f(t, \psi(t), \psi'(t))](\psi - x)(t)\mathrm{d}t = 0 \end{aligned} \tag{12}$$

since on the set $\{t \in T : x(t) \leq \psi(t)\}$, we have $\tau(x)(t) = \psi(t)$ and $\tau(x)'(t) = \psi'(t)$. Using (11) and (12) in (6), we have that

$$\begin{aligned} 0 &\leq \int_0^b u(t, x(t))(\psi - x)_+(t)\mathrm{d}t = \int_{\{x \leq \psi\}} u(t, x(t))(\psi - x)(t)\mathrm{d}t \\ &= \int_0^b -(\psi - x)_+^2(t)\mathrm{d}t \leq 0 \end{aligned}$$

(recall the definition of $u(t,x)$). So $\psi(t) \leq x(t)$ for all $t \in T$. In a similar way we show that $x(t) \leq \phi(t)$ for all $t \in T$. Therefore $S \subseteq K$ as claimed.
Claim # 2. S is nonempty. This will be proved by means of Schauder's fixed point theorem. To this end let $D = \{x \in W^{2,1}(T) : (B_0x)(0) = \nu_0,\ (B_1x)(b) = \nu_1\}$ and let $\hat{L} : D \subseteq L^1(T) \to L^1(T)$ be defined by $\hat{L}x = -x''$ for every $x \in D$. First note that for every $h \in L^1(T)$ the boundary value problem

$$\left\{ \begin{array}{l} -x''(t) + x(t) = h(t) \text{ a.e. on } T \\ (B_0x)(0) = \nu_0,\ (B_1x)(b) = \nu_1 \end{array} \right\} \tag{13}$$

has a unique solution $x \in W^{2,1}(T)$. Indeed uniqueness of the solution is clear. For the existence, note that if $h \in C(T)$, then it follows from corollary 3.1 of Mönch [15]. In the general case, let $h \in L^1(T)$ and take $h_n \in C(T)$ such that $h_n \to h$ in $L^1(T)$ as $n \to \infty$. For each h_n, $n \geq 1$, the solution $x_n(\cdot)$ of (13) is given by $x_n(t) = u(t) + \int_0^b G(t,s)(x_n(s) - h_n(s))\mathrm{d}s$, where $u \in C^2(T)$ is the unique solution of $x''(t) = 0$ $t \in T$, $(B_0x)(0) = \nu_0$, $(B_1x)(b) = \nu_1$ and $G(t,s)$ is the Green's function for the problem $x'' = g(t)$ $t \in T$, $(B_0x)(0) = 0$, $(B_1x)(b) = 0$ for $g \in C(T)$ given. From the proof of corollary 3.1 (b) of Mönch [15], we know that $\sup_{n \geq 1} ||x_n||_\infty \leq \sup_{n \geq 1} ||\eta_n||_\infty$, where $\eta_n \in C^2(T)$ is the unique solution of $\eta''(t) = -h_n(t)$ $t \in T$, $(B_0\eta)(0) = |\nu_0|$, $(B_1\eta)(b) = |\nu_1|$. We know that

$\eta_n(t) = u(t) - \int_0^b G(t,s)h_n(s)\mathrm{d}s$ and so it follows that $\sup_{n\geq 1} \|\eta_n\|_\infty < \infty$. Hence $\{x_n\}_{n\geq 1}$ is bounded in $C(T)$. Since $-x_n''(t) = h_n(t) - x_n(t)$, $t \in T$, it follows that $\{x_n''\}_{n\geq 1}$ is uniformly integrable. From Brezis [3] (p. 132) we know that the norm $\|\cdot\|_{W^{2,1}(T)}$ is equivalent to the norm $\|x\| = \|x\|_1 + \|x''\|_1$. Therefore $\{x_n''\}_{n\geq 1}$ is bounded in $W^{2,1}(T)$. Since $W^{2,1}(T)$ embeds continuously in $C^1(T)$ and compactly in $L^1(T)$ and by the Dunford–Pettis compactness criterion, by passing to a subsequence if necessary, we may assume that $x_n \xrightarrow{w} x$ in $C^1(T)$ (hence $x_n'(t) \to x'(t)$ for all $t \in T$), $x_n \to x$ in $L^1(T)$ and $x_n'' \xrightarrow{w} y$ in $L^1(T)$ as $n \to \infty$. Evidently $y = x''$. So in the limit as $n \to \infty$, we have $-x''(t) + x(t) = h(t)$ a.e. on T, $(B_0x)(0) = \nu_0$, $(B_1x)(b) = \nu_1$. Therefore we have proved that $R(I + \hat{L}) = L^1(T)$.

Next let $x_1, x_2 \in D$ and $x = x_1 - x_2$. Define

$$T_+ = \{t \in T : x(t) > 0\} \quad \text{and} \quad T_- = \{t \in T : x(t) < 0\}$$

both open sets in T. For $\lambda > 0$ we have

$$\begin{aligned}
\int_0^b |x(t) - \lambda x''(t)|\mathrm{d}t &\geq \int_{T_+} |x(t) - \lambda x''(t)|\mathrm{d}t + \int_{T_-} |x(t) - \lambda x''(t)|\mathrm{d}t \\
&\geq \int_{T_+} (x(t) - \lambda x''(t))\mathrm{d}t - \int_{T_-} (x(t) - \lambda x''(t))\mathrm{d}t \\
&= \int_{T_+} x(t)\mathrm{d}t - \int_{T_-} x(t)\mathrm{d}t - \lambda \int_{T_+} x''(t)\mathrm{d}t + \lambda \int_{T_-} x''(t)\mathrm{d}t \\
&= \int_0^b |x(t)|\mathrm{d}t - \lambda\left[\int_{T_+} x''(t)\mathrm{d}t - \int_{T_-} x''(t)\mathrm{d}t\right].
\end{aligned}$$

Let (a,c) be a connected component of T_+. Then $x(a) = x(c) = 0$ and $x(t) > 0$ for all $t \in (a,c)$. Thus $x'(a) \geq 0$ and $x'(c) \leq 0$ and from this it follows that $\int_a^c x''(t)\mathrm{d}t = x'(c) - x'(a) \leq 0$. Therefore we deduce that $\int_{T_+} x''(t)\mathrm{d}t \leq 0$. Similarly we show that $\int_{T_-} x''(t)\mathrm{d}t \geq 0$. So finally we have $-\lambda[\int_{T_+} x''(t)\mathrm{d}t - \int_{T_-} x''(t)\mathrm{d}t] \geq 0$ and thus we obtain

$$\begin{aligned}
&\int_0^b |x(t) - \lambda x''(t)|\mathrm{d}t \geq \int_0^b |x(t)|\mathrm{d}t \\
&\Rightarrow \|x_1 + \lambda\hat{L}x_1 - (x_2 + \lambda\hat{L}x_2)\|_1 \geq \|x_1 - x_2\|_1.
\end{aligned}$$

This last inequality together with the fact that $R(I + \hat{L}) = L^1(T)$, implies that $(I + \hat{L})^{-1}$: $L^1(T) \to D \subseteq L^1(T)$ is well-defined and nonexpansive (is the resolvent of the m-accretive operator $\hat{L}$; see Vrabie [21], Lemma 1.1.5, p. 20). For $k > 0$ consider the set

$$\Gamma_k = \{x \in D : \|x\|_1 + \|x''\|_1 \leq k\}.$$

Recalling that $\|x\|_1 + \|x''\|_1$ is an equivalent norm on $W^{2,1}(T)$ see Brezis [3], p. 132), it follows that Γ_k is bounded in $W^{2,1}(T)$ and since the latter embeds compactly in $L^1(T)$, we conclude that Γ_k is relatively compact in $L^1(T)$. So from Vrabie [21] (Proposition 2.2.1, p. 56), we have that $(I + \hat{L})^{-1}$ is a compact operator. If $C \subseteq L^1(T)$ is bounded and $u \in C$, let $x = (I + \hat{L})^{-1}(u)$. Then $-x'' + x = u$ and from what we proved we have

$$\|x\|_1 \leq \| - x'' + x\|_1 \leq \sup[\|u\|_1 : u \in C] = |C| < \infty.$$

So $\|x''\|_1 \leq 2|C|$ and thus we conclude that $(I + \hat{L})^{-1}(C)$ is bounded in $W^{2,1}(T)$. Since the latter embeds compactly in $W^{1,1}(T)$, we infer that $(I + \hat{L})^{-1}(C)$ is relatively compact in $W^{1,1}(T)$. Moreover, if $u_n \to u$ in $L^1(T)$ as $n \to \infty$ and $x_n = (I + \hat{L})^{-1}(u_n)$, then

$x_n \to x = (I+\hat{L})^{-1}(u)$ in $L^1(T)$ as $n \to \infty$ (recall that $(I+\hat{L})^{-1}$ is continuous on $L^1(T)$) and $\{x_n\}_{n\geq 1}$ is bounded in $W^{2,1}(T)$. Exploiting the compact embedding of $W^{2,1}(T)$ in $W^{1,1}(T)$, we have that $x_n \to x$ in $W^{1,1}(T)$, i.e. $(I+\hat{L})^{-1} : L^1(T) \to D \subseteq W^{1,1}(T)$ is continuous, hence a compact operator.

Now let $H : W^{1,1}(T) \to L^1(T)$ be defined by

$$H(x)(\cdot) = f(\cdot, \tau(x)(\cdot), q_N(\tau(x)'(\cdot))) - u(\cdot, x(\cdot)) + x(\cdot).$$

We will show that $H(\cdot)$ is bounded and continuous. Boundedness is a straightforward consequence of hypothesis $\mathrm{H(f)}_1$ (iii) and of the definition of the penalty function $u(t,x)$. So we need to show that $H(\cdot)$ is continuous. To this end let $x_n \to x$ in $W^{1,1}(T)$ as $n \to \infty$. By passing to a subsequence if necessary, we may assume that $x_n(t) \to x(t)$ and $x_n'(t) \to x'(t)$ a.e. on T as $n \to \infty$. Hence we have $\tau(x_n)(t) \to \tau(x)(t)$ for every $t \in T$ and $q_N(\tau(x_n)'(t)) \to q_N(\tau(x)'(t))$ a.e. on T as $n \to \infty$. Note that $\{x_n\}_{n\geq 1}$ is bounded in $C(T)$ (since $W^{1,1}(T)$ embeds continuously in $C(T)$) and so by virtue of hypotheses $H(f)_1$, the continuity of $u(t,\cdot)$ and the dominated convergence theorem, we have that $H(x_n) \to H(x)$ in $L^1(T)$ as $n \to \infty$ and so we have proved the continuity of $H : W^{1,1}(T) \to L^1(T)$.

Then consider the operator $(I+\hat{L})^{-1}H : W^{1,1}(T) \to W^{1,1}(T)$. Evidently this operator is continuous (in fact compact), $(I+\hat{L})^{-1}H(D) \subseteq D$ and $\overline{(I+\hat{L})^{-1}H(D)}$ is compact in $W^{1,1}(T)$ (since for every $x \in W^{1,1}(T)$, $||H(x)||_1 \leq k^*$ with $k^* = ||\gamma_r||_1 + b\max\{||\phi||_\infty, ||\psi||_\infty\}$ and $r = \max\{||\phi||_\infty, ||\psi||_\infty, N\}$). Since $D \subseteq W^{1,1}(T)$ is closed, convex, we can apply Schauder's fixed point theorem (see Gilbarg–Trudinger [8], Corollary 10.2, p. 222), to obtain $x = (I+\hat{L})^{-1}H(x)$. Then $-x'' + x = H(x)$, $x \in D$; i.e. $x \in W^{2,1}(T)$ is a solution of (3). This proves the nonemptiness of S.

To conclude the proof of the theorem, note that if $x \in S$, then from claim # 1 we have $\psi(t) \leq x(t) \leq \phi(t)$ for all $t \in T$. So we have $\tau(x)(t) = x(t)$, $\tau(x)'(t) = x'(t)$ and $u(t, x(t)) = 0$. Also recalling that $|x'(t)| \leq N$ for all $t \in T$, we also have that $q_N(x'(t)) = x'(t)$. Therefore finally

$$\left\{ \begin{array}{ll} -x''(t) = f(t, x(t), x'(t)) & \text{a.e. on } T \\ (B_0 x)(0) = \nu_0, \ (B_1 x)(b) = \nu_1 & \end{array} \right\}$$

i.e., $x \in W^{2,1}(T)$ solves problem (1) and $x \in [\psi, \phi]$.

Now we will improve the conclusion of theorem 1, by showing that problem (1) has extremal solutions in the order interval $K = [\psi, \phi]$; i.e. there exist a least solution $x_* \in K$ and a greatest solution $x^* \in K$ of (1), such that if $x \in W^{2,1}(T)$ is any other solution of (1) in K, we have $x_*(t) \leq x(t) \leq x^*(t)$ for all $t \in T$.

Theorem 2. *If hypotheses* $\mathrm{H(f)}_1$ *and* H_0 *hold, then problem* (1) *has extremal solutions in the order interval* $K = [\psi, \phi]$.

Proof. Let S_1 be the set of solutions of (1) contained in the order interval $K = [\psi, \phi]$. From theorem 1 we have that $S_1 \neq \phi$. First we will show that S_1 is a directed set (i.e. if $x_1, x_2 \in S_1$, then there exists $x \in S_1$ such that $x_1(t) \leq x(t)$ and $x_2(t) \leq x(t)$ for all $t \in T$). To this end let $x_1, x_2 \in S_1$ and let $x_3 = \max\{x_1, x_2\}$. Since $x_1, x_2 \in W^{2,1}(T)$, we have that $x_3 \in W^{1,1}(T)$ (see Gilbarg–Trudinger [8], Lemma 7.6, p. 145). Let $\tau_k : W^{1,1}(T) \to W^{1,1}(T)$ be defined by

$$\tau_k(x)(t) = \left\{ \begin{array}{lll} \phi(t) & \text{if} & \phi(t) \leq x(t) \\ x(t) & \text{if} & x_k(t) \leq x(t) \leq \phi(t) \\ x_k(t) & \text{if} & x(t) \leq x_k(t) \end{array} \right. \qquad k = 1, 2, 3\,.$$

Also we introduce the penalty function $u_3 : T \times \mathbb{R} \to \mathbb{R}$ and the truncation function $q_N : \mathbb{R} \to \mathbb{R}$ ($N = 1 + \max\{N_1, ||\psi'||_\infty, ||\phi'||_\infty\}$) defined by

$$u_3(t,x) = \begin{cases} x - \phi(t) & \text{if} \quad \phi(t) \leq x \\ 0 & \text{if} \quad x_3(t) \leq x \leq \phi(t) \\ x - x_3(t) & \text{if} \quad x \leq x_3(t) \end{cases}$$

and

$$q_N(x) = \begin{cases} N & \text{if} \quad N \leq x \\ x & \text{if} \quad -N \leq x \leq N\,. \\ -N & \text{if} \quad x \leq -N \end{cases}$$

Then we consider the following boundary value problem:

$$\left\{ \begin{array}{l} -x''(t) = f(t, \tau_3(x)(t), q_N(\tau_3(x)'(t))) + \sum\limits_{k=1}^{2} |f(t, \tau_k(x)(t), q_N(\tau_k(x)'(t))) \\ \qquad\qquad -f(t, \tau_3(x)(t), q_N(\tau_3(x)'(t)))| - u_3(t, x(t)) \qquad \text{a.e. on } T \\ (B_0x)(0) = \nu_0, (B_1x)(b) = \nu_1 \end{array} \right\}. \tag{14}$$

Arguing as in the proof of theorem 1, we establish that problem (14) has a nonempty solution set. We will show that this solution set is in the order interval $[x_3, \phi]$. So let $x \in W^{2,1}(T)$ be a solution of (14). We have

$$\begin{aligned} x_1''(t) - x''(t) &= f(t, \tau_3(x)(t), q_N(\tau_3(x)'(t))) - f(t, x_1(t), x_1'(t)) \\ &\quad + \sum_{k=1}^{2} |f(t, \tau_k(x)(t), q_N(\tau_k(x)'(t))) - f(t, \tau_3(x)(t), q_N(\tau_3(x)'(t)))| \\ &\quad - u_3(t, x(t)) \quad \text{a.e on } T. \end{aligned}$$

Multiply with $(x_1 - x)_+(t)$ and then integrate over $T = [0, b]$. Using the definition of the truncation functions $r_k (k = 1, 2, 3), q_N$ and boundary conditions, we obtain

$$\begin{aligned} &\int_0^b u_3(t, x(t))(x_1 - x)_+(t) \mathrm{d}t \geq 0 \\ &\Rightarrow \int_0^b (x_1 - x)_+^2(t) \mathrm{d}t = 0 \qquad \text{(recall the definition of } u_3\text{)} \\ &\Rightarrow x_1(t) \leq x(t) \quad \text{for all } t \in T. \qquad \text{a.e. on } I. \end{aligned}$$

In a similar way we show that $x_2(t) \leq x(t)$ and $x(t) \leq \phi(t)$ for all $t \in T$. Therefore we conclude that every solution $x(\cdot) \in W^{2,1}(T)$ of (14) is located in the order interval $[x_3, \phi]$. Hence $\tau_k(x)(t) = x(t)$ and $\tau_k(x)'(t) = x'(t)$ for all $t \in T$ and all $k \in \{1, 2, 3\}$ and $u_3(t, x(t)) = 0$. Thus

$$\left\{ \begin{array}{ll} -x''(t) = f(t, x(t), q_N(x'(t))) & \text{a.e. on } T \\ (B_0x)(0) = \nu_0, (B_1x)(b) = \nu_1 & \end{array} \right\}.$$

As we already mentioned the Nagumo growth condition (see (H_0)) guarantees that $|x'(t)| \leq N$ for all $t \in T$ and so $q_N(x'(t)) = x'(t)$. Therefore $x \in S_1$ and we have proved that S_1 is a directed set.

Now let C be a chain in S_1. Then since $C \subseteq L^1(T)$, according to Dunford–Schwartz [5] (Corollary IV.II.7, p. 336), we can find $\{x_n\}_{n\geq 1} \subseteq C$ such that $\sup C = \sup_{n\geq 1} x_n$. Then by the monotone convergence theorem, we have that $x_n \to x$ in $L^1(T)$ as $n \to \infty$ and so $\psi(t) \leq x(t) \leq \phi(t)$ a.e. on T. For every $n \geq 1$ we know that $||x_n||_\infty \leq \max\{||\psi||_\infty, ||\phi||_\infty\} = r_0$ and $\sup_{n\geq 1} ||x_n'||_\infty \leq N_1$. So if $r = \max\{r_0, N_1\}$, by virtue of hypothesis $\mathrm{H(f)}_1$ (iv) we have that $||x_n''(t)|| \leq \gamma_r(t)$ a.e. on T. Thus $\{x_n\}_{n\geq 1}$ is bounded in $W^{2,1}(T)$ and $\{x_n''\}_{n\geq 1}$ is uniformly integrable. So as before exploiting the compact embedding of $W^{2,1}(T)$ in $W^{1,1}(T)$, the continuous embedding of $W^{2,1}(T)$ in $C^1(T)$ and invoking the Dunford–Pettis theorem, we may assume that $x_n \to x$ in $W^{1,1}(T)$, $x_n(t) \to x(t)$, $x_n'(t) \to x'(t)$ for all $t \in T$ and $x_n'' \xrightarrow{w} y$ in $L^1(T)$ as $n \to \infty$. It is easy to see that $y = x''$ and $(B_0x)(0) = \nu_0$, $(B_1x)(b) = \nu_1$. Also from the dominated convergence theorem, we have that $-x''(\cdot) = f(\cdot, x(\cdot), x'(\cdot))$ in $L^1(T)$. Hence $-x''(t) = f(t, x(t), x'(t))$ a.e. on T, $(B_0x)(0) = \nu_0$, $(B_1x)(b) = \nu_1$. Thus $x = \sup C \in S_1$. Using Zorn's lemma, we infer that S_1 has a maximal element $x^* \in S_1$. Since S_1 is directed, it follows that x^* is unique and is the greatest element of S_1 in $[\psi, \phi]$. Similarly we can prove the existence of a least solution x_* of (1) in $[\psi, \phi]$. Therefore (1) has extremal solutions in $K = [\psi, \phi]$.

3. Periodic problems

In this section, we focus our attention on the 'periodic problem':

$$\left\{\begin{array}{ll} -x''(t) = f(t, x(t), x'(t)) & \text{a.e. on } T \\ x(0) = x(b),\ x'(0) = x'(b) & \end{array}\right\}. \tag{15}$$

This problem was studied using the method of upper and lower solutions by Gaines–Mawhin [6], Leela [14], Lakshmikantham–Leela [13], Nieto [16], Cabada–Nieto [4], Omari–Trombetta [20] and Gao–Wang [7]. From these works only Gaines–Mawhin, Cabada–Nieto, Omari–Trombetta and Gao–Wang had a vector field depending also on x' and moreover, among these papers only Cabada–Nieto and Gao–Wang used Caratheodory type conditions on $f(t, x, y)$ with Lipschitz continuity in the y-variable in Cabada–Nieto (see Theorem 2.2 in Cabada–Nieto [4]). Theorem 3 below extends all these results. A similar result using a different method of proof, was obtained by Gao–Wang [7].

DEFINITION

A function $\psi \in W^{2,1}(T)$ is said to be a 'lower solution' of (18) if

$$\left\{\begin{array}{ll} -\psi''(t) \leq f(t, \psi(t), \psi'(t)) & \text{a.e. on } T \\ \psi(0) = \psi(b),\ \psi'(0) \geq \psi'(b) & \end{array}\right\}.$$

A function $\phi \in W^{2,1}(T)$ is said to be an 'upper solution' of (18) if it satisfies the reverse inequalities.

Theorem 3. *If hypotheses* $\mathrm{H(f)}_1$ *and* H_0 *hold, then problem* (18) *has a solution* $x \in W^{2,1}(T)$ *within the order interval* $K = [\psi, \phi]$.

Proof. The proof is the same as that of theorem 1, with some minor modifications. Note that in this case $D = \{x \in W^{2,1}(T) : x(0) = x(b), x'(0) = x'(b)\}$ and $\hat{L} : D \subseteq L^1(T) \to L^1(T)$ is defined by $\hat{L}x = -x''$ for all $x \in D$. The rest of the proof is identical and only in the applications of the integration by parts formula (Green's identity), we use the periodic conditions instead of the Sturm–Liouville boundary conditions.

Next we look for the extremal solutions in the order interval $[\psi, \phi]$ of the periodic problem (18). For this we introduce a different set of hypotheses on the vector field $f(t,x,y)$.

$H(f)_2$: $f : T \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a function such that

(i) for every $x, y \in \mathbb{R}$, $t \to f(t,x,y)$ is measurable;
(ii) there exists $M > 0$ such that for almost all $t \in T$ and all $y \in [-N,N]$, $x \to f(t,x,y) + Mx$ is strictly increasing (recall that $N = 1 + \max\{N_1, \|\psi'\|_\infty, \|\phi'\|_\infty\}$);
(iii) there exists $k \in L^1(T)$ such that $|f(t,x,y_1) - f(t,x,y_2)| \leq k(t)|y_1 - y_2|$ a.e. on T for all $x, y_1, y_2 \in \mathbb{R}$;
(iv) for every $r > 0$, there exists $\gamma_r \in L^1(T)$ such that $|f(t,x,y)| \leq \gamma_r(t)$ a.e. on T for all $x, y \in \mathbb{R}$, $|x|$, $|y| \leq r$.

Remark. Hypothesis $H(f)_2$ (ii) allows for jump discontinuities (countably many) in the x–variable. However note that for every $x : T \to \mathbb{R}$ measurable, $t \to f(t,x(t),y)$ is measurable. This is an immediate consequence of Theorem 1.9, p. 32 of Appell–Zabrejko [1]. Moreover since $(t,y) \to f(t,x(t),y)$ is a Caratheodory function, is jointly measurable and so in particular superpositionally measurable; if $y : T \to \mathbb{R}$ is measurable, then so is $t \to f(t,x(t),y(t))$.

Theorem 4. *If hypotheses* $H(f)_2$ *and* H_0 *hold, then problem* (18) *has extremal solutions in the order interval* $K = [\psi, \phi]$.

Proof. Without any loss of generality, we may assume that $M > 1$. Then for any $z \in K = [\psi, \phi]$, we consider the following periodic problem

$$\left\{ \begin{array}{ll} -x''(t) = f(t,z(t),q_N(\tau(x)'(t))) - u(t,x(t)) + M(z(t) - x(t)) & \text{a.e. on } T \\ x(0) = x(b),\ x'(0) = x'(b) & \end{array} \right\}. \tag{16}$$

We will establish the existence of solutions for problem (18). So let $D = \{x \in W^{2,1}(T) : x(0) = x(b), x'(0) = x'(b)\}$ and let $L : D \subseteq L^1(T) \to L^1(T)$ be defined by $Lx = -x'' + (M-1)x$. As in the proof of theorem 1, we can check that L is invertible and $L^{-1} : L^1(T) \to D \subseteq W^{1,1}(T)$ is a compact, linear operator. Also as before we define $H : W^{1,1}(T) \to L^1(T)$ by

$$H(x)(t) = f(t,z(t),q_N(\tau(x)'(t))) - u(t,x(t)) - x(t) + Mx(t).$$

This map is bounded and continuous. Note that $x \in D$ solves (19) if and only if $x = L^{-1}H(x)$. As in the proof of theorem 1, the existence of a fixed point of $L^{-1}H$ is implied by corollary 10.2, p. 222 of Gilbarg–Trudinger [8], since $L^{-1}(D) \subseteq D$ and $L^{-1}H(D)$ is compact in $W^{1,1}(T)$. So problem (19) has solutions.

Now we will show that any solution of (19) is within $K = [\psi, \phi]$. Indeed we have:

$$\left\{ \begin{array}{l} \psi''(t) - x''(t) \geq f(t,z(t),q_N(\tau(x)'(t))) - f(t,\psi(t),\psi'(t)) \\ \qquad -u(t,x(t)) + M(z(t) - x(t)) \quad \text{a.e. on } T \\ (\psi - x)(0) = (\psi - x)(b),\ (\psi - x)'(0) \geq (\psi - x)'(b) \end{array} \right\}.$$

Multiplying the above inequality with $(\psi - x)_+(t)$ and integrating over $T = [0, b]$ as in the proof of theorem 2, using the definitions of r, q_N and the boundary conditions for ψ and x, we obtain that

$$\begin{aligned}0 &\leq \int_0^b u(t, x(t))(\psi - x)_+(t)\mathrm{d}t = \int_0^b (x(t)\\&\quad - \psi(t))(\psi - x)_+(t)\mathrm{d}t = -\int_0^b [(\psi - x)_+(t)]^2\mathrm{d}t\\&\Rightarrow 0 = \int_0^b [(\psi - x)_+(t)]^2\mathrm{d}t\\&\Rightarrow \psi(t) \leq x(t) \qquad \text{for all } t \in T.\end{aligned}$$

In a similar fashion, we show that $x(t) \leq \phi(t)$ for all $t \in T$. Therefore every solution $x \in W^{2,1}(T)$ of (19) is located in $K = [\psi, \phi]$. Thus recalling the definitions of $\tau(x), q_N$ and u, we see that $-x''(t) = f(t, z(t), x'(t)) + M(x(t) - x(t))$ a.e. on T, $x(0) = x(b)$, $x'(0) = x'(b)$. Now we will show that this solution is unique. To this end, on $L^1(T)$ we consider an equivalent norm $|\cdot|_1$ given by

$$|x|_1 = \int_0^b \exp\left(-\lambda \int_0^t k(s)\mathrm{d}s\right)|x(t)|\mathrm{d}t, \quad \lambda > 0.$$

Similarly on $W^{2,1}(T)$ we consider the equivalent norm given by

$$|x|_{2,1} = |x|_1 + |x'|_1 + |x''|_1.$$

Suppose that $x_1, x_2 \in W^{2,1}(T)$ are two solutions of (19). Then

$$x_1 = L_M^{-1}H_0(x_1) \quad \text{and} \quad x_2 = L_M^{-1}H_0(x_1),$$

where $L_M^{-1} = (MI + \hat{L})^{-1}$ with $\hat{L}x = -x''$ for all $x \in D = \{x \in W^{2,1}(T) : x(0) = x(b), x'(0) = x'(b)\}$ and $H_0(x)(\cdot) = f(\cdot, z(\cdot), q_N(\tau(x)'(\cdot)))$. Recall that $L_M^{-1} : L^1(T) \to D \subseteq W^{1,1}(T)$ is linear compact. So $L_M^{-1} : (L^1(T), |\cdot|_1) \to (W^{2,1}(T), |\cdot|_{2,1})$ is linear continuous. Moreover, using hypotheses H(f)$_2$ we can easily check as before that $H_0 : (W^{2,1}(T), |\cdot|_{2,1}) \to (L^1(T), |\cdot|_1)$ is continuous. Then we have

$$\begin{aligned}|x_1 - x_2|_{2,1} &\leq \|L_M^{-1}\|_{\mathcal{L}}|H_0(x_1) - H_0(x_2)|_1\\&= \|L_M^{-1}\|_{\mathcal{L}} \int_0^b \exp\left(-\lambda \int_0^t k(s)\mathrm{d}s\right)|H_0(x_1)(t) - H_0(x_2)(t)|\mathrm{d}t\\&\leq \|L_M^{-1}\|_{\mathcal{L}} \int_0^b \exp\left(-\lambda \int_0^t k(s)\mathrm{d}s\right)k(t)|x_1'(t) - x_2'(t)|\mathrm{d}t\\&= -\frac{1}{\lambda}\|L_M^{-1}\|_{\mathcal{L}} \int_0^b |x_1'(t) - x_2'(t)|d\left(\exp\left(-\lambda \int_0^t k(s)\mathrm{d}s\right)\right)\\&\leq \frac{1}{\lambda}\|L_M^{-1}\|_{\mathcal{L}} \int_0^b \exp\left(-\lambda \int_0^t k(s)\mathrm{d}s\right)|x_1''(t) - x_2''(t)|\mathrm{d}t\\&= \frac{1}{\lambda}\|L_M^{-1}\|_{\mathcal{L}}|x_1'' - x_2''|_1.\end{aligned}$$

So if $\lambda > \|L_M^{-1}\|_{\mathcal{L}}$, we infer that $x_1''(t) = x_2''(t)$ a.e. on T. Hence $x_1'(t) - x_2'(t) = c_1$ for all $t \in T$, with $c_1 \in \mathbb{R}$. Since $x_1'(0) = x_1'(b)$ and $x_2'(0) = x_2'(b)$, from the mean value theorem, we deduce that there exists $\xi \in (0,b)$ such that $x_1'(\xi) = x_2'(\xi)$. Therefore $c_1 = 0$ and so $x_1'(t) = x_2'(t)$ for all $t \in T$, which implies that $x_1(t) - x_2(t) = c_2$ for all $t \in T$, with $c_2 \in \mathbb{R}$. But for almost all $t \in T$, we have

$$\begin{aligned} f(t,z(t),q_N(x_1'(t))) + M(z(t)-x_1(t)) &= f(t,z(t),q_N(x_2'(t))) \\ &\quad + M(z(t)-x_2(t)) \\ \Rightarrow x_1(t) = x_1(t) + c_2 \quad \text{a.e. on } T; \ \text{i.e. } c_2 = 0 \text{ and so } \ x_1 = x_2. \end{aligned}$$

Then define $R : [\psi,\phi] \to [\psi,\phi]$ where $R(z)(\cdot)$ is the unique solution of (19). We claim that $R(\cdot)$ is increasing. Indeed let $z_1, z_2 \in [\psi,\phi]$, $z_1 \le z_2$, $z_1 \ne z_2$ and set $x_1 = R(z_1)$, $x_2 = R(z_2)$. We have

$$-x_1''(t) = f(t,z_1(t),q_N(x_1'(t))) + M(z_1(t)-x_1(t)) \quad \text{a.e. on } T$$

and

$$-x_2''(t) = f(t,z_2(t),q_N(x_2'(t))) + M(z_2(t)-x_2(t)) \quad \text{a.e. on } T.$$

Suppose that $\max_{t\in T}[x_1(t)-x_2(t)] = \varepsilon > 0$ and suppose that this maximum is attained at $t_0 \in T$. First we assume that $0 < t_0 < b$. Then we have $x_1'(t_0) = x_2'(t_0) = \nu_0$ and we can find $\delta > 0$ such that for every $t \in T_\delta = [t_0, t_0+\delta]$ we have $x_2(t) < x_1(t)$. So we obtain

$$\begin{aligned} -x_1''(t) &= f(t,z_1(t),q_N(x_1'(t))) + M(z_1(t)-x_1(t)) \\ &< f(t,z_2(t),q_N(x_1'(t))) + M(z_2(t)-x_2(t)) \\ &= f(t,z_2(t),q_N(x_1'(t))) + Mw(t) \\ &\qquad \text{a.e. on } T_\delta, \ \text{with } w(t) = z_2(t) - x_2(t), \end{aligned}$$

and

$$-x_2''(t) = f(t,z_2(t),q_N(x_2'(t))) + Mw(t) \quad \text{a.e. on } T_\delta.$$

Since $x_1'(t_0) = x_2'(t_0) = \nu_0$, from a well–known differential inequality (see for example Hale [9], theorem 6.1, p. 31), we obtain that $0 \le x_1'(t) - x_2'(t)$ for all $t \in T_\delta$. So after integration we see that $x_1(t_0) - x_2(t_0) \le x_1(t) - x_2(t)$ for every $t \in T_\delta$. Since $t_0 \in T$ is the point at which $(x_1 - x_2)(\cdot)$ attains its maximum on T, we have that $x_1(t) = x_2(t) + \varepsilon$ for every $t \in T_\delta$ and so $x_1'(t) = x_2'(t)$ for every $t \in T_\delta$. Thus we have

$$\begin{aligned} 0 = x_1''(t) - x_2''(t) &\ge f(t,z_2(t),q_N(x_2'(t))) - f(t,z_2(t),q_N(x_1'(t))) \\ &+ M(x_1(t)-x_2(t)) > 0 \quad \text{a.e. on } T_\delta, \end{aligned}$$

a contradiction.

Next assume $t_0 = 0$. Then $\varepsilon = x_1(0) - x_2(0) \ge x_1(h) - x_2(h)$ for all $h \in [0,\delta]$ and $\varepsilon = x_1(b) - x_2(b) \ge x_1(h) - x_2(h)$ for all $h \in [b-\delta, b]$. From the first inequality we infer that $(x_1-x_2)'(0) \le 0$ while from the second we have $(x_1-x_2)'(b) \ge 0$ and so $(x_1-x_2)'(0) \ge 0$. Therefore $x_1'(0) = x_2'(0) = \nu_0$ and so we can proceed as in the previous case and derive a contradiction. Similarly we treat the case $t_0 = b$. Therefore $x_1 \le x_2$ and so $R(\cdot)$ is increasing as claimed.

Now let $\{y_n\}_{n\ge1}$ be an increasing sequence in $[\psi,\phi]$. Set $x_n = R(y_n)$, $n \ge 1$. The sequence $\{x_n\}_{n\ge1} \subseteq [\psi,\phi]$ is increasing. From the monotone convergence theorem, we have that $y_n \to y$ and $x_n \to x$ in $L^1(T)$ as $n \to \infty$. Also by hypothesis $H(f)_2$ (iii), $|x_n''(t)| \le \gamma_r(t)$ a.e. on T with $r = \max\{N, \|\phi\|_\infty, \|\psi\|_\infty\}$, with $\gamma_r \in L^1(T)$. So $\{x_n\}_{n\ge1}$ is bounded in $W^{2,1}(T)$ and $\{x_n''\}_{n\ge1}$ is uniformly integrable. From the compact embedding of

$W^{2,1}(T)$ in $W^{1,1}(T)$ and the Dunford–Pettis theorem, we have that $x_n \to x$ in $W^{1,1}(T)$ and at least for a subsequence we have $x_n'' \xrightarrow{w} g$ in $L^1(T)$ as $n \to \infty$. Clearly $x'' = g$ and so for the original sequence we have $x_n'' \xrightarrow{w} x''$ in $L^1(T)$ as $n \to \infty$. So finally $x_n \xrightarrow{w} x$ in $W^{2,1}(T)$. Invoking theorem 3.1 of Heikkila–Lakshmikantham–Sun [10], we deduce that $R(\cdot)$ has extremal fixed points in $K = [\psi, \phi]$. But note these extremal fixed points of $R(\cdot)$, are the extremal solutions in $K = [\psi, \phi]$ of the periodic problem (19).

Next we consider the situation where the vector field f is independent of x'. This is the case studied by Nieto [16]. However here we are more general than Nieto, since the dependence of f on x can be splitted into a continuous and a discontinuous part. So we will be studying the following periodic problem:

$$\left\{\begin{array}{l} -x''(t) = f(t,x(t),x(t)) \ \text{ a.e. on } T \\ x(0) = x(b), \quad x'(0) = x'(b) \end{array}\right\}. \tag{17}$$

H_0': There exist $\psi \in W^{2,1}(T)$ a lower solution and $\phi \in W^{2,1}(T)$ an upper solution such that $\psi(t) \le \phi(t)$ for all $t \in T$.

$H(f)_3$: $f : T \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a function such that

(i) for every $y \in W^{2,1}(T)$ and every $x \in \mathbb{R}$, $t \to f(t,x,y(t))$ is measurable;
(ii) for almost all $t \in T$ and all $y \in \mathbb{R}$, $x \to f(t,x,y)$ is continuous;
(iii) there exists $M \in L^1(T)_+$ such that for almost all $t \in T$ and all $x \in [\psi(t), \phi(t)]$, $y \to f(t,x,y) + M(t)y$ is increasing;
(iv) for every $r > 0$ there exists $\gamma_r \in L^1(T)$ such that if $|f(t,x,y)| \le \gamma_r(t)$ a.e. on T for all $x, y \in \mathbb{R}$ with $|x|, |y| \le r$.

Remark. The superpositional measurability hypothesis $H(f)_3$ (i) is satisfied, if for every $x \in \mathbb{R}$, there exists $g_x : T \times \mathbb{R} \to \mathbb{R}$ a Borel measurable function such that $g_x(t,y) = f(t,x,y)$ for almost all $t \in T$ and all $y \in \mathbb{R}$. This follows from the monotonicity hypothesis $H(f)_3$ (iii) and theorem 1.9 of Appell–Zabrejko [1].

Theorem 5. *If hypotheses* H_0' *and* $H(f)_3$ *hold, then problem* (23) *has a solution* $x \in W^{2,1}(T)$ *in the order interval* $K = [\psi, \phi]$.

Proof. Let $y \in K = [\psi, \phi] = \{y \in W^{2,1}(T) : \psi(t) \le y(t) \le \phi(t)$ for all $t \in T\}$ and consider the following periodic problem

$$\left\{\begin{array}{l} -x''(t) = f(t,x(t),x(t)) + M(t)(y(t) - x(t)) \ \text{ a.e. on } T \\ x(0) = x(b), \ x'(0) = x'(b) \end{array}\right\}. \tag{18}$$

Problem (24) has at least one solution in K (see Nieto [16]). By $S(y)$ we denote the solutions of (24) in K. Let $y_1, y_2 \in K$, $y_1 \le y_2$, $x_1 \in S(y_1)$ and $y_1 \le x_1$. Consider the following problem:

$$\left\{\begin{array}{l} -x''(t) = f(t,\tau_1(t,x(t)),y_2(t)) + M(t)(y_2(t) - \tau_1(t,x(t))) \\ \qquad\qquad -u_1(t,x(t)) \ \text{ a.e. on } T \\ x(0) = x(b), \ x'(0) = x'(b) \end{array}\right\}. \tag{19}$$

Here $\tau_1 : T \times \mathbb{R} \to \mathbb{R}$ (the truncation function) is defined by

$$\tau_1(t,x) = \left\{\begin{array}{lll} \phi(t) & \text{if} & \phi(t) \le x \\ x & \text{if} & x_1(t) \le x \le \phi(t) \\ x_1(t) & \text{if} & x \le x_1(t) \end{array}\right.$$

and $u_1 : T \times \mathbb{R} \to \mathbb{R}$ (the penalty function) is defined by

$$u_1(t,x) = \begin{cases} x - \phi(t) & \text{if} \quad \phi(t) \le x \\ 0 & \text{if} \quad x_1(t) \le x \le \phi(t) \\ x - x_1(t) & \text{if} \quad x \le x_1(t) \end{cases}.$$

Both are Caratheodory functions. As before we let $D = \{x \in W^{2,1}(T) : x(0) = x(b),$ $x'(0) = x'(b)\}$ and define $L : D \subseteq L^1(T) \to L^1(T)$ by $Lx = -x''$ for all $x \in D$. Again we can check that $\hat{L} = (I + L)$ is invertible and $\hat{L}^{-1} : L^1(T) \to D \subseteq W^{1,1}(T)$ is compact. Also $H : W^{1,1}(T) \to L^1(T)$ is given by

$$H(x)(\cdot) = f(\cdot, \tau_1(\cdot, x(\cdot)), y_2(\cdot)) + M(\cdot)(y_2(\cdot) - \tau_1(\cdot, x(\cdot))) - u_1(\cdot, x(\cdot)) + x(\cdot)$$

This map is continuous and there exists $k^* > 0$ such that $||H(x)||_1 \le k^*$ for all $x \in W^{1,1}$ (T). So $\hat{L}^{-1}H(D)$ is relatively compact in $W^{1,1}(T)$ and thus we can apply corollary 10.2, p. 222, of Gilbarg–Trudinger [8] and obtain $x \in D$ such that $x = \hat{L}^{-1}H(x)$. Therefore problem (25) has a solution.

Note that by virtue of hypothesis H(f)$_3$ (iii) and the fact that $\tau_1(t, x_1(t)) = x_1(t)$ and $u_1(t, x_1(t)) = 0$, we have

$$\left\{ \begin{array}{l} -x_1''(t) = f(t, x_1(t), y_1(t)) + M(y_1(t) - x_1(t)) \le f(t, x_1(t), y_2(t)) \\ \qquad\qquad\qquad + M(y_2(t) - x_1(t)) \quad \text{a.e. on } T \\ x_1(0) = x_1(b), \ x_1'(0) = x_1'(b) \end{array} \right\}.$$

So $x_1 \in W^{2,1}(T)$ is a lower solution of (25). Similarly since $y_2 \le \phi$, we have

$$\left\{ \begin{array}{l} -\phi''(t) \ge f(t, \phi(t), \phi(t)) \ge f(t, \phi(t), y_2(t)) + M(y_2(t) - \phi(t)) \quad \text{a.e. on } T \\ \phi(0) = \phi(b), \ \phi_1'(0) \le \phi'(b) \end{array} \right\}$$

and so we see that $\phi \in W^{2,1}(T)$ is an upper solution of (25).

Now we will show that the solutions of (25) are within the order interval $K_1 = [x_1, \phi]$ Indeed we have

$$\begin{aligned} x_1''(t) - x''(t) = & f(t, \tau_1(t, x(t)), y_2(t)) + M(t)y_2(t) - f(t, x_1(t), y_1(t)) \\ & - M(t)y_1(t) + M(t)(x_1(t) - \tau_1(t, x(t))) - u_1(t, x(t)) \\ & \text{a.e. on } T. \end{aligned}$$

Multiply the above equation with $(x_1 - x)_+(\cdot)$ and then integrate over $T = [0, b]$. As in previous proofs we obtain

$$\int_0^b u_1(t, x(t))(x_1 - x)_+(t)\mathrm{d}t \ge 0$$

$$\Rightarrow -\int_0^b [(x_1 - x)_+(t)]^2 \mathrm{d}t \ge 0; \quad \text{i.e. } x_1(t) \le x(t) \quad \text{for all } t \in T.$$

Similarly we show that $x(t) \le \phi(t)$ for all $t \in T$. Therefore every solution of (25) is in the order interval $K_1 = [x_1, \phi]$. Because of this fact, equation (25) becomes

$$\left\{ \begin{array}{l} -x''(t) = f(t, x(t), y_2(t)) + M(t)(y_2(t) - x(t)) \quad \text{a.e. on } T \\ x(0) = x(b), \ x'(0) = x'(b) \end{array} \right\}$$

and so $x \in S(y_2)$ and $x_1 \le x$.

Next we will show that for every $y \in K = [\psi, \phi]$, the set $S(y)$ is compact in $L^1(T)$. To this end let $x \in S(y)$. Then $||x||_\infty \leq \max\{||\phi||_\infty, ||\psi||_\infty\} = r$. Hence $||x''(t)|| \leq \gamma_r(t) + 2M(t)r$ a.e. on T. Hence $S(y)$ is bounded in $W^{2,1}(T)$ and since the latter embeds compactly in $L^1(T)$, we have that $S(y)$ is relatively compact in $L^1(T)$. Then let $\{x_n\}_{n\geq 1} \subseteq S(y)$ and assume that $x_n \to x$ in $L^1(T)$ as $n \geq \infty$. Since $\{x''_n\}_{n\geq 1}$ is uniformly integrable, by passing to a subsequence if necessary we may assume that $x''_n \xrightarrow{w} g$ in $L^1(T)$ as $n \to \infty$. Because $W^{2,1}(T)$ embeds continuously in $C^1(T)$, $\{x'_n\}_{n\geq 1}$ is bounded in $C(T)$ and for all $0 \leq s \leq t \leq b$ and all $n \geq 1$, $|x'_n(t) - x'_n(s)| \leq \int_s^t (\gamma_r(\tau) + 2M(\tau)r)\mathrm{d}\tau$ from which it follows that $\{x'_n\}_{n\geq 1}$ is equicontinuous. So by the Arzela–Ascoli theorem we have that $x'_n \to x'$ in $C(T)$ as $n \to \infty$ and so $g = x''$. Then via the dominated convergence theorem, as before, we can check that

$$\left\{\begin{array}{ll} -x''(t) = f(t, x(t), y(t)) + M(t)(y(t) - x(t)) & \text{a.e. on } T \\ x(0) = x(b),\ x'(0) = x'(b) & \end{array}\right\}.$$

Hence $x \in S(y)$ and this proves that $S(y)$ is closed, hence compact in $L^1(T)$. Since the positive cone $L^1(T)_+ = \{x \in L^1(T) : x(t) \geq 0 \text{ a.e. on } T\}$ is regular (in fact fully regular; see Krasnoselskii [12]), from proposition 2 of Heikkila–Hu [11], we infer that $S(\cdot)$ has a fixed point in K; i.e. there exists $x \in K = [\psi, \phi]$ such that $x \in S(x)$. Therefore

$$\left\{\begin{array}{ll} -x''(t) = f(t, x(t), x(t)) & \text{a.e. on } T \\ x(0) = x(b),\ x'(0) = x'(b) & \end{array}\right\}$$

and so problem (23) has a solution in $K = [\psi, \phi]$.

4. Properties of the solutions

For problems linear in x', we can say something about the structure of the solution set of the periodic problem. Our result extends theorem 4.2 of Nieto [16].

The problem under consideration is the following:

$$\left\{\begin{array}{ll} -x''(t) = f(t, x(t)) + Mx'(t) & \text{a.e. on } T \\ x(0) = x(b),\ x'(0) = x'(b). & \end{array}\right\}. \tag{20}$$

Our hypotheses on the vector field $f(t,x)$ are the following:

$\mathrm{H(f)}_4$: $f : T \times \mathbb{R} \to \mathbb{R}$ is a function such that

(i) for every $x \in \mathbb{R}$, $t \to f(t,x)$ is measurable;
(ii) for almost all $t \in T$, $x \to f(t,x)$ is continuous and decreasing;
(iii) for every $r > 0$ there exists $\gamma_r \in L^\infty(T)$ such that $|f(t,x)| \leq \gamma_r(t)$ a.e. on T for all $x \in \mathbb{R}$, $|x| \leq r$.

Remark. Under these hypotheses the Nagumo growth condition is automatically satisfied since for $k = \max\{||\phi||_\infty, ||\psi||_\infty\}$, we have $|f(t,x) + My| \leq \gamma_k(t) + M|y|$ a.e. on T for all $x \in [\psi(t), \phi(t)]$, and so if $h(r) = ||\gamma_k||_\infty + Mr$, we have for all

$$\lambda > 0 \int_\lambda^\infty \frac{r}{h(r)} \mathrm{d}r = \int_\lambda^\infty \frac{r}{||\gamma_k||_\infty + Mr} \mathrm{d}r = +\infty.$$

Theorem 6. *If hypotheses* H'_0 *and* $\mathrm{H(f)}_4$ *hold and* $M > 0$, *then the solution set S of* (30) *in* $K = [\psi, \phi]$ *is nonempty, w-compact and convex in* $W^{2,1}(T)$.

Proof. From theorem 1 we know that $S \neq \phi$. Let $x \in S$ and define $\hat{x}(t) = x(t) - \frac{1}{b}\int_0^b x(t)\mathrm{d}t$ $t \in T$. Let $T_0 = \{x \in \mathbb{R} : \hat{x} + c \in S\}$. Note that $T_0 \neq \phi$, since $c = \frac{1}{b}\int_0^b x(t)\mathrm{d}t \in T_0$. We claim that T_0 is an interval. Indeed let $c_1, c_2 \in T_0$, $c_1 < c_2$ and take $c \in (c_1, c_2)$. Set $y = x + c$. We have

$$\begin{aligned}-y''(t) = -\hat{x}''(t) &= f(t, (\hat{x} + c_1)(t)) + M(\hat{x} + c_1)'(t)\\ &= f(t, (\hat{x} + c_2)(t)) + M(\hat{x} + c_2)'(t) \text{ a.e. on } T.\end{aligned}$$

By hypothesis H(f)$_4$ (ii), we have

$$\begin{aligned}&f(t, (\hat{x} + c_1)(t)) \geq f(t, y(t)) \geq f(t, (\hat{x} + c_2)(t)) \qquad \text{a.e. on } T\\ &\Rightarrow -y''(t) = f(t, y(t)) + My'(t) \qquad \text{a.e. on } T.\end{aligned}$$

Also it is clear that $y(0) = y(b)$ and $y'(0) = y'(b)$. Therefore $y \in S$ and so $c \in T_0$, which proves that T_0 is an interval.

Next we will show that $S = \{\hat{x} + c : c \in T_0\}$. Indeed if ν, $x \in S$, then we have

$$\begin{aligned}&(x''(t) - \nu''(t))(x(t) - \nu(t))\\ &\quad = (f(t, \nu(t)) + M\nu'(t) - f(t, x(t)) - Mx'(t))(x(t) - \nu(t))\\ &\quad = (f(t, \nu(t)) - f(t, x(t)))(x(t) - \nu(t)) + M(\nu'(t) - x'(t))(x(t) - \nu(t))\\ &\quad \geq M(\nu'(t) - x'(t))(x(t) - \nu(t)) \quad \text{a.e. on } T.\end{aligned}$$

Integrating over $T = [0, b]$, we obtain

$$\begin{aligned}&\int_0^b (x''(t) - \nu''(t))(x(t) - \nu(t))\mathrm{d}t = -\int_0^b (x'(t) - \nu'(t))^2\mathrm{d}t\\ &\geq M\int_0^b (\nu'(t) - x'(t))(x(t) - \nu(t))\mathrm{d}t\\ &= -M\int_0^b (x(t) - \nu(t))d(x - \nu)(t) = 0\end{aligned}$$

$\Rightarrow x'(t) = \nu'(t)$ for every $t \in T$
$\Rightarrow (x - \nu)(\cdot) =$ constant.

So indeed $S = \{\hat{x} + c : c \in T_0\}$ and since as we saw earlier T_0 is an interval, we deduce that S is convex.

Finally we will prove that S is w-compact in $W^{2,1}(T)$. To this end, let $y \in S$. Then there exists $k \in T_0$ such that $y = \hat{x} + k$, hence $||y||_{2,1} = ||\hat{x} + k||_{2,1}$. Since $y \in K = [\psi, \phi]$, we have $|k| \leq \max\{||\psi||_\infty + ||x_1||_\infty, ||\phi||_\infty + ||x_1||_\infty\} = \eta$. Therefore $||y||_{2,1} \leq ||\hat{x}||_1 + b|k| + ||\hat{x}'||_1 + ||\hat{x}''||_1 \leq ||\hat{x}||_{2,1} + b\eta$ and so S is bounded in $W^{2,1}(T)$. We will show that S is closed in $W^{2,1}(T)$. Let $\{y_n\}_{n\geq 1} \subseteq S$ and assume that $y_n \to y$ in $W^{2,1}(T)$. We have

$$-y_n''(t) = f(t, y_n(t)) + My_n'(t) \quad \text{a.e. on } T,\ n \geq 1. \tag{21}$$

Since $W^{2,1}(T)$ embeds continuously in $C^1(T)$, by passing to a subsequence if necessary, we may assume that $y_n''(t) \to y''(t)$ a.e. on T, $y_n'(t) \to y'(t)$ and $y_n(t) \to y(t)$ for all $t \in T$. So $f(t, y_n(t)) \to f(t, y(t))$ a.e. on T. Thus passing to the limit as $n \to \infty$ in (31), we obtain

$$\begin{aligned}&-y''(t) = f(t, y(t)) + My'(t) \quad \text{a.e. on } T,\ y(0) = y(b),\ y'(0) = y'(b)\\ &\Rightarrow y \in S.\end{aligned}$$

So S is closed, hence weakly closed since it is convex. To show that S is weakly compact in $W^{2,1}(T)$, we need to show that given $\{x_n\}_{n\geq 1} \subseteq S$, we can find a weakly convergent subsequence. Since $\{x_n\}_{n\geq 1}$ is bounded in $W^{2,1}(T)$ and the latter embeds compactly in $W^{1,1}(T)$, by passing to a subsequence if necessary, we may assume that $x_n \to x$ in $W^{1,1}(T)$ as $n \to \infty$. Also $x_n'' = \hat{x}''$ and so $||x_n''(t)|| = ||\hat{x}''(t)||$ a.e. on T. Therefore by the Dunford–Pettis theorem, we may assume that $x_n'' \xrightarrow{w} g$ in $L^1(T)$ and $g = x''$. So $x \in W^{2,1}(T)$ and $x_n \xrightarrow{w} x$ in $W^{2,1}(T)$. Since S is weakly closed in $W^{2,1}(T)$, $x \in S$ and so S is weakly compact in $W^{2,1}(T)$.

In general if the vector field f is decreasing in the x-variable, then the upper and lower solutions of the problem, as well as the solutions exhibit some interesting properties.

First we consider the general periodic problem (18), with the following hypotheses on the vector field $f(t,x,y)$.

$\mathrm{H(f)}_5$: $f : T \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a function such that

(i) for every $x, y \in \mathbb{R}$, $t \to f(t,x,y)$ is measurable;
(ii) for almost all $t \in T$ and all $y \in \mathbb{R}$, $x \to f(t,x,y)$ is strictly decreasing;
(iii) for all $x, y, y' \in \mathbb{R}$ $|f(t,x,y) - f(t,x,y')| \leq k(t)|y - y'|$ a.e. on T with $k \in L^1(T)$;
(iv) for every $r > 0$ there exists $\gamma_r \in L^1(T)$ such that $|f(t,x,y)| \leq \gamma_r(t)$ a.e. on T for all $x, y \in \mathbb{R}$, $|x|, |y| \leq r$.

PROPOSITION 7

If $\mathrm{H(f)}_5$ *holds,* $\phi \in W^{2,1}(T)$ *is an upper solution and* $\psi \in W^{2,1}(T)$ *a lower solution for problem* (18), *then for all* $t \in T$, $\psi(t) \leq \phi(t)$.

Proof. Suppose not. Let $t_0 \in T$ be such that $\max_{t\in T}(\psi - \phi)(t) = (\psi - \phi)(t_0) = \varepsilon > 0$. First assume that $0 < t_0 < b$. Then $\psi'(t_0) = \phi'(t_0) = \nu_0$ and we can find $\delta > 0$ such that for all $t \in T_\delta = [t_0, t_0 + \delta]$, we have $\phi(t) < \psi(t)$. Then we have

$$-\psi''(t) \leq f(t, \psi(t), \psi'(t)) < f(t, \phi(t), \psi'(t)) \quad \text{a.e. on } T_\delta$$
$$\text{and} - \phi''(t) \geq f(t, \phi(t), \phi'(t)) \quad \text{a.e. on } T.$$

Consider the following initial value problem

$$\left\{ \begin{array}{ll} -y'(t) = f(t, \phi(t), y(t)) & \text{a.e. on } T_\delta = [t_0, t_0 + \delta] \\ y(t_0) = \nu_0 & \end{array} \right\}. \tag{22}$$

Because of hypothesis $\mathrm{H(f)}_5$ (iii), problem (32) has a unique solution $y \in W^{1,1}(T_\delta)$. Moreover, from the definitions of upper and lower solutions and a well-known differential inequality (see Hale [9], p. 31), we infer that $\phi'(t) \leq y(t) \leq \psi'(t)$ for all $t \in T_\delta$ and so $(\psi - \phi)'(t) \geq 0$ for all $t \in T_\delta$. Integrating, we have $(\psi - \phi)(t_0) \leq (\psi - \phi)(t)$ for all $t \in T_\delta$. Recalling the choice of t_0, we see that $(\psi - \phi)(t) =$ constant for all $t \in T_\delta$, hence $\psi'(t) = \phi'(t)$ for all $t \in T_\delta$. Thus for almost all $t \in T_\delta$, we have

$$-\psi''(t) < f(t, \phi(t), \psi'(t)) = f(t, \phi(t), \phi'(t)) \leq -\phi''(t),$$

a contradiction to the fact that $(\psi - \phi)''(t) = 0$ for all $t \in T_\delta$.

If $t_0 = 0$, then since $(\psi - \phi)(0) = (\psi - \phi)(b)$, we can find $\delta > 0$ such that $(\psi - \phi)(0) \geq (\psi - \phi)(t) > 0$ for all $t \in [0, \delta]$ and $0 < (\psi - \phi)(t) \leq (\psi - \phi)(b)$ for all $t \in [b - \delta, b]$. From the first inequality we have that $(\psi - \phi)'(0) \leq 0$, while from the second it follows that $(\psi-\phi)'(b)\geq 0$. But from the definitions of the upper and lower solutions we have $(\psi-\phi)'(0)\geq(\psi-\phi)'(b)\geq 0$, therefore we conclude that $\psi'(0)=\phi'(0)=\nu_0$ and we can proceed as in the previous case.

The case $t_0=b$ is treated in a similar fashion.

Our second observation concerning ϕ,ψ, refers to problem (30) where the vector field depends linearly in x'.

PROPOSITION 8

If $\mathrm{H(f)}_4$ *holds,* $\phi\in W^{2,1}(T)$ *is an upper solution of* (30), $\psi\in W^{2,1}(T)$ *is a lower solution of* (30) *and for all* $t\in T$ $\phi(t)\leq\psi(t)$, *then* $(\psi-\phi)(\cdot)$ *is constant.*

Proof. By definition we have

$$\begin{aligned}&-\psi''(t)\leq f(t,\psi(t))+M\psi'(t) &&\text{a.e. on } T,\ \psi(0)=\psi(b),\ \psi'(0)\geq\psi'(b)\\ &-\phi''(t)\geq f(t,\phi(t))+M\phi'(t) &&\text{a.e. on } T,\ \phi(0)=\phi(b),\ \phi'(0)\leq\phi'(b).\end{aligned}$$

Hence we have

$$\psi''(t)-\phi''(t)\geq f(t,\phi(t))-f(t,\psi(t))+M(\phi'(t)-\psi'(t))\quad\text{a.e. on } T.$$

Multiplying with $(\psi-\phi)(t)$ and then integrating over $T=[0,b]$, we obtain

$$\begin{aligned}\int_0^b(\psi''-\phi'')(t)(\psi-\phi)(t)\mathrm{d}t\geq&\int_0^b(f(t,\phi(t))-f(t,\psi(t)))(\psi-\phi)(t)\mathrm{d}t\\ &+M\int_0^b(\phi'-\psi')(t)(\psi-\phi)(t)\mathrm{d}t.\end{aligned}\tag{23}$$

By Green's formula, we have

$$\begin{aligned}&\int_0^b(\psi''-\phi'')(t)(\psi-\phi)(t)\mathrm{d}t=(\psi-\phi)'(b)-(\psi-\phi)'()0)\\ &\quad-\int_0^b[(\psi'-\phi')(t)]^2\mathrm{d}t\leq-\int_0^b[(\psi'-\phi')(t)]^2\mathrm{d}t.\end{aligned}\tag{24}$$

Also from hypothesis $\mathrm{H(f)}_4$ (ii) it follows that

$$\int_0^b(f(t,\phi(t))-f(t,\psi(t)))(\psi-\phi)(t)\mathrm{d}t\geq 0.\tag{25}$$

Finally note that

$$\begin{aligned}&\int_0^b M(\phi'-\psi')(t)(\psi-\phi)(t)\mathrm{d}t=-M\int_0^b(\psi'-\phi')(t)(\psi-\phi)(t)\mathrm{d}t\\ &\quad=-M\int_0^b(\psi-\phi)(t)d(\psi-\phi)(t)=-M(\psi-\phi)(b)+M(\psi-\phi)(0)=0.\end{aligned}\tag{26}$$

Using (34), (35) and (36) in (33), we obtain

$$\int_0^b[(\psi'-\phi')(t)]^2\mathrm{d}t\leq 0$$

$$\Rightarrow\psi'(t)=\phi'(t)\quad\text{for all } t\in T\quad\text{and so } (\psi-\phi)(\cdot)\text{ is constant.}$$

An immediate consequence of proposition 8, is the following result:

COROLLARY 9

If H(f)$_4$ *holds and* $x_1, x_2 \in W^{2,1}(T)$ *are two solutions of* (30) *such that* $x_1(t) \leq x_2(t)$ *for all* $t \in T$, *then* $(x_1 - x_2)(\cdot)$ *is constant.*

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Boundary controllability of integrodifferential systems in Banach spaces

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Abstract. Sufficient conditions for boundary controllability of integrodifferential systems in Banach spaces are established. The results are obtained by using the strongly continuous semigroup theory and the Banach contraction principle. Examples are provided to illustrate the theory.

Keywords. Boundary controllability; integrodifferential system; semigroup theory; fixed point theorem.

1. Introduction

Controllability of nonlinear systems represented by ordinary differential equations in Banach spaces has been extensively studied by several authors. Balachandran *et al* [1] studied the controllability of nonlinear integrodifferential systems whereas in [2] they have investigated the local null controllability of nonlinear functional differential systems in Banach spaces by using the Schauder fixed point theorem. Controllability of nonlinear functional integrodifferential systems in Banach spaces has been studied by Park and Han [10].

Several abstract settings have been developed to describe the distributed control systems on a domain Ω in which the control is acted through the boundary Γ. But in these approaches one can encounter the difficulty for the existence of sufficiently regular solution to state space system, the control must be taken in a space of sufficiently smooth functions. Balakrishnan [3] showed that the solution of a parabolic boundary control equation with L^2 controls can be expressed as a mild solution to an operator equation. Fattorini [6] discussed the general theory of boundary control systems. Barbu and Precupanu [4] studied a class of convex control problems governed by linear evolution systems covering the principal boundary control systems of parabolic type. In [5] Barbu investigated a class of boundary-distributed linear control systems in Banach spaces. Lasiecka [8] established the regularity of optimal boundary controls for parabolic equations with quadratic cost criterion. Recently Han and Park [7] derived a set of sufficient conditions for the boundary controllability of a semilinear system with a nonlocal condition. The purpose of this paper is to study the boundary controllability of nonlinear integrodifferential systems in Banach spaces by using the Banach fixed point theorem.

2. Preliminaries

Let E and U be a pair of real Banach spaces with norms $||\cdot||$ and $|\cdot|$, respectively. Let σ be a linear closed and densely defined operator with $D(\sigma) \subseteq E$ and let τ be a linear operator with $D(\tau) \subseteq E$ and $R(\tau) \subseteq X$, a Banach space.

f u is continuously differentiable on $[0, b]$, then z can be defined as a mild solution to the Cauchy problem

$$\dot{z}(t) = Az(t) + \sigma Bu(t) - B\dot{u}(t) + f\left(t, x(t), \quad \int_0^t g(t, s, x(s))\mathrm{d}s\right),$$
$$z(0) = x_0 - Bu(0)$$

and the solution of (1) is given by

$$x(t) = T(t)[x_0 - Bu(0)] + Bu(t)$$
$$+ \int_0^t T(t-s)\left[\sigma Bu(s) - B\dot{u}(s) + f\left(s, x(s), \int_0^s g(s, \tau, x(\tau)\mathrm{d}\tau)\right)\right]\mathrm{d}s. \quad (3)$$

Since the differentiability of the control u represents an unrealistic and severe requirement, it is necessary to extend the concept of the solution for the general inputs $u \in L^1$ (J, U). Integrating (3) by parts, we get

$$x(t) = T(t)x_0 + \int_0^t [T(t-s)\sigma - AT(t-s)]Bu(s)\mathrm{d}s$$
$$+ \int_0^t T(t-s)f\left(s, x(s), \int_0^s g(s, \tau, x(\tau))\mathrm{d}\tau\right)\mathrm{d}s. \quad (4)$$

Thus (4) is well defined and it is called a mild solution of the system (1).

DEFINITION

The system (1) is said to be controllable on the interval J if for every $x_0, x_1 \in E$, there exists a control $u \in L^2(J, U)$ such that the solution $x(.)$ of (1) satisfies $x(b) = x_1$.

We further consider the following additional conditions:

(vii) There exists a constant $K_1 > 0$ such that $\int_0^b \nu(t)\mathrm{d}t \leq K_1$.
(viii) The linear operator W from $L^2(J, U)$ into E defined by

$$Wu = \int_0^b [T(b-s)\sigma - AT(b-s)]Bu(s)\mathrm{d}s$$

induces an invertible operator $\tilde{W}$ defined on $L^2(J, U)/\ker W$ and there exists a positive constant $K_2 > 0$ such that $\|\tilde{W}^{-1}\| \leq K_2$. The construction of the bounded inverse operator $\tilde{W}^{-1}$ in general Banach space is outlined in the Remark.

(ix) $M\|x_0\| + [bM\|\sigma B\| + K_1]\ K_2[\|x_1\| + M\|x_0\| + N] + N \leq r$, where $N = bM[M_1[r + b(L_1 r + L_2)] + M_2]$.
(x) Let $q = bMM_1K_2[1 + bL_1](bM\|\sigma B\| + K_1)$ be such that $0 \leq q < 1$.

3. Main result

Theorem. *If the hypotheses* (i)–(x) *are satisfied, then the boundary control integrodifferential system* (1) *is controllable on J.*

Proof. Using the hypothesis (viii), for an arbitrary function $x(.)$ define the control

$$u(t) = \tilde{W}^{-1}\left[x_1 - T(b)x_0 - \int_0^b T(b-s)f(s,x(s),\int_0^s g(s,\tau,x(\tau))\mathrm{d}\tau)\mathrm{d}s\right](t). \tag{5}$$

Let $Y = C(J,B_r)$. Using this control, we shall show that the operator Φ defined by

$$\begin{aligned}\Phi x(t) = T(t)x_0 &+ \int_0^t [T(t-s)\sigma - AT(t-s)]B\tilde{W}^{-1}[x_1 - T(b)x_0 \\ &- \int_0^b T(b-\tau)f(\tau,x(\tau),\int_0^\tau g(\tau,\theta,x(\theta))\mathrm{d}\theta)\mathrm{d}\tau](s)\mathrm{d}s \\ &+ \int_0^t T(t-s)f(s,x(s),\int_0^s g(s,\theta,x(\theta))\mathrm{d}\theta)\mathrm{d}s\end{aligned}$$

has a fixed point. First we show that Φ maps Y into itself. For $x \in Y$,

$$\begin{aligned}\|\Phi x(t)\| \leq{} & \|T(t)x_0\| + \left\|\int_0^t [T(t-s)\sigma - AT(t-s)]B\tilde{W}^{-1}\left[x_1 - T(b)x_0\right.\right. \\ & \left.\left.- \int_0^b T(b-\tau)f(\tau,x(\tau),\int_0^\tau g(\tau,\theta,x(\theta))\mathrm{d}\theta)\mathrm{d}\tau\right](s)\mathrm{d}s\right\| \\ & + \left\|\int_0^t T(t-s)f(s,x(s),\int_0^s g(s,\theta,x(\theta))\mathrm{d}\theta)\mathrm{d}s\right\| \\ \leq{} & \|T(t)x_0\| + \int_0^t \|T(t-s)\|\|\sigma B\|\|\tilde{W}^{-1}\|\left[\|x_1\| + \|T(b)x_0\|\right. \\ & + \int_0^b \|T(b-\tau)\|\left[\left\|f(\tau,x(\tau),\int_0^\tau g(\tau,\theta,x(\theta))\mathrm{d}\theta)\right.\right. \\ & \left.\left.\left.- f(\tau,0,0)\right\| + \|f(\tau,0,0)\|\right]\mathrm{d}\tau\right]\mathrm{d}s \\ & + \int_0^t \|AT(t-s)B\|\|\tilde{W}^{-1}\|\left[\|x_1\| + \|T(b)x_0\|\right. \\ & + \int_0^b \|T(b-\tau)\|\left[\left\|f(\tau,x(\tau),\int_0^\tau g(\tau,\theta,x(\theta))\mathrm{d}\theta)\right.\right. \\ & \left.\left.\left.- f(\tau,0,0)\right\| + \|f(\tau,0,0)\|\right]\mathrm{d}\tau\right]\mathrm{d}s \\ & + \int_0^t \|T(t-s)\|\left[\left\|f(s,x(s),\int_0^s g(s,\theta,x(\theta))\mathrm{d}\theta)\right.\right. \\ & \left.\left.- f(s,0,0)\right\| + \|f(s,0,0)\|\right]\mathrm{d}s \\ \leq{} & M\|x_0\| + bM\|\sigma B\|K_2[\|x_1\| + M\|x_0\| \\ & + bM[M_1[r + b(L_1 r + L_2)] + M_2] \\ & + K_1K_2[\|x_1\| + M\|x_0\| + bM[M_1[r + b(L_1 r + L_2)] + M_2] \\ & + bM[M_1[r + b(L_1 r + L_2)] + M_2] \\ \leq{} & M\|x_0\| + [bM\|\sigma B\| + K_1]K_2[\|x_1\| + M\|x_0\| + N] + N \\ \leq{} & r.\end{aligned}$$

Thus Φ maps Y into itself. Now, for $x_1, x_2 \in Y$ we have

$$\begin{aligned}
\|\Phi x_1(t) - \Phi x_2(t)\| \leq & \int_0^t [\|T(t-s)\|\|\sigma B\| + \|AT(t-s)B\|]\|\tilde{W}^{-1}\| \\
& \left[\int_0^b \|T(b-\tau)\| \left\| f(\tau, x_1(\tau), \int_0^\tau g(\tau, \theta, x_1(\theta))\mathrm{d}\theta) \right.\right. \\
& \left.\left. -f(\tau, x_2(\tau), \int_0^\tau g(\tau, \theta, x_2(\theta))\mathrm{d}\theta) \right\| \mathrm{d}\tau \right] \mathrm{d}s \\
& + \int_0^t \|T(t-s)\| \left\| f(s, x_1(s), \int_0^s g(s, \theta, x_1(\theta))\mathrm{d}\theta) \right. \\
& \left. -f(s, x_2(s), \int_0^s g(s, \theta, x_2(\theta))\mathrm{d}\theta) \right\| \mathrm{d}s \\
\leq & \int_0^t [M\|\sigma B\| + \nu(t)]K_2[bMM_1(\|x_1(\tau) - x_2(\tau)\| \\
& + bL_1\|x_1(\theta) - x_2(\theta)\|)]\mathrm{d}s \\
& + bMM_1(\|x_1(s) - x_2(s)\| + bL_1\|x_1(\theta) - x_2(\theta)\|) \\
\leq & q\|x_1(t) - x_2(t)\|.
\end{aligned}$$

Therefore, Φ is a contraction mapping and hence there exists a unique fixed point $x \in Y$ such that $\Phi x(t) = x(t)$. Any fixed point of Φ is a mild solution of (1) on J which satisfies $x(b) = x_1$. Thus the system (1) is controllable on J.

4. Applications

Example 1. Let Ω be a bounded and open subset of R^n and let Γ be a sufficiently smooth boundary of Ω (say of class C^∞).

Consider the boundary control integrodifferential system,

$$\begin{aligned}
\frac{\partial y(t,x)}{\partial t} - \Delta y(t,x) &= \mu(t, y(t,x), \int_0^t \eta(t,s,y(s,x))\mathrm{d}s), \quad \text{in} \quad Y = (0,b) \times \Omega, \\
y(t,0) &= u(t,0), \quad \text{on} \quad \Sigma = (0,b) \times \Gamma, \quad t \in [0,b], \\
y(0,x) &= y_0(x), \quad \text{for} \quad x \in \Omega,
\end{aligned} \tag{6}$$

where $u \in L^2(\Sigma)$, $y_0 \in L^2(\Omega)$, $\mu \in L^2(Y)$ and $\eta \in Y$.

The above problem can be formulated as a boundary control problem of the form (1) by suitably taking the spaces E, X, U and the operators B_1, σ and τ as follows:

Let $E = L^2(\Omega)$, $X = H^{-\frac{1}{2}}(\Gamma)$, $U = L^2(\Gamma)$, $B_1 = I$, the identity operator and $D(\sigma) = \{y \in L^2(\Omega);\ \Delta y \in L^2(\Omega)\}$, $\sigma = \Delta$. The operator τ is the 'trace' operator such that $\tau y = y|_\Gamma$ is well defined and belongs to $H^{-\frac{1}{2}}(\Gamma)$ for each $y \in D(\sigma)$ (see [5]) and the operator A is given by

$$A = \Delta, \quad D(A) = H_0^1(\Omega) \cup H^2(\Omega).$$

(Here $H^k(\Omega), H^\alpha(\Gamma)$ and $H_0^1(\Omega)$ are usual Sobolev spaces on Ω, Γ.)

Let us assume that the nonlinear functions μ and η satisfy the following Lipschitz condition:

$$\|\mu(t,v_1,w_1)-\mu(t,v_2,w_2)\| \le K_1[\|v_1-v_2\|+\|w_1-w_2\|],$$
$$\|\eta(t,s,v_1)-\eta(t,s,v_2)\| \le K_2\|v_1-v_2\|,$$

where K_1, $K_2 > 0$, v_1, $v_2 \in B_r$ and w_1, $w_2 \in \Omega$.

Define the linear operator $B : L^2(\Gamma) \to L^2(\Omega)$ by $Bu = w_u$ where w_u is the unique solution to the Dirichlet boundary value problem,

$$\Delta w_u = 0 \quad \text{in} \quad \Omega,$$
$$w_u = u \quad \text{in} \quad \Gamma.$$

In other words (see [9])

$$\int_\Omega w_u \Delta\psi \mathrm{d}x = \int_\Gamma u\frac{\partial\psi}{\partial n}\mathrm{d}x, \quad \text{for all} \quad \psi \in H_0^1(\Omega)\cup H^2(\Omega), \tag{7}$$

where $\partial\psi/\partial n$ denotes the outward normal derivative of ψ which is well defined as an element of $H^{\frac{1}{2}}(\Gamma)$. From (7), it follows that,

$$\|w_u\|_{L^2(\Omega)} \le C_1\|u\|_{H^{-\frac{1}{2}}(\Gamma)}, \quad \text{for all} \quad u \in H^{-\frac{1}{2}}(\Gamma)$$

and

$$\|w_u\|_{H^1(\Omega)} \le C_2\|u\|_{H^{\frac{1}{2}}(\Gamma)}, \quad \text{for all} \quad u \in H^{\frac{1}{2}}(\Gamma),$$

where C_i, $i = 1, 2$ are positive constants independent of u.

From the above estimates it follows by an interpolation argument [12] that

$$\|AT(t)B\|_{L(L^2(\Gamma),L^2(\Gamma))} \le Ct^{-\frac{3}{4}}, \quad \text{for all} \quad t > 0 \quad \text{with} \quad \nu(t) = Ct^{-\frac{3}{4}}.$$

Further assume that the bounded invertible operator $\tilde{W}$ exists. Choose b and other constants such that the conditions (ix) and (x) are satisfied. Hence, we see that all the conditions stated in the theorem are satisfied and so the system (6) is controllable on $[0, b]$.

Example 2. Consider the boundary control system,

$$\begin{aligned}
&\frac{\partial y(t,x)}{\partial t} - \Delta y(t,x) = f\left(t, y(t,x), \int_0^t g(t,s,y(s,x))\mathrm{d}s\right) \quad \text{in} \quad Q = (0,b)\times\Omega,\\
&\frac{\partial y(t,0)}{\partial n} + \beta y(t,0) = u(t,0), \quad \text{in} \quad (0,b)\times\Gamma, \quad t \in [0,b],\\
&\qquad y(0,x) = y_0(x), \quad x \in \Omega,
\end{aligned} \tag{8}$$

where $y_0 \in L^2(\Omega), f \in L^2(Q)$, $g \in Q$ and $u \in L^2(\Gamma)$. Here β is a nonnegative constant. Let us assume that the nonlinear functions f and g satisfy the Lipschitz condition:

$$\|f(t,v_1,w_1)-f(t,v_2,w_2)\| \le M_1[\|v_1-v_2\|+\|w_1-w_2\|],$$
$$\|g(t,s,v_1)-g(t,s,v_2)\| \le M_2\|v_1-v_2\|,$$

where $M_1, M_2 > 0$, $v_1, v_2 \in B_r$ and $w_1, w_2 \in \Omega$.

Take $E = L^2(\Omega)$, $U = X = L^2(\Gamma)$, $B_1 = I$, $\sigma y = \Delta y$, $\tau y = \beta y + (\partial y/\partial n)$ and $D(\sigma) = H^2(\Omega)$.

The operator A is given by

$$Ay = \Delta y, D(A) = \left\{ y \in H^2(\Omega);\quad \beta y + \frac{\partial y}{\partial n} = 0 \right\}.$$

Now the problem (8) becomes an abstract formulation of (1).

Define the linear operator $B : L^2(\Gamma) \to L^2(\Omega)$ by $Bu = z_u$ where $z_u \in H^1(\Omega)$ is the unique solution to the Neumann boundary value problem,

$$z_u - \Delta z_u = 0 \quad \text{in} \quad \Omega,$$

$$\beta z_u + \frac{\partial z_u}{\partial n} = u \quad \text{in} \quad \Gamma.$$

Consider on the product space $H^1(\Omega) \times H^1(\Omega)$, the bilinear functional

$$h(y, \psi) = \int_\Omega (y\psi + \operatorname{grad} y \operatorname{grad} \psi)\mathrm{d}x - \int_\Gamma (u - \beta y)\psi \mathrm{d}\sigma, \tag{9}$$

where $u \in H^{-\frac{1}{2}}(\Gamma)$ (here $\int_\Gamma u\psi\,\mathrm{d}\sigma$ is the value of u at $\psi \in H^{\frac{1}{2}}(\Gamma)$. Since h is coercive, there is a $z_u \in H^1(\Omega)$ satisfying $h(z_u, \psi) = 0$ for all $\psi \in H^1(\Omega)$. Hence $z_u = Bu$ is the solution to (8). From (9) we see that

$$\|w_u\|_{H^1(\Omega)} \leq C_1 \|u\|_{H^{-\frac{1}{2}}(\Gamma)}.$$

Since the operator $-A$ is self-adjoint and positive, we have

$$\int_0^b \|AT(t)y_0\|^2_{L^2(\Omega)}\mathrm{d}t \leq C\|y_0\|^2_{D((-A)^{\frac{1}{2}})} \quad \text{for all} \quad y_0 \in D((-A)^{\frac{1}{2}}) = H^1(\Omega). \tag{10}$$

Let δ be the scalar function defined by

$$\delta(t) = \liminf_{n\to\infty} \|A_n T(t)\|_{L(H^1(\Omega),L^2(\Omega))}, \quad t \in [0, b],$$

where $A_n = A(I + n^{-1}A)^{-1}$ for $n = 1, 2, \ldots$. Obviously,

$$\|AT(t)\|_{L(H^1(\Omega),L^2(\Omega))} \leq \delta(t) \quad \text{for} \quad t \in (0, b]. \tag{11}$$

Also we find that (10) implies that

$$\int_0^b \|A_n T(t)y_0\|^2_{L(H^1(\Omega),L^2(\Omega))}\mathrm{d}t \leq C \quad \text{for all } n.$$

Therefore by Fatou's lemma it follows that $\delta \in L^2(0, b)$ and hence from (10) and (11) we have

$$\|AT(t)Bu\|_{L^2(\Omega)} \leq C\delta(t)\|u\|_{L^2(\Gamma)}, \quad \text{for all} \quad t \in (0, b), \quad u \in L^2(\Gamma)$$

with $\nu(t) = C\delta(t) \in L^2(0, b)$. Further assume that the bounded invertible operator $\tilde{W}$ exists. Choose b and other constants in such a way that the conditions (ix) and (x) are satisfied. Thus we find that all the conditions stated in the theorem are satisfied. Hence the system (8) is controllable on $[0, b]$.

Remark (see also [11]). *Construction of* $\tilde{W}^{-1}$.

Let $Y = L^2[J,U]/\ker W$. Since ker W is closed, Y is a Banach space under the norm

$$\|[u]\|_Y = \inf_{u\in[u]} \|u\|_{L^2[J,U]} = \inf_{W\hat{u}=0} \|u+\hat{u}\|_{L^2[J,U]},$$

where $[u]$ are the equivalence classes of u.

Define $\tilde{W} : Y \to X$ by

$$\tilde{W}[u] = Wu, \quad u \in [u].$$

Now $\tilde{W}$ is one-to-one and

$$\|\tilde{W}[u]\|_X \leq \|W\| \|[u]\|_Y.$$

We claim that V = Range W is a Banach space with the norm

$$\|v\|_V = \|\tilde{W}^{-1}v\|_Y.$$

This norm is equivalent to the graph norm on $D(\tilde{W}^{-1})$ = Range W, $\tilde{W}$ is bounded and since $D(\tilde{W}) = Y$ is closed, $\tilde{W}^{-1}$ is closed and so the above norm makes Range $W = V$, a Banach space.

Moreover,

$$\begin{aligned}\|Wu\|_V &= \|\tilde{W}^{-1}Wu\|_Y = \|\tilde{W}^{-1}\tilde{W}[u]\| \\ &= \|[u]\| = \inf_{u\in[u]} \|u\| \leq \|u\|,\end{aligned}$$

so

$$W \in \mathcal{L}(L^2[J,U], V).$$

Since $L^2[J,U]$ is reflexive and ker W is weakly closed, so that the infimum is actually attained. For any $v \in V$, we can therefore choose a control $u \in L^2[J,U]$ such that $u = \tilde{W}^{-1}v$.

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Errata

Steady-state response of a micropolar generalized thermoelastic half-space to the moving mechanical/thermal loads

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(Proc. Indian Acad. Sci. (Math. Sci.), Vol. 110, No. 4, pp. 449–465, November 2000)

1. On page 451, in eq. (13) following two expressions have been left out and should be included:

$$u_x' = \frac{\rho\omega^* c_2}{vT_0} u_x, \qquad u_z' = \frac{\rho\omega^* c_2}{vT_0} u_z.$$

2. On page 454, a typographical error has been found out in eq. (48). The expression for Δ_0 should read as:

$$\Delta_0 = b_2'^2 \xi^2 \xi_1 \xi_2 \xi_3' (q_2 - q_1) - f_3' (f_1' \xi_2 q_2 - f_2' \xi_1 q_1).$$

All the analytical expressions and numerical results do not change due to these errors.

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Descent principle in modular Galois theory

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Abstract. We propound a descent principle by which previously constructed equations over $\mathrm{GF}(q^n)(X)$ may be deformed to have incarnations over $\mathrm{GF}(q)(X)$ without changing their Galois groups. Currently this is achieved by starting with a vectorial (= additive) q-polynomial of q-degree m with Galois group $\mathrm{GL}(m, q)$ and then, under suitable conditions, enlarging its Galois group to $\mathrm{GL}(m, q^n)$ by forming its generalized iterate relative to an auxiliary irreducible polynomial of degree n. Elsewhere this was proved under certain conditions by using the classification of finite simple groups, and under some other conditions by using Kantor's classification of linear groups containing a Singer cycle. Now under different conditions we prove it by using Cameron-Kantor's classification of two-transitive linear groups.

Keywords. Galois group; iteration; transitivity.

1. Introduction

In this paper we make some progress towards understanding which finite groups are Galois groups of coverings of the affine line over a ground field of characteristic $p \neq 0$, having at most one branch point other than the point at infinity. We are specially interested in the case when the ground field is not algebraically closed. In particular we realize some of the matrix groups $\mathrm{GL}(m, q^n)$, where $q = p^u > 1$ is a power of p and $m > 0$ and $n > 0$ are integers, over smaller fields of characteristic p than had previously been accomplished. For a tie-up with the geometric case of an algebraically closed ground field and the arithmetic case of a finite ground field see Remark 5.1 at the end of the paper. Likewise, for a tie-up with Drinfeld module theory see Remark 5.2 at the end of the paper.

To describe the contents of the paper in greater detail, henceforth let $q = p^u > 1$ be a power of a prime p, let $m > 0$ and $n > 0$ be integers, and let $\mathrm{GF}(q) \subset k_q \subset K \subset \Omega$ be fields where Ω is an algebraic closure of K; note that there are no assumptions on the field k_q other than for it to contain $\mathrm{GF}(q)$. Also let $E = E(Y)$ be a monic separable vectorial q-polynomial of q-degree m in Y over K, i.e.,

$$E = E(Y) = Y^{q^m} + \sum_{i=1}^{m} X_i Y^{q^{m-i}} \quad \text{with} \quad X_i \in K \text{ and } X_m \neq 0, \tag{1.1}$$

where the elements $X_1, \ldots, X_m$ need not be algebraically independent over k_q. When we want to assume that, for a subset J^* of $\{1, \ldots, m\}$, the elements $\{X_i : i \in J^*\}$ are algebraically independent over k_q and $K = k_q(\{X_i : i \in J^*\})$ with $X_i = 0$ for all $i \notin J^*$, we may express this by saying that we are in the *generic* case of type J^*, and we may indicate it by writing $E^*_{m,q}$ for E and K^* for K. When J^* is the singleton $J^\flat = \{m\}$

we may say that we are in the *binomial* case. When J^* is the pair $J_\mu^\dagger = \{m-\mu, m\}$ with $1 \le \mu < m$ we may say that we are in the μ-*trinomial* case. When J^* is the set $J^\ddagger = \{m-\nu : \nu = 0 \text{ or } \nu = \text{a divisor of } m\}$, we may say that we are in the *divisorial* case. Note that the Y-derivative of $E(Y)$ is X_m and hence if $m \in J^*$ then in the generic case of type J^*, the equation $E(Y) = 0$ gives a covering of the affine line over $k_q(\{X_i : m \neq i \in J^*\})$ having $X_m = 0$ as the only possible branch point other than the point at infinity.

In the *general* (= not necessarily generic) case, let V be the set of all roots of E in Ω, and note that then V is an m-dimensional GF(q)-vector-subspace of Ω. Moreover, since GF(q) is assumed to be a subfield of k_q and hence of K, every K-automorphism of the splitting field $K(V)$ of E over K induces a GF(q)-linear transformation of V. Consequently $\mathrm{Gal}(E, K) < \mathrm{GL}(V)$, i.e., the Galois group of E over K may be regarded as a subgroup of GL(V) (see [Ab3]). If we do not assume $\mathrm{GF}(q) \subset k_q$ then we only get $\mathrm{Gal}(E, K) < \Gamma\mathrm{L}(V)$, where $\Gamma\mathrm{L}(V)$ is the group of all semilinear transformations of V (see [Ab6]). By fixing a basis of V we may identify GL(V) with GL(m, q), and ΓL(V) with ΓL(m, q). If $J_1^\dagger \subset J^*$ then in the generic case of type J^*, as shown in [Ab2] to [Ab4], we have $\mathrm{Gal}(E^*_{m,q}, K^*) = \mathrm{GL}(m, q)$ but over GF(p), as shown in [Ab6], we have $\mathrm{Gal}(E^*_{m,q}, \mathrm{GF}(p)(\{X_i : i \in J^*\})) = \Gamma\mathrm{L}(m, q)$; for applications of these results see [Ab1] and [Ab5]. To mitigate this bloating we take recourse to generalized iteration as defined in Remark 3.30 of [Ab7] and repeated below. Here bloating refers to the fact that a more direct approach would give a Galois group which is larger than desired, when working over a smaller ground field, and the goal is to modify the covering in order to shrink the group from semilinear to general linear.

DEFINITION 1.2

For every nonnegative integer j we inductively define the jth *iterate* $E^{[[j]]}$ of E by putting $E^{[[0]]} = E^{[[0]]}(Y) = Y$, $E^{[[1]]} = E^{[[1]]}(Y) = E(Y)$, and $E^{[[j]]} = E^{[[j]]}(Y) = E(E^{[[j-1]]}(Y))$ for all $j > 1$. Next we define the *generalized* rth *iterate* $E^{[r]}$ of E for any $r = r(T) = \sum r_i T^i \in \Omega[T]$ with $r_i \in \Omega$ (and $r_i = 0$ for all except a finite number of i), where T is an indeterminate, by putting $E^{[r]} = E^{[r]}(Y) = \sum r_i E^{[[i]]}(Y)$. Note that, for the Y-derivative $E_Y^{[r]}(Y)$ of $E^{[r]}(Y)$ we clearly have

$$E_Y^{[r]}(Y) = E_Y^{[r]}(0) = r(X_m) \tag{1.2.1}$$

and hence if $r(X_m) \neq 0$ then $E^{[r]}$ is a separable vectorial q-polynomial over Ω whose q-degree in Y equals m times the T-degree of r. Also note that the definition of $E^{[r]}$ remains valid for any vectorial E without assuming it to be monic or separable. Moreover, in such a general set-up, this makes the additive group of all vectorial q-polynomials $E = E(Y)$ in Y over Ω into a $\Omega[T]$-*premodule* having all the properties of a module except the left distributive law and the associativity of multiplication, i.e., for all $r, r' \in \Omega[T]$ we have $E^{[r+r']} = E^{[r]} + E^{[r']}$, but for all E, E' over Ω we need not have $(E+E')^{[r]} = E^{[r]} + E'^{[r]}$, and in general $E^{[rr']}$ need not be equal to $(E^{[r]})^{[r']}$. Reverting to the fixed monic separable vectorial E exhibited in (1.1), the said premodule structure makes Ω into a GF$(q)[T]$-*module* when for every $r \in \mathrm{GF}(q)[T]$ and $z \in \Omega$ we define the 'product' of r and z to be $E^{[r]}(z)$; we denote this GF$(q)[T]$-module by Ω_E. Now let us fix

$$s = s(T) \in R = \mathrm{GF}(q)[T] \text{ of } T\text{-degree } n \text{ with } s(X_m) \neq 0 \tag{1.2.2}$$

and note that then $E^{[s]}$ is a separable vectorial q-polynomial of q-degree mn in Y over K, and the coefficient of its highest degree term equals the coefficient of the highest degree of $s(T)$. Let $V^{[s]}$ be the set of all roots of $E^{[s]}$ in Ω, and note that then $V^{[s]}$ is an (mn)-dimensional $\mathrm{GF}(q)$-vector-subspace of Ω. Let $\mathrm{GF}(q,s) = R/sR$ where sR is the ideal generated by s in $R = \mathrm{GF}(q)[T]$, and let $\omega : R \to \mathrm{GF}(q,s)$ be the canonical epimorphism. Now $V^{[s]}$ is a submodule of Ω_E and as such it is annihilated by sR and hence we may regard it as a $\mathrm{GF}(q,s)$-module; note that then, for every $r \in R$ and $z \in \Omega$, the 'product' of $\omega(r)$ and z is given by $\omega(r)z = E^{[r]}(z) = \sum r_i E^{[i]}(z)$, and for every $g \in \mathrm{Gal}(K(V^{[s]}), K)$ we have $g(\omega(r)z) = \sum g(r_i)E^{[i]}(g(z)) = (\omega(r))g(z)$; also note that for all $r \in R$ and $z \in \Omega$ we have $rz = \omega(r)z = E^{[r]}(z) = \theta(r,z)$ with $\theta(r,z) \in (\mathrm{GF}(q)[X_1, \dots, X_m])[z]$. It follows that, in a natural manner,

$$\mathrm{Gal}(E^{[s]}, K) < \mathrm{GL}(V^{[s]}), \tag{1.2.3}$$

where $\mathrm{GL}(V^{[s]})$ is the *group of all* $\mathrm{GF}(q,s)$*-linear automorphisms* of $V^{[s]}$, by which we mean all additive isomorphisms $\sigma : V^{[s]} \to V^{[s]}$ such that for all $\eta \in \mathrm{GF}(q,s)$ and $z \in V^{[s]}$ we have $\sigma(\eta z) = \eta\sigma(z)$. Note that

$$s \text{ irreducible in } R \Rightarrow \mathrm{GL}(V^{[s]}) \approx \mathrm{GL}(m, q^n), \tag{1.2.4}$$

where $\approx$ denotes isomorphism. Also note that the Y-derivative of $E^{[s]}(Y)$ is $s(X_m)$ and hence if $m \in J^*$ and s is irreducible in R then in the generic case of type J^*, the equation $E^{[s]}(Y) = 0$ gives a covering of the affine line over $k_q(\{X_i : m \neq i \in J^*\})$ having $s(X_m) = 0$ as the only possible branch point other than the point at infinity; this branch point is rational if and only if $n = 1$.

Now part of what was proved in [Ab7] can be stated as follows:

Trinomial Lemma 1.3. *If* $J_1^\dagger \subset J^*$ *then in the generic case of type* J^* *we have* $\mathrm{Gal}(E^*_{m,q}, K^*) = \mathrm{GL}(m,q)$.

In Note 3.37 of [Ab7] the following problem about generalized iterations was posed.

Problem. Show that if $J^* = \{1, 2, \dots, m\}$ then in the generic case of type J^* we have $\mathrm{Gal}(E^{*[s]}_{m,q}, K^*) = \mathrm{GL}(V^{[s]})$.

In [AS1] this was proved when $s = T^n$ and in Theorem 3.25 of [Ab7] that result was semilinearized. Likewise in [AS2] it was proved under the assumptions that s is irreducible and m is a square-free integer with $\mathrm{GCD}(m,n) = 1$ and $\mathrm{GCD}(mnu, 2p) = 1$, where we recall that u is the exponent of p in q, i.e., u is the positive integer defined by $q = p^u$. Actually, what was proved in (1.18) of [AS2] was the following slightly more general result.

Weak divisorial Theorem 1.4. *Assume that s is irreducible in R, and* $J^\ddagger \subset J^*$. *Also assume that m is a square-free integer with* $\mathrm{GCD}(m,n) = 1$, *and* $\mathrm{GCD}(mnu, 2p) = 1$. *Then in the generic case of type* J^* *we have* $\mathrm{Gal}(E^{*[s]}_{m,q}, K^*) = \mathrm{GL}(V^{[s]}) \approx \mathrm{GL}(m, q^n)$.

Now CPT (= the classification of projectively transitive permutation groups, i.e., subgroups of GL acting transitively on nonzero vectors) is a remarkable consequence of CT (= the classification theorem of finite simple groups). The implication CT $\Rightarrow$ CPT was mostly proved by Hering [He1, He2]; it is also discussed by Cameron [Cam], Kantor [Ka2], and Liebeck [Lie]. The proof of (1.4) given in [AS2] makes essential use of the following weaker version of CPT, which follows by scanning the list of projectively transitive permutation groups given in [Ka2] or [Lie].

Weak CPT 1.5. Let d be an odd positive integer, and let $G < \mathrm{GL}(d,p)$ be transitive on the nonzero vectors $\mathrm{GF}(p)^d \setminus \{0\}$. Then there exist positive integers b, c with $bc = d$ and a group G_0 with $\mathrm{SL}(b, p^c) < G_0 < \Gamma\mathrm{L}(b, p^c)$ such that $G \approx G_0$.

The $m = 1$ case of (1.4), without the hypothesis $\mathrm{GCD}(mnu, 2p) = 1$, was proved by Carlitz [Car] (also see Hayes [Hay]) in connection with his explicit class field theory. In our proof of (1.4) we used the following variation of Carlitz's result which we reproved as Theorem 1.20 in [AS2]; recall that a univariate polynomial $\widetilde{F}(Y) = \sum_{i=0}^{N} \widetilde{F}_i Y^i$ of positive degree N in Y is said to be *Eisenstein* relative $(\widetilde{R}, \widetilde{M})$, where $\widetilde{M}$ is a prime ideal in a ring $\widetilde{R}$, if $\widetilde{F}_N \in \widetilde{R} \setminus \widetilde{M}$, $\widetilde{F}_i \in \widetilde{M}$ for $1 \le i \le N-1$, and $\widetilde{F}_0 \in \widetilde{M} \setminus \widetilde{M}^2$.

Carlitz irreducibility lemma 1.6. *Assume that s is irreducible in R, and $J^\flat \subset J^*$. Let $s^*(T)$ be a nonconstant irreducible factor of $s(T)$ in $k_q[T]$, and let M^* be the ideal in $R^* = k_q[\{X_i : i \in J^*\}]$ generated by $\{X_i : i \in J^* \setminus J^\flat\} \cup \{s^*(X_m)\}$. Then, for $m = 1$, in the generic case of type J^* we have that $M^* = s^*(X_m)R^*$ is a maximal ideal in $R^* = k_q[X_m]$, $Y^{-1}E_{1,q}^{*[s]}(Y)$ is Eisenstein relative to (R^*, M^*), $Y^{-1}E_{1,q}^{*[s]}(Y)$ is irreducible in $K^*[Y]$, and* $\mathrm{Gal}(E_{1,q}^{*[s]}, K^*) = \mathrm{GL}(V^{[s]}) \approx \mathrm{GL}(1, q^n)$. *Moreover, without assuming $m = 1$, but assuming* $\mathrm{GCD}(m, n) = 1$, *in the generic case of type J^* we have that M^* is a maximal ideal in R^*, $Y^{-1}E_{m,q}^{*[s]}(Y)$ is Eisenstein relative to (R^*, M^*), $Y^{-1}E_{m,q}^{*[s]}(Y)$ is irreducible in $K^*[Y]$, and* $\mathrm{Gal}(E_{m,q}^{*[s]}, K^*)$ *has an element of order $q^{mn} - 1$.*

In proving (1.4), in addition to items (1.5) and (1.6), we also used the first part of the following well-known versatile lemma which was initiated by Singer in [Sin] and which was stated as Lemma 1.23 in [AS2]; for an elementary proof of a supplemented version of this see Lemma 5.13 and §6 of [Ab8].

Singer cycle lemma 1.7. *Let $A \in$* $\mathrm{GL}(m, q)$ *have order $e = q^m - 1$. Then* $\det(A)$ *has order $\epsilon = q - 1$, and A acts transitively on the nonzero vectors* $\mathrm{GF}(q)^m \setminus \{0\}$, *i.e., it is an e-cycle in the symmetric group S_e (and as such it is called a Singer cycle). Moreover, in* $\mathrm{GL}(m, q)$ *all subgroups generated by such elements, i.e., all cyclic subgroups of order e, form a nonempty complete set of conjugates.*

Now the last assertion of (1.6) says that if s is irreducible in R and $J^\flat \subset J^*$ with $\mathrm{GCD}(m, n) = 1$ then $\mathrm{Gal}(E_{m,q}^{*[s]}, K)$, as a subgroup of $\mathrm{GL}(m, q^n)$, contains a Singer cycle. In his 1980 paper [Ka1], without using CT, Kantor proved the following variation (1.8) of (1.5) by replacing the hypothesis of G acting transitively on nonzero vectors by the stronger hypothesis that G contains a Singer cycle.

Kantor's Singer cycle theorem 1.8. *If $G <$* $\mathrm{GL}(m, q^n)$ *contains an element of order $q^{mn} - 1$ then for some divisor m' of m we have* $\mathrm{GL}(m', q^{nm/m'}) \lhd G$, *where* $\mathrm{GL}(m', q^{nm/m'})$ *is regarded as a subgroup of* $\mathrm{GL}(m, q)$ *in a natural manner.*

As a consequence of (1.6) and (1.8), but without using (1.5), and hence without using CT, in (5.18) of [Ab8] we proved the following stronger version (1.9) of (1.4) in which the assumption $\mathrm{GCD}(mnu, 2p) = 1$ is replaced by the weaker assumption $\mathrm{GCD}(m, p) = 1$.

Strong divisorial theorem 1.9. *Assume that s is irreducible in R, and $J^\ddagger \subset J^*$. Also assume that m is a square-free integer with* $\mathrm{GCD}(m, n) = 1$, *and* $\mathrm{GCD}(m, p) = 1$. *Then in the generic case of type J^* we have* $\mathrm{Gal}(E_{m,q}^{*[s]}, K^*) = \mathrm{GL}(V^{[s]}) \approx \mathrm{GL}(m, q^n)$.

In (1.14) of [Ab9] we settled another case of the above Problem by proving the following Theorem without using the above results (1.4) to (1.9).

Two step theorem 1.10. *Assume that s is irreducible in R, and $J_1^\dagger = J^*$. Also assume that $m = n = 2$. Then in the generic case of type J^* we have* $\mathrm{Gal}(E_{m,q}^{*[s]}, K^*) = \mathrm{GL}(V^{[s]}) \approx \mathrm{GL}(m, q^n)$.

The proof of (1.10) was based on the following lemma which was stated as Lemma 1.16 in [Ab9] and established in §3 of that paper.

Packet throwing lemma 1.11. *Let $\widetilde{M}$ be the maximal ideal in a regular local domain $\widetilde{R}$ of dimension $d > 0$ with quotient field $\widetilde{K}$. Let $\widetilde{F}(Y) = \sum_{0 \le i \le N} \widetilde{F}_i Y^i$ be a polynomial of degree $N > 0$ in Y which is Eisenstein relative to $(\widetilde{R}, \widetilde{M})$.* [Note that then for some elements $F_2, \dots, F_d$ in $\widetilde{R}$ we have $(\widetilde{F}_0, F_2, \dots, F_d)\widetilde{R} = \widetilde{M}$.] *Let $\widehat{K} = \widetilde{K}(\eta)$ where η is an element in an overfield of $\widetilde{K}$ with $\widetilde{F}(\eta) = 0$, and let $\widehat{R} = \widetilde{R}[\eta]$ and $\widehat{M} = \eta\widehat{R} + \widetilde{M}\widehat{R}$. Then $\widehat{R}$ is the integral closure of $\widetilde{R}$ in $\widehat{K}$, $\widehat{R}$ is a d dimensional regular local domain with maximal ideal $\widehat{M}$, $\widehat{M} \cap \widetilde{R} = \widetilde{M}$, and for any $\widehat{\eta} \in \widehat{K}$ with $\widetilde{F}(\widehat{\eta}) = 0$ and any $F_2, \dots, F_d$ in $\widetilde{R}$ with $(\widetilde{F}_0, F_2, \dots, F_d)\widetilde{R} = \widetilde{M}$ we have $(\widehat{\eta}, F_2, \dots, F_d)\widehat{R} = \widehat{M}$, and hence for any $\widehat{\eta} \in \widehat{K}$ with $\widetilde{F}(\widehat{\eta}) = 0$ we have $\widehat{\eta} \in \widehat{M} \setminus \widehat{M}^2$. Moreover, if for some positive integer $D < N - 1$ we have $\widetilde{F}_D \notin \widetilde{M}^2 + \widetilde{F}_0\widetilde{R}$ and $\widetilde{F}_i \in \widetilde{M}^{D+2-i} + \widetilde{F}_0\widetilde{R}$ for $1 \le i \le D - 1$, and $\eta_1, \dots, \eta_D$ are pairwise distinct elements in $\widehat{K}$ with $\widetilde{F}(\eta_j) = 0$ for $1 \le j \le D$, then $\widetilde{F}(Y) = \widehat{F}(Y)\prod_{1 \le j \le D}(Y - \eta_j)$ where $\widehat{F}(Y)$ is a polynomial of degree $N - D$ in Y which is Eisenstein relative to $(\widehat{R}, \widehat{M})$.*

In proving (1.10), the following consequence of (1.11) was implicitly used; in §2 we shall explicitly deduce it from (1.11).

Two transitivity lemma 1.12. *Assume that s is irreducible in R, and we are in the generic case of type J^* with $J^\flat \subset J^*$ and $m > 1$. [Note that by (1.2) we know that then* $\mathrm{Gal}(E_{q,m}^{*[s]}, K^*) < \mathrm{GL}(V^{[s]}) \approx \mathrm{GL}(m, q^n)$ *and hence we may regard* $\mathrm{Gal}(E_{q,m}^{*[s]}, K^*)$ *to be acting on the $(m-1)$-dimensional projective space $\mathcal{P}(m-1, q^n)$ over* $\mathrm{GF}(q^n)$ *(where the action is not faithful unless $q^n = 2$).] Let $N = q^{mn} - 1$ and $\widetilde{F}(Y) = Y^{-1}E_{q,m}^{*[s]}(Y) = \sum_{0 \le i \le N} \widetilde{F}_i Y^i$ with $\widetilde{F}_i \in R^* = k_q[\{X_j : j \in J^*\}]$. Assume that the localization of R^* at some nonzero prime ideal in it is a regular local domain $\widetilde{R}$ with maximal ideal $\widetilde{M}$ such that $\widetilde{F}(Y)$ is Eisenstein relative to $(\widetilde{R}, \widetilde{M})$. Let $D = q^n - 1$ and assume that $\widetilde{F}_D \notin \widetilde{M}^2 + \widetilde{F}_0\widetilde{R}$ and $\widetilde{F}_i \in \widetilde{M}^{D+2-i} + \widetilde{F}_0\widetilde{R}$ for $1 \le i \le D - 1$. Then* $\mathrm{Gal}(E_{q,m}^{*[s]}, K^*)$ *is two transitive on the $(m-1)$-dimensional projective space $\mathcal{P}(m-1, q^n)$ over* $\mathrm{GF}(q^n)$.

In Theorem I of [CKa], Cameron–Kantor proved the following:

Cameron-Kantor's two transitivity theorem 1.13. *If $m > 2$ and $G < \Gamma\mathrm{L}(m, q)$ is two transitive on the projective space $\mathcal{P}(m-1, q)$, then either* $\mathrm{SL}(m, q) < G$ *or G = the alternating group A_7 inside* $\mathrm{SL}(4, 2)$.

As a consequence of (1.6), (1.7), (1.12), (1.13), and the coefficient computations of §3, but *without using* (1.5) or (1.8) to (1.10), in §4 we shall prove the following theorem. With an eye on further applications, the computations of §3 are more extensive than what we need here.

Main theorem 1.14. *Assume that s is irreducible in R, and $n < m$ with* $\mathrm{GCD}(m, n) = 1$ *and $J_n^\dagger \subset J^*$. Then in the generic case of type J^* we have* $\mathrm{Gal}(E_{m,q}^{*[s]}, K^*) = \mathrm{GL}(V^{[s]}) \approx \mathrm{GL}(m, q^n)$.

In §5 we shall make some motivational and philosophical remarks.

2. Proof of two transitivity lemma

To continue with the discussion of (1.2), for a moment assume that s is irreducible in R
$s(X_m) \neq 0$ and $m > 1$. Then by (1.2.3) and (1.2.4) we have $\mathrm{Gal}(E^{[s]}, K) < \mathrm{GL}(V^{[s}$
$\mathrm{GL}(m, q^n)$ and hence we may regard $\mathrm{Gal}(E^{[s]}, K)$ to be acting on the $(m-1)$-dimensi
projective space $\mathcal{P}(m-1, q^n)$ over $\mathrm{GF}(q^n)$ (where the action is not faithful unless $q^n =$
Let $N = q^{mn} - 1$ and $F(Y) = Y^{-1}E^{[s]}(Y)$. Then $F(Y) \in K[Y]$ is of Y-degree N.
a moment assume that $F(Y)$ is irreducible in $K[Y]$ and let $\widehat{K} = K(\eta)$ where η is a
of $F(Y)$ in Ω. Then $[\widehat{K} : K] = N$ and $\mathrm{Gal}(E^{[s]}, K)$ is transitive on $\mathcal{P}(m-1,$
Let R_0 be the set of all nonzero members of R of T-degree less than n. Then, in
notation of (1.2), $(\omega(r)\eta)_{r \in R_0}$ are all the distinct 'nonzero scalar multiples' of η in
(R/s)-vector space $V^{[s]}$, and clearly R_0 is the set of all $\alpha_0 + \alpha_1 T + \cdots + \alpha_{n-1}T^{n-1}$
$(\alpha_0, \alpha_1, \ldots, \alpha_{n-1}) \in \mathrm{GF}(q)^n \setminus \{(0, 0, \ldots, 0)\}$. This gives us D distinct roots of $F(Y)$
where $D = q^n - 1$. Therefore $F(Y) = \widehat{F}^*(Y)\prod_{r \in R_0}(Y - \omega(r)\eta)$ where $\widehat{F}^*(Y) \in$
is of Y-degree $N - D = q^{mn} - q^n > 1$. Now $(\omega(r)\eta)_{r \in R_0}$ is the inverse image of a
in $\mathcal{P}(m-1, q^n)$ under the natural surjection $\mathrm{GF}(q^n)^m \setminus \{0\} \to \mathcal{P}(m-1, q^n)$ obta
by identifying $V^{[s]}$ with $\mathrm{GF}(q^n)^m$ via a basis. It follows that if $\widehat{F}^*(Y)$ is irredu
in $\widehat{K}$ then $\mathrm{Gal}(E^{[s]}, K)$ is two transitive on $\mathcal{P}(m-1, q^n)$. It is also clear that if $F($
$\widehat{F}(Y)\prod_{1 \le i \le D}(Y - \eta_i)$ where $\eta_1, \ldots, \eta_D$ are distinct roots of $F(Y)$ in $\widehat{K}$ and $\widehat{F}(Y) \in$
is irreducible then we must have $\widehat{F}^*(Y) = \widehat{F}(Y)$. Therefore we get the following:

Projective action lemma 2.1. *In the situation of* (1.2) *assume that s is irreducible*
with $s(X_m) \neq 0$ and $m > 1$. Let $F(Y) = Y^{-1}E^{[s]}(Y)$ and note that then $F(Y) \in K[$
of Y-degree $N = q^{mn} - 1$. Assume that $F(Y)$ is irreducible in $K[Y]$ and let $\widehat{K} =$
where η is a root of $F(Y)$ in Ω. Then $[\widehat{K} : K] = N$ and $\mathrm{Gal}(E^{[s]}, K)$ *is transiti*
$\mathcal{P}(m-1, q^n)$. Moreover, if upon letting $D = q^n - 1$ we have $F(Y) = \widehat{F}(Y)\prod_{i \le i \le D}(Y$
where $\eta_1, \ldots, \eta_D$ are distinct roots of $F(Y)$ in $\widehat{K}$ and $\widehat{F}(Y) \in \widehat{K}[Y]$ is irreducible
$\mathrm{Gal}(E^{[s]}, K)$ *is two transitive on $\mathcal{P}(m-1, q^n)$.*

Since Eisenstein polynomials are irreducible, upon taking $E = E^*_{m,q}$ with $F = \widehat{F}$
$K = K^* = \widetilde{K}$ in (2.1), by (1.11) we get (1.12).

3. Coefficient computations

Let $R^{\natural} = \mathrm{GF}(q)[X_1, \ldots, X_m]$. Then clearly for every $\nu > 0$ we have

$$E^{[[\nu]]}(Y) = Y^{q^{m\nu}} + \sum_{i=1}^{m\nu} D_{\nu,i}Y^{q^{m\nu-i}} \quad \text{with } D_{\nu,i} \in R^{\natural}.$$

Also

$$E^{[[1]]}(Y) = E(Y) = Y^{q^m} + \sum_{i=1}^{m} X_i Y^{q^{m-i}}$$

and hence for every integer $\nu > 1$ we have

$$E^{[[\nu]]}(Y) = E(E^{[[\nu-1]]}(Y)) = \left(Y^{q^{m\nu-m}} + \sum_{i=1}^{m\nu-m} D_{\nu-1,i}Y^{q^{m\nu-m-i}}\right)^{q^m}$$

$$+ \sum_{v=1}^{m} X_v \left(Y^{q^{m\nu-m}} + \sum_{w=1}^{m\nu-m} D_{\nu-1,w}Y^{q^{m\nu-m-w}}\right)^{q}$$

$$
\begin{aligned}
&= \left(Y^{q^{m\nu}} + \sum_{i=1}^{m\nu-m} D_{\nu-1,i}^{q^m} Y^{q^{m\nu-i}} \right) \\
&\quad + \left(\sum_{v=1}^{m} X_v Y^{q^{m\nu-v}} + \sum_{v=1}^{m} \sum_{w=1}^{m\nu-m} X_v D_{\nu-1,w}^{q^{m-v}} Y^{q^{m\nu-v-w}} \right)
\end{aligned}
$$

and therefore, for any positive integer i, upon letting

$$
Q(i) = \begin{cases} \text{the set of all pairs of integers } (v, w) \\ \text{with } 1 \le v \le m \text{ and } 1 \le w \le m\nu - m \\ \text{such that } v + w = i \end{cases} \tag{3.3}
$$

we get

$$
D_{\nu,i} = \sum_{(v,w)\in Q(i)} X_v D_{\nu-1,w}^{q^{m-v}} \quad \text{if} \quad m\nu - m < i \le m\nu \tag{3.4}
$$

and

$$
D_{\nu,i} = X_i + D_{\nu-1,i}^{q^m} + \sum_{(v,w)\in Q(i)} X_v D_{\nu-1,w}^{q^{m-v}} \quad \text{if} \quad 1 \le i \le m. \tag{3.5}
$$

By induction we shall show that for every $\nu > 0$ we have

$$
D_{\nu,m\nu} = X_m^{\nu} \tag{3.6}
$$

and

$$
\begin{cases} \text{if } l \text{ is an integer with } 1 \le l < m \\ \text{such that } X_i = 0 \text{ whenever } m - l < i < m \\ \text{then } D_{\nu,i} = 0 \text{ whenever } m\nu - l < i < m\nu \\ \text{and } D_{\nu,m\nu-l} = X_{m-l} \sum_{\lambda=0}^{\nu-1} X_m^{(\nu-1)+\lambda(q^l-1)} \end{cases} \tag{3.7}
$$

and

$$
\begin{cases} \text{if } j \text{ is an integer with } 1 \le j \le m \\ \text{such that } X_i = 0 \text{ whenever } 1 \le i < j \\ \text{then for } 1 \le i \le \min(m, 2j-1) \text{ we have} \\ D_{\nu,i} = \sum_{\lambda=0}^{\nu-1} X_i^{q^{m\lambda}} \\ \text{which we know to be zero if } 1 \le i < j. \end{cases} \tag{3.8}
$$

By (3.2), this is obvious for $\nu = 1$. So let $\nu > 1$ and assume true for $\nu - 1$. Then clearly $Q(m\nu) = \{(m, m\nu - m)\}$, and hence by (3.4) and the $\nu - 1$ version of (3.6) we get

$$
\begin{aligned}
D_{\nu,m\nu} &= X_m D_{\nu-1,m\nu-m} \\
&= X_m X_m^{\nu-1} \\
&= X_m^{\nu}.
\end{aligned}
$$

Likewise, if l is an integer with $1 \le l < m$ such that $X_i = 0$ whenever $m - l < i < m$, then by (3.4) we get

$$
D_{\nu,i} = \begin{cases} X_m D_{\nu-1,m\nu-m-i} & \text{if } m\nu - l < i < m\nu \\ X_m D_{\nu-1,m\nu-m-l} + X_{m-l} D_{\nu-1,m\nu-m}^{q^l} & \text{if } m\nu - l = i \end{cases}
$$

and hence by the $\nu - 1$ versions of (3.6) and (3.7) we get

$$D_{\nu,i} = 0 \quad \text{if} \quad m\nu - l < i < m\nu$$

and

$$\begin{aligned} D_{\nu,m\nu-l} &= X_{m-l}\left(X_m^{(\nu-1)q^l} + \sum_{\lambda=0}^{\nu-1} X_m^{(\nu-2)+\lambda(q^l-1)}\right) \\ &= X_{m-l}\sum_{\lambda=0}^{\nu-1} X_m^{(\nu-1)+\lambda(q^l-1)}. \end{aligned}$$

Similarly, if j is an integer with $1 \le j \le m$ such that $X_i = 0$ whenever $1 \le i < j$, then for all i, v, w with $1 \le i \le 2j - 1$ and $(v, w) \in Q(i)$ we have either $v < j$ or $w < j$, and hence by (3.5) and the $\nu - 1$ version of (3.8) we see that for $1 \le i \le \min(m, 2j - 1)$ we have

$$D_{\nu,i} = X_i + D_{\nu-1,i}^{q^m} = X_i + \left(\sum_{\lambda=0}^{\nu-2} X_i^{q^{m\lambda}}\right)^{q^m} = \sum_{\lambda=0}^{\nu-1} X_i^{q^{m\lambda}}.$$

4. Proof of main Theorem

To prove the Main Theorem 1.14, assume that s is irreducible in R and $n < m$ with $\mathrm{GCD}(m, n) = 1$. Also assume that we are in the generic case of type J^* with $J_n^\dagger \subset J^*$. In view of (1.2.3) and (1.2.4), after identifying $V^{[s]}$ with $\mathrm{GF}(q^n)^m$ via a basis, we have $\mathrm{Gal}(E_{q,m}^{*[s]}, K^*) < \mathrm{GL}(m, q^n)$ and we may regard $\mathrm{Gal}(E_{q,m}^{*[s]}, K^*)$ as acting on $\mathcal{P}(m-1, q^n)$ (where the action is not faithful unless $q^n = 2$). We want to show that $\mathrm{Gal}(E_{q,m}^{*[s]}, K^*) = \mathrm{GL}(m, q^n)$.

Let $N = q^{mn} - 1$ and $\widetilde{F}(Y) = Y^{-1}E_{q,m}^{*[s]}(Y) = \sum_{0\le i\le N} \widetilde{F}_i Y^i$ with $\widetilde{F}_i Y^i \in R^* = k_q[\{X_j : j \in J^*\}]$. Let $D = q^n - 1$. Note that $s = s(T) = \sum_{0\le\nu\le n} s_\nu T^\nu$ with $s_\nu \in \mathrm{GF}(q)$ and $s_n \ne 0$. Let $\overline{k}_q$ be an algebraic closure of k_q in Ω, and let ζ be a root of $s(T)$ in $\overline{k}_q$. Since $s(T)$ is irreducible in R, we get $\zeta^{q^n-1} = 1$ and $s'(\zeta) \ne 0$ where $s'(T)$ is the T-derivative of $s(T)$. Let $\widetilde{R}$ be the localization of $\overline{k}_q[X_n, X_m]$ at the maximal ideal generated by X_n and $X_m - \zeta$. Then $\widetilde{R}$ is two dimensional regular local domain with maximal ideal $\widetilde{M} = (X_n, X_m - \zeta)\widetilde{R}$.

For a moment suppose that $k_q = \overline{k}_q$ and $J_n^\dagger = J^*$, and let us write $K^\dagger$ for K^* and $E_{m,q}^\dagger$ for $E_{m,q}^*$. Now by (1.6) and (1.7) we see that $\widetilde{F}(Y)$ is Eisenstein relative to $(\widetilde{R}, \widetilde{M})$, and the determinantal map $\mathrm{Gal}(E_{m,q}^{\dagger[s]}, K^\dagger) \to \mathrm{GF}(q^n) \setminus \{0\}$ is surjective. By (1.2.1) we have

$$\widetilde{F}_0 = s(X_m).$$

By taking $l = n$ in (3.7) we see that

$$\widetilde{F}_i = 0 \text{ for } 1 \le i \le D - 1$$

and

$$\widetilde{F}_D = X_{m-n}\Theta(X_m),$$

where

$$\Theta(X_m) = \sum_{0\le\nu\le n} s_\nu \sum_{0\le\lambda\le\nu-1} X_m^{(\nu-1)+\lambda(q^n-1)}.$$

Since $\zeta^{q^n-1} = 1$, we get

$$\sum_{0\le\lambda\le\nu-1} \zeta^{(\nu-1)+\lambda(q^n-1)} = \nu\zeta^{\nu-1}$$

and therefore

$$\Theta(\zeta) = \sum_{0\le\nu\le n} s_\nu\nu\zeta^{\nu-1} = s'(\zeta) \neq 0.$$

It follows that

$$\widetilde{F}_D \not\in \widetilde{M}^2 + \widetilde{F}_0\widetilde{R}$$

and hence by (1.12) we conclude that $\mathrm{Gal}(E_{m,q}^{\dagger[s]}, K^\dagger)$ is two transitive on $\mathcal{P}(m-1, q^n)$. If $n > 1$ then by (1.13) we see that $\mathrm{SL}(m, q^n) < \mathrm{Gal}(E_{m,q}^{\dagger[s]}, K^\dagger)$ and hence, because the determinantal map $\mathrm{Gal}(E_{m,q}^{\dagger[s]}, K^\dagger) \to \mathrm{GF}(q^n)\setminus\{0\}$ is surjective, we must have $\mathrm{Gal}(E_{m,q}^{\dagger[s]}, K^\dagger) = \mathrm{GL}(m, q^n)$. If $n = 1$ then by (1.3) we get $\mathrm{Gal}(E_{m,q}^{\dagger[s]}, K^\dagger) = \mathrm{GL}(m, q^n)$. Thus in both the cases we have $\mathrm{Gal}(E_{m,q}^{\dagger[s]}, K^\dagger) = \mathrm{GL}(m, q^n)$.

Now let us return to the case when the field k_q need not be algebraically closed. Since $\overline{k}_q$ is an overfield of k_q and $E_{m,q}^{\dagger[s]}$ is obtained from $E_{m,q}^{*[s]}$ by putting $X_i = 0$ for all $i \in J^* \setminus J_n^\dagger$, in view of the extension principle (cf. p. 93 of [Ab2]) and the specialization principle (cf. p. 1894 of [AbL]), see that $\mathrm{Gal}(E_{m,q}^{\dagger[s]}, K^\dagger) < \mathrm{Gal}(E_{m,q}^{*[s]}, K^*)$. Therefore $\mathrm{Gal}(E_{m,q}^{*[s]}, K^\dagger) = \mathrm{GL}(m, q^n)$.

5. Concluding remarks

Let us end with some remarks on motivation and philosophy.

Remark 5.1 (*Algebraic fundamental groups*). The algebraic fundamental group $\pi_A(L_k)$ of the affine line L_k over a field k is defined to be the set of all Galois groups of finite unramified Galois coverings of the affine line L_k over k. Similarly we define $\pi_A(L_{k,t})$ for $L_{k,t} = L_k$ punctured at t points, and more generally we define $\pi_A(C_{g,w})$ for a nonsingular projective genus g curve C over k punctured at $w + 1$ points. Let $Q(p)$ be the set of all quasi-p groups, i.e., finite groups G such that $G = p(G)$ where $p(G)$ is the subgroup of G generated by all of its p-Sylow subgroups, and more generally let $Q_t(p)$ be the set of all quasi-(p, t) groups, i.e., those G for which $G/p(G)$ is generated by t generators. In [Ab1], as *geometric conjectures* it was predicted that if k is an algebraically closed field of characteristic p then $\pi_A(L_k) = Q(p)$, and more generally $\pi_A(L_{k,t}) = Q_t(p)$ and $\pi_A(C_{g,w}) = Q_{2g+w}(p)$. In 1994, these were settled affirmatively by Raynaud [Ray] and Harbater [Har]. For higher dimensional versions of the geometric conjectures see [Ab5]. Then, mostly inspired by Fried–Guralnick–Saxl [FGS] and Guralnick–Saxl [GuS], we turned our attention to coverings defined over finite fields. In [Ab6] this led to the *arithmetical question* asking whether $\pi_A(L_{\mathrm{GF}(q)}) = Q_1(p)$, the philosophy behind this being that dropping from an algebraically closed field to a finite field is somewhat like adding a branch point. In particular we may ask whether $\pi_A(L_{k,1})$ contains $Q_1(p)$ where

k is an overfield of GF(q). As indicated in the introduction, in doing this arithmetic problem, the linear groups got bloated towards their semilinear versions and the attempt t unbloat them led us to generalized iterations.

Remark 5.2 (*Division points and Drinfeld modules*). The generalized iterations themselve came out of the theory of Drinfeld modules as developed in his paper [Dri]. This work c Drinfeld seems to have been inspired by Serre's work [Se1] on division points of ellipti curves which was later generalized by him [Se2] to abelian varieties. In turn, our descriptio of the module $E^{[s]}$ in (1.2) is based on the ideas of Drinfeld modules. For a discussion c Drinfeld modules and their relationship with division points of elliptic curves and abelia varieties see Goss [Gos]. Very briefly, the roots of the separable vectorial q-polynomi E of q-degree $2m$ exhibited in (1.1) form a $2m$ dimensional GF(q)-vector-space on whic the Galois group of E acts. The said Galois group also acts on the roots of $E^{[s]}$ discusse in (1.2) which are the analogues of 's-division points of E.' Indeed, we have used th letter E to remind ourselves of elliptic curves in case of $m = 1$ and more generally of 2 dimensional abelian varieties. We hope that the present descent principle can someho be 'lifted' to characteristic zero. Before that it should be made to work in the symplecti situation, the bloated semilinear equations for which can be found in [Ab7]. Prior to tha the GL work of this paper should be completed.

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Obstructions to Clifford system extensions of algebras

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Abstract. In this paper we do phrase the obstruction for realization of a generalized group character, and then we give a classification of Clifford systems in terms of suitable low-dimensional cohomology groups.

Keywords. Clifford system; character; cohomology groups; obstructions.

1. Introduction

The problem of Clifford system extensions resides in the classification and the construction of the manifold of all Clifford systems over a commutative ring k, $S = \bigoplus_{\sigma \in G} S_\sigma$, the type being given group G and with 1-component S_1 isomorphic to a given k-algebra R. Each such G-graded Clifford system extension realizes a *generalized collective character* of G in R, that is a group homomorphism $\Phi : G \to \mathrm{Pic}_k(R)$ of G into the group of isomorphism classes of invertible left $R \otimes_k R^\circ$-modules, and this leads to a problem of obstruction. When a generalized collective character is specified, it is possible that no Clifford system extensions realizing the specified homomorphism can exist. The main result in this paper is to obtain a necessary and sufficient condition for the existence of such a Clifford system extension, formulated in terms of a certain 3-dimensional group cohomology class $T(\Phi)$, referred to here as the *Teichmüller obstruction* of Φ. The construction of $T(\Phi)$ is closely analogous to a construction by Kanzaki [9], for a description of the Chase-Harrison-Rosenberg seven term exact sequence [2] about the Brauer group. In the case where a generalized collective character Φ has an extension, the manifold of such strongly graded extensions is shown as a principal and homogeneous space under a 2nd cohomology group.

This paper has been strongly influenced by the work on the classification of crossed-product rings by Hacque in [7, 8], where he makes a systematic analysis of the important phenomenon bound to the existence of obstructions. Clifford systems, also called *strongly graded algebras*, are a direct generalization of crossed product algebras and they were introduced and applied by Dade in several important papers [3, 4], where he develops Clifford's theory axiomatically, and which can be referred to for general background.

In §2, we state a minimum of needed notation and terminology. Section 3 contains the main results of the paper, namely the construction of the Teichmüller obstruction map and the obstruction theorems. We conclude in §4 by exhibiting a non-realizable collective character.

2. Clifford system extensions and generalized collective characters

Throughout the paper k is a commutative ring with identity and G is a group.

A *G-graded Clifford system over k S* is a k-algebra with identity, also denoted by together with a family of k-submodules $S_\sigma, \sigma \in G$, such that $S = \bigoplus_{\sigma \in G} S_\sigma$ and $S_\sigma S_\tau$ $S_{\sigma\tau}$ for all $\sigma, \tau \in G$, where the product $S_\sigma S_\tau$ consists of all finite sums of ring produc xy of elements $x \in S_\sigma$ and $y \in S_\tau$. Note that the 1-component S_1 is a k-subalgebra of and each σ-component $S_\sigma, \sigma \in G$, is a two-sided S_1-submodule of S.

By a Clifford system extension of a k-algebra R we mean a Clifford system k-algebra whose 1-component S_1 is isomorphic to R. More precisely, we have the following:

DEFINITION 2.1

Let R be a k-algebra and G a group. A G-graded Clifford system extension of R is a pa (S, j), where $S = \bigoplus_{\sigma \in G} S_\sigma$ is a G-graded Clifford system k-algebra and $j : R \hookrightarrow S$ is k-algebra embedding with $j(R) = S_1$.

If (S, j), (S', j') are two G-graded Clifford system extensions of R, by a morphism b tween them $f : (S, j) \to (S', j')$, we mean a grade-preserving k-algebra homomorphis $f : S \to S'$ that respects the embeddings of R, that is, such that $fj = j'$.

The most striking example is the group algebra $R[G]$, but also crossed products of and G yield examples of G-graded Clifford system extensions of a k-algebra R.

From ([4], Corollary 2.10) it follows that any Clifford system extension morphis $f : (S, j) \to (S', j')$ is necessarily an isomorphism. Therefore the existence of a mo phism is an equivalence relation between G-graded Clifford system extensions of R an in this case, we usually say that the extensions are equivalent. Then

$$\mathrm{Cliff}_k(G, R) \qquad (\!$$

denotes the set of equivalence classes of G-graded Clifford system extensions of the algebra R.

If (S, j) is a G-graded Clifford system extension of R, then each S_σ, $\sigma \in G$, is invertible $R \otimes_k R^\circ$-module and, for every $\sigma, \tau \in G$, the canonical morphism $S_\sigma \otimes_R S_\tau$ – $S_{\sigma\tau}$, $x_\sigma \otimes x_\tau \mapsto x_\sigma x_\tau$, is an $R \otimes_k R^\circ$-isomorphism. Hence, there is a canonical map

$$\chi : \mathrm{Cliff}_k(G, R) \longrightarrow \mathrm{Hom}_{Gp}(G, \mathrm{Pic}_k(R)), \qquad (2$$

where $\mathrm{Hom}_{Gp}(G, \mathrm{Pic}_k(R))$ is the set of group homomorphisms of G into $\mathrm{Pic}_k(R)$, th group of isomorphism classes of invertible $R \otimes_k R^\circ$-modules, which carries the class a G-graded Clifford system extension (S, j) to the group homomorphism $\chi_{[S,\,j]} : G$ – $\mathrm{Pic}_{k(R)}$, given by

$$\chi_{[S,\,j]}(\sigma) = [S_\sigma], \quad \sigma \in G. \qquad (3$$

We have the Baer notion of *Kollectivcharakter* in mind, and we define a *genera ized collective character* of the group G in the k-algebra R as a group homomorphis $\Phi : G \to \mathrm{Pic}_k(R)$. Let us recall the exact group sequence ([1], Chapter II, (5.4)),

$$1 \to \mathrm{InAut}(R) \longrightarrow \mathrm{Aut}_k(R) \xrightarrow{\delta} \mathrm{Pic}_k(R), \qquad (4$$

in which δ maps a k-algebra automorphism of R, $\alpha \in \mathrm{Aut}_k(R)$, to the class of the invertibl $R \otimes_k R^\circ$-module R_α, which is the same left R-module as R with right action given b $x \cdot y = x\alpha(y)$, $x, y \in R$. Then, there is a canonical embedding $\mathrm{Out}_k(R) \overset{\delta}{\hookrightarrow} \mathrm{Pic}_k(R)$, the group of *outer automorphisms* of the k-algebra R, $\mathrm{Out}_k(R) = \mathrm{Aut}_k(R)/\mathrm{InAut}(R)$, int the Picard group $\mathrm{Pic}_k(R)$. A group homomorphism $\Phi : G \to \mathrm{Out}_k(R)$ has been called *collective character* (cf. Hacque [7, 8]); so that collective characters of G in R are those generalized ones factoring through the embedding $\mathrm{Out}_k(R) \hookrightarrow \mathrm{Pic}_k(R)$. Of course, by *character* we understand a group homomorphism $G \to \mathrm{Aut}_k(R)$.

Hence $\mathrm{Hom}_{Gp}(G, \mathrm{Pic}_k(R))$ is the set of generalized collective characters of G in R, and the map χ associates with each equivalence class of G-graded Clifford system extensions of R a generalized collective character. We refer to a generalized collective character $\Phi : G \to \mathrm{Pic}_k(R)$ as *realizable* if it is in the image of χ, that is, if it is induced as explained above from a G-graded Clifford system extension of R. The map χ produces a partitioning of the set of equivalence classes of G-graded Clifford system extensions of R,

$$\mathrm{Cliff}_k(G, R) = \coprod_{\Phi} \mathrm{Cliff}_k(G, R; \Phi), \tag{5}$$

where, for any generalized collective character $\Phi \in \mathrm{Hom}_{Gp}(G, \mathrm{Pic}_k(R))$, we denote by $\mathrm{Cliff}_k(G, R; \Phi) = \chi^{-1}(\Phi)$ the fiber of χ over Φ . Thus a generalized collective character Φ is realizable if the set $\mathrm{Cliff}_k(G, R; \Phi)$ is not empty. We refer to $\mathrm{Cliff}_k(G, R; \Phi)$ as the *set of equivalence classes of realizations* of the generalized collective character Φ.

3. The Teichmüller cocycle and the obstruction theorems

If R is a k-algebra, let $C(R) = \{r \in R \mid rx = xr, x \in R\}$ denote its center. Then $C(R)$ is a k-algebra whose group of units we denote by $C(R)^*$.

We will often use the following elementary fact, which is a consequence of ([1], Chapter II, (3.5)).

Lemma 3.1. *If P, Q are invertible $R \otimes_k R^\circ$-modules, then for any two $R \otimes_k R^\circ$-isomorphisms $\alpha, \beta : P \to Q$, there exists a unique $u \in C(R)^*$ such that $\beta = u\alpha = \alpha u$ (i.e., $\beta(x) = u\alpha(x) = \alpha(ux)$ for all $x \in P$).*

Proof. Given $\alpha : P \to Q$, an $R \otimes_k R^\circ$-isomorphism, the map $C(R)^* \to \mathrm{Isom}_{R\otimes_k R^\circ}(P, Q)$, $u \mapsto u\alpha$, is bijective since it can be obtained as the composite map of the canonical group isomorphism $C(R)^* \cong \mathrm{Aut}_{R\otimes_k R^\circ}(R)$, the group isomorphism $- \otimes_R P : \mathrm{Aut}_{R\otimes_k R^\circ}(R) \cong \mathrm{Aut}_{R\otimes_k R^\circ}(P)$ and the bijection induced by α, $\alpha_* : \mathrm{Aut}_{R\otimes_k R^\circ}(P) \cong \mathrm{Isom}_{R\otimes_k R^\circ}(P, Q)$. □

If P is any invertible $R \otimes_k R^\circ$-module and $u \in C(R)^*$, since $x \mapsto xu$ is an $R \otimes_k R^\circ$-automorphism of P, there exists a unique element $\alpha_P(u) \in C(R)^*$ such that $\alpha_P(u)x = xu$ for all $x \in P$. Clearly $\alpha_P : C(R)^* \to C(R)^*$ is an automorphism and

$$\mathrm{Pic}_k(R) \xrightarrow{\rho} \mathrm{Aut}(C(R)^*), \quad \rho([P]) = \alpha_P \tag{6}$$

is a group homomorphism (note that ρ is the restriction to $C(R)^*$ of Bass' homomorphism $h : \mathrm{Pic}_k(R) \to \mathrm{Aut}_k(C(R))$ ([1], Chap. II, (5.4)). Hence $C(R)^*$ is a $\mathrm{Pic}_k(R)$-module.

By composition with the homomorphism (6) we have for any group G a map

$$\mathrm{Hom}_{Gp}(G, \mathrm{Pic}_k(R)) \xrightarrow{\rho_*} \mathrm{Hom}_{Gp}(G, \mathrm{Aut}(C(R)^*)) \tag{7}$$

that to each generalized collective character of G in the k-algebra R, $\Phi : G \to \mathrm{Pic}_k(R)$, associates a character $\Phi^* = \rho\Phi : G \to \mathrm{Aut}(C(R)^*)$ from group G in the abelian group $C(R)^*$. Of course, the set of characters $\mathrm{Hom}_{Gp}(G, \mathrm{Aut}(C(R)^*))$ is the set of G-module structures on $C(R)^*$. Hence every generalized collective character $\Phi : G \to \mathrm{Pic}_k(R)$, of

G in R, determines a G-module structure on $C(R)^*$ for which the corresponding G-action of an element $\sigma \in G$ on an element $u \in C(R)^*$ is given by ${}^{\sigma}u = \alpha_P(u)$ for any $P \in \Phi(\sigma)$. In particular,

$$xu = {}^{\sigma}ux \tag{8}$$

for any $\sigma \in G$, $u \in C(R)^*$, $x \in P$ and $P \in \Phi(\sigma)$. We will denote by $H^n_{\Phi}(G, C(R)^*)$, $n \geq 0$, the nth cohomology group of G with coefficients in this G-module.

We will now show how every generalized collective character $\Phi : G \to \mathrm{Pic}_k(R)$ has a cohomology class $T(\Phi) \in H^3_{\Phi}(G, C(R)^*)$ canonically associated with it, whose construction has several precedents: the *Teichmüller cocycle* homomorphism $H^0(G, Br(R)) \to H^3(G, R^*)$ [10,6], defined when R/k is a field Galois extension with group G; the *Eilenberg-Mac Lane obstruction* defined by a G-kernel, defined in [5] for the study of group extensions with a non-abelian kernel; the description by Kanzaki [9] of the homomorphism $H^1(G, \mathrm{Pic}_R(R)) \to H^3(G, R^*)$, in the Chase-Harrison-Rosenberg seven term exact sequence [2], about the Brauer group relative to a Galois extension of commutative rings R/k; the *Teichmüller obstruction* associated to a collective character $\Phi : G \to \mathrm{Out}(R)$, by Hacque in [7,8] for the study of obstructions to the existence of crossed product rings.

Let $\Phi : G \to \mathrm{Pic}_k(R)$ be a generalized collective character of a group G in a k-algebra R. In each isomorphism class $\Phi(\sigma) \in \mathrm{Pic}_k(R)$, choose an invertible $R \otimes_k R^{\circ}$-module $P_\sigma \in \Phi(\sigma)$; in particular, select $P_1 = R$. Since Φ is a homomorphism, the modules $P_\sigma \otimes_R P_\tau$ and $P_{\sigma\tau}$ must be $R \otimes_k R^{\circ}$-isomorphic for each pair $\sigma, \tau \in G$. Then we can select $R \otimes_k R^{\circ}$-isomorphisms

$$\Gamma_{\sigma,\tau} : P_\sigma \otimes_R P_\tau \to P_{\sigma\tau} \tag{9}$$

with $\Gamma_{\sigma,1}(x \otimes r) = xr$ and $\Gamma_{1,\sigma}(r \otimes x) = rx$, $r \in R, x \in P_\sigma$.

For any three elements $\sigma, \tau, \gamma \in G$, the diagram

$$\begin{tikzcd}
P_\sigma \otimes_R P_\tau \otimes_R P_\gamma \arrow[r, "\Gamma_{\sigma,\tau} \otimes P_\gamma"] \arrow[d, "P_\sigma \otimes \Gamma_{\tau,\gamma}"'] & P_{\sigma\tau} \otimes_R P_\gamma \arrow[d, "\Gamma_{\sigma\tau,\gamma}"] \\
P_\sigma \otimes_R P_{\tau\gamma} \arrow[r, "\Gamma_{\sigma,\tau\gamma}"'] & P_{\sigma\tau\gamma}
\end{tikzcd} \tag{10}$$

need not be commutative but, by Lemma 3.1, there exists a unique element $T^{\Phi}_{\sigma,\tau,\gamma} \in C(R)^*$ such that

$$\Gamma_{\sigma\tau,\gamma}(\Gamma_{\sigma,\tau} \otimes P_\gamma) = T^{\Phi}_{\sigma,\tau,\gamma}(\Gamma_{\sigma,\tau\gamma}(P_\sigma \otimes \Gamma_{\tau,\gamma})). \tag{11}$$

Clearly $T^{\Phi}_{1,\tau,\gamma} = T^{\Phi}_{\sigma,1,\gamma} = T^{\Phi}_{\sigma,\tau,1} = 1$ so that the choices of P_σ and $\Gamma_{\sigma,\tau}$ determine a normalized 3-dimensional cochain of G with coefficients in $C(R)^*$.

Lemma 3.2. *The cochain $T = T^{\Phi} : G^3 \to C(R)^*$ is a 3-cocycle of G with coefficients in the G-module $C(R)^*$.*

Proof. We must prove the identity

$$T_{\sigma,\tau,\gamma}T_{\sigma,\tau\gamma,\delta}\ {}^{\sigma}T_{\tau,\gamma,\delta} = T_{\sigma\tau,\gamma,\delta}T_{\sigma,\tau,\gamma\delta} \tag{12}$$

for any $(\sigma, \tau, \gamma, \delta) \in G^4$. To see this, we compute the isomorphism

$$J = (\Gamma_{\sigma\tau\gamma,\delta})(\Gamma_{\sigma\tau,\gamma} \otimes 1)(\Gamma_{\sigma,\tau} \otimes 1 \otimes 1) : P_\sigma \otimes_R P_\tau \otimes_R P_\gamma \otimes_R P_\delta \longrightarrow P_{\sigma\tau\gamma\delta}$$

in two ways. On one hand, for all $x \in P_\sigma$, $y \in P_\tau$, $z \in P_\gamma$ and $t \in P_\delta$, we have

$$\begin{aligned}
J(x \otimes y \otimes z \otimes t) &= \Gamma_{\sigma\tau\gamma,\delta}(\Gamma_{\sigma\tau,\gamma}(\Gamma_{\sigma,\tau}(x \otimes y) \otimes z) \otimes t) \\
&\overset{(11)}{=} T_{\sigma,\tau,\gamma}\Gamma_{\sigma\tau\gamma,\delta}(\Gamma_{\sigma,\tau\gamma}(x \otimes \Gamma_{\tau,\gamma}(y \otimes z)) \otimes t) \\
&\overset{(11)}{=} T_{\sigma,\tau,\gamma}T_{\sigma,\tau\gamma,\delta}\Gamma_{\sigma,\tau\gamma\delta}(x \otimes \Gamma_{\tau\gamma,\delta}(\Gamma_{\tau,\gamma}(y \otimes z) \otimes t)) \\
&\overset{(11)}{=} T_{\sigma,\tau,\gamma}T_{\sigma,\tau\gamma,\delta}\Gamma_{\sigma,\tau\gamma\delta}(x \otimes T_{\tau,\gamma,\delta}\Gamma_{\tau,\gamma\delta}(y \otimes \Gamma_{\gamma,\delta}(z \otimes t)) \\
&\overset{(8)}{=} T_{\sigma,\tau,\gamma}T_{\sigma,\tau\gamma,\delta}\ {}^{\sigma}T_{\tau,\gamma,\delta}\Gamma_{\sigma,\tau\gamma\delta}(x \otimes (\Gamma_{\tau,\gamma\delta}(y \otimes \Gamma_{\gamma,\delta}(z \otimes t))),
\end{aligned}$$

and on the other hand

$$\begin{aligned}
J(x \otimes y \otimes z \otimes t) &= T_{\sigma\tau,\gamma,\delta}\Gamma_{\sigma\tau,\gamma\delta}(\Gamma_{\sigma,\tau}(x \otimes y) \otimes \Gamma_{\gamma,\delta}(z \otimes t)) \\
&\overset{(11)}{=} T_{\sigma\tau,\gamma,\delta}T_{\sigma,\tau,\gamma\delta}\Gamma_{\sigma,\tau\gamma\delta}(x \otimes (\Gamma_{\tau,\gamma\delta}(y \otimes \Gamma_{\gamma,\delta}(z \otimes t)))
\end{aligned}$$

and comparing the two expressions together with Lemma 3.1 gives (12). □

We now observe the effect of different choices of P_σ and $\Gamma_{\sigma,\tau}$ in the construction of the 3-cocycle T^Φ for a given generalized collective character $\Phi : G \to \mathrm{Pic}_k(R)$.

Lemma 3.3. (i) *If the choice of* Γ *in* (9) *is changed, then* T^Φ *is changed to a cohomologous cocycle. By suitably changing* Γ, T^Φ *may be changed to any cohomologous cocycle.*

(ii) *If the choice of the invertible* $R \otimes_k R^\circ$*-modules* P *is changed, then a suitable new selection of* Γ *leaves cocycle* T^Φ *unaltered.*

Proof. (i) By Lemma 3.1, any other choice of $\Gamma_{\sigma,\tau}$ in (9) has the form $\Gamma'_{\sigma,\tau} = h_{\sigma,\tau}\Gamma_{\sigma,\tau}$, where $h : G^2 \to C(R)^*$ is a normalized 2-cochain of G in $C(R)^*$.

For any $\sigma, \tau, \gamma \in G$ we have the following expressions for the isomorphism $J = \Gamma'_{\sigma\tau,\gamma}(\Gamma'_{\sigma,\tau} \otimes P_\gamma)$ from $P_\sigma \otimes_R P_\tau \otimes_R P_\gamma$ onto $P_{\sigma\tau\gamma}$:

$$\begin{aligned}
J(x \otimes y \otimes z) &= \Gamma'_{\sigma\tau,\gamma}(\Gamma'_{\sigma,\tau}(x \otimes y) \otimes z) \\
&\overset{(8)}{=} h_{\sigma\tau,\gamma}\Gamma_{\sigma\tau,\gamma}(h_{\sigma,\tau}\Gamma_{\sigma,\tau}(x \otimes y) \otimes z) \\
&= h_{\sigma\tau,\gamma}h_{\sigma,\tau}\Gamma_{\sigma\tau,\gamma}(\Gamma_{\sigma,\tau}(x \otimes y) \otimes z) \\
&\overset{(11)}{=} h_{\sigma\tau,\gamma}h_{\sigma,\tau}f_{\sigma,\tau,\gamma}\Gamma_{\sigma,\tau\gamma}(x \otimes \Gamma_{\tau,\gamma}(y \otimes z))
\end{aligned}$$

and

$$\begin{aligned}
J(x \otimes y \otimes z) &\overset{(11)}{=} T'_{\sigma,\tau,\gamma}\Gamma'_{\sigma,\tau\gamma}(x \otimes \Gamma'_{\tau,\gamma}(y \otimes z)) \\
&= T'_{\sigma,\tau,\gamma}h_{\sigma,\tau\gamma}\Gamma_{\sigma,\tau\gamma}(x \otimes h_{\tau,\gamma}\Gamma_{\tau,\gamma}(y \otimes z)) \\
&\overset{(11)}{=} T'_{\sigma,\tau,\gamma}h_{\sigma,\tau\gamma}\ {}^{\sigma}h_{\tau,\gamma}\Gamma_{\sigma,\tau\gamma}(x \otimes \Gamma_{\tau,\gamma}(y \otimes z))
\end{aligned}$$

and comparing the two expressions together with Lemma 3.1 yield

$$T'_{\sigma,\tau,\gamma}h_{\sigma,\tau\gamma}\ {}^{\sigma}h_{\tau,\gamma} = h_{\sigma\tau,\gamma}h_{\sigma,\tau}T_{\sigma,\tau,\gamma}, \tag{13}$$

an identity that asserts that the 3-cocycles T and T' are cohomologous.

(ii) If $P'_\sigma \in \Phi(\sigma)$, $\sigma \in G$, is another selection of invertible $R \otimes_k R^\circ$-modules, then we can select $R \otimes_k R^\circ$-isomorphisms $\varphi_\sigma : P'_\sigma \to P_\sigma$ and choose $\Gamma'_{\sigma,\tau} : P'_\sigma \otimes_R P'_\tau \to P'_{\sigma\tau}$, the isomorphism making the following diagram commutative:

$$\begin{tikzcd}
P_\sigma \otimes_R P_\tau \otimes_R P_\gamma \arrow[r, "\Gamma_{\sigma,\tau}\otimes P_\gamma"] \arrow[d, "P_\sigma\otimes\Gamma_{\tau,\gamma}"'] & P_{\sigma\tau} \otimes_R P_\gamma \arrow[d, "\Gamma_{\sigma\tau,\gamma}"] \\
P_\sigma \otimes_R P_{\tau\gamma} \arrow[r, "\Gamma_{\sigma,\tau\gamma}"'] & P_{\sigma\tau\gamma}
\end{tikzcd} \tag{14}$$

for each $\sigma, \tau \in G$. Thus we have

$$\begin{aligned}
\varphi_{\sigma\tau\gamma}(\Gamma'_{\sigma\tau,\gamma}(\Gamma'_{\sigma,\tau}(x\otimes y)\otimes z)) &= \Gamma_{\sigma\tau,\gamma}(\varphi_{\sigma\tau}(\Gamma'_{\sigma,\tau}(x\otimes y))\otimes\varphi_\gamma(z)) \\
&= \Gamma_{\sigma\tau,\gamma}((\Gamma_{\sigma,\tau}(\varphi_\sigma(x)\otimes\varphi_\tau(y))\otimes\varphi_\gamma(z)) \\
&= T_{\sigma,\tau,\gamma}\Gamma_{\sigma,\tau\gamma}(\varphi_\sigma(x)\otimes\Gamma_{\tau,\gamma}(\varphi_\tau(y))\otimes\varphi_\gamma(z))) \\
&= T_{\sigma,\tau,\gamma}\Gamma_{\sigma,\tau\gamma}(\varphi_\sigma)(x)\otimes\varphi_{\tau\gamma}(\Gamma'_{\tau,\gamma}(y\otimes z)) \\
&= T_{\sigma,\tau,\gamma}\varphi_{\sigma\tau\gamma}(\Gamma'_{\sigma,\tau\gamma}(x\otimes\Gamma'_{\tau,\gamma}(y\otimes z)) \\
&= \varphi_{\sigma\tau\gamma}(T_{\sigma,\tau,\gamma}\Gamma'_{\sigma,\tau\gamma}(x\otimes\Gamma'_{\tau,\gamma}(y\otimes z))),
\end{aligned}$$

for all $x \in P'_\sigma$, $y \in P'_\tau$ and $z \in P'_\gamma$.

Hence $\Gamma'_{\sigma\tau,\gamma}(\Gamma'_{\sigma,\tau}(x\otimes y)\otimes z) = T_{\sigma,\tau,\gamma}\Gamma'_{\sigma,\tau\gamma}(x\otimes\Gamma'_{\tau,\gamma}(y\otimes z))$ and the 3-cocycle T is unchanged. □

These lemmas show that each generalized collective character $\Phi : G \to \mathrm{Pic}_k(R)$ determines in invariant fashion a 3-dimensional cohomology class $T(\Phi) = [T^\Phi] \in H^3_\Phi(G, C(R)^*)$. We refer to the map $\Phi \mapsto T(\Phi)$ as the *Teichmüller obstruction map* (see [8] for background).

Next we prove the main objective of this paper.

Theorem 3.4. *A generalized collective character* $\Phi : G \to \mathrm{Pic}_k(R)$ *is realizable if and only if its Teichmüller obstruction* $T(\Phi) \in H^3_\Phi(G, C(R)^*)$ *vanishes.*

Proof. Suppose first that $(S = \bigoplus_{\sigma\in G} S_\sigma,\ j)$ is a realization of Φ. Then, in the construction of the Teichmüller 3-cocycle T^Φ of G with coefficients in the G-module $C(R)^*$, one can take just the invertible $R\otimes_k R^\circ$-modules S_σ, $\sigma \in G$, $\sigma \neq 1$, and the canonical $R\otimes_k R^\circ$-isomorphisms $\Gamma_{\sigma,\tau} : S_\sigma \otimes_R S_\tau \to S_{\sigma\tau}$, $\Gamma_{\sigma,\tau}(x\otimes y) = xy$, $\Gamma_{\sigma,1}(x\otimes r) = xj(r)$ and $\Gamma_{1,\sigma}(r\otimes x) = j(r)x$ for each $\sigma, \tau \in G$. Since multiplication in the k-algebra S is associative $\Gamma_{\sigma\tau,\gamma}(\Gamma_{\sigma,\tau}\otimes S_\gamma) = \Gamma_{\sigma,\tau\gamma}(S_\sigma\otimes\Gamma_{\tau,\gamma})$ for all $\sigma, \tau, \gamma \in G$, and then $T^\Phi_{\sigma,\tau,\gamma} = 1$ in (11). Therefore, $T(\Phi) = [T^\Phi]$ is the zero cohomology class.

Conversely, suppose that the generalized collective character Φ has a vanishing cohomology class $f(\Phi)$. Select any invertible $R\otimes_k R^\circ$-modules $P_\sigma \in \Phi(\sigma)$, $\sigma \in G$, with $P_1 = R$. By Lemma 3.3(i), there is a choice of $R\otimes_k R^\circ$-isomorphisms $\Gamma_{\sigma,\tau} : P_\sigma\otimes_R P_\tau \to P_{\sigma\tau}$ with $\Gamma_{1,\sigma}$ and $\Gamma_{\sigma,1}$ the canonical ones, such that the Teichmüller 3-cocycle T^Φ is identically 1. This means that (10) is commutative for any $\sigma, \tau, \gamma \in G$. Hence, the family $(P_\sigma, \Gamma_{\sigma,\tau})$ gives rise to a *generalized crossed product algebra* in the sense of Kanzaki [9] $\Delta = \bigoplus_{\sigma\in G} P_\sigma$, where the product of elements $x \in P_\sigma$ and $y \in P_\tau$ is defined by $xy = \Gamma_{\sigma,\tau}(x\otimes y)$, which is a G-graded Clifford system over k, extension of R by the canonical injection $j : R = P_1 \hookrightarrow \Delta$. Since $\chi_{[\Delta,j]}(\sigma) = [P_\sigma] = \Phi(\sigma)$, Φ is realized, that is, $\mathrm{Cliff}_k(G, R; \Phi) \neq \emptyset$. □

Now, to complete the classification of G-graded Clifford system extensions of a k-algebra R, we have the following result.

Theorem 3.5. *If a generalized collective character $\Phi : G \to \mathrm{Pic}_k(R)$ is realizable, then the set of isomorphism classes of realizations of Φ, $\mathrm{Cliff}_k(G, R; \Phi)$, is a principal homogeneous space under the abelian group $H^2_\Phi(G, C(R)^*)$. In particular, there is a (non-canonical) bijection*

$$\mathrm{Cliff}_k(G, R; \Phi) \cong H^2_\Phi(G, C(R)^*).$$

Proof. We will describe an action

$$H^2_\Phi(G, C(R)^*) \times \mathrm{Cliff}_k(G, R; \Phi) \longrightarrow \mathrm{Cliff}_k(G, R; \Phi) \tag{15}$$

below.

Let $h : G^2 \to C(R)^*$ be a normalized 2-cocycle representative of an element $[h] \in H^2_\Phi(G, C(R)^*)$ and $(S = \bigoplus_{\sigma \in G} S_\sigma,\ j : R \cong S_1)$ be a G-graded Clifford system extension of R, representative of an element $[S,\ j] \in \mathrm{Cliff}_k(G, R; \Phi)$. A new G-graded Clifford system extension of R, $({}^hS,\ j)$ is defined by considering the k-algebra ${}^hS$ which is the same G-graded k-algebra as $S = \bigoplus_{\sigma \in G} S_\sigma$, where the product of elements $x \in S_\sigma$ and $y \in S_\tau$ is now defined by

$$x \star y = j(h_{\sigma,\tau})xy.$$

Since for any $x \in S_\sigma$, $y \in S_\tau$ and $z \in S_\gamma$ we have

$$\begin{aligned} x \star (y \star z) &= x \star (j(h_{\sigma,\tau})yz) = j(h_{\sigma,\tau\gamma})xj(h_{\tau,\gamma})yz \\ &\overset{(8)}{=} j(h_{\sigma,\tau\gamma})j({}^{\sigma}h_{\tau,\gamma})xyz = j(h_{\sigma\tau,\gamma})j(h_{\sigma,\tau})xyz \\ &= j(h_{\sigma\tau,\gamma})(x \star y)z = (x \star y) \star z, \end{aligned}$$

the multiplication is associative and so ${}^hS$ is a k-algebra. Furthermore, $S_\sigma \star S_\tau = j(h_{\sigma,\tau})S_\sigma S_\tau = j(h_{\sigma,\tau})S_{\sigma\tau} = S_{\sigma\tau}$, since $h_{\sigma,\tau}$ is invertible for all $\sigma, \tau \in G$.

Therefore $({}^hS,\ j)$ is actually a G-graded Clifford system over k extension of R, clearly representing an element $[{}^hS,\ j] \in \mathrm{Cliff}_k(G, R; \Phi)$, which we maintain depends only on $[h]$ and $[S,\ j]$. To see this, let us suppose that h' is another representative of $[h]$ and $(S',\ j')$ is another representative of $[S,\ j]$. Then, there must exist a 1-cochain $\psi : G \to C(R)^*$ such that $h'_{\sigma,\tau}{}^{\sigma}\psi_\tau\psi_\sigma = \psi_{\sigma\tau}h_{\sigma,\tau}$, $\sigma, \tau \in G$, and a grade-preserving isomorphism $f : S \to S'$ such that $fj = j'$, from which we build the grade-preserving k-isomorphism ${}^{\psi}f : {}^hS \to {}^{h'}S'$, ${}^{\psi}f(x) = f(j(\psi_\sigma)x)$ if $x \in S_\sigma$. For each $x \in S_\sigma$ and $y \in S_\tau$, we have

$$\begin{aligned} {}^{\psi}f(x \star y) &= f(j(\psi_{\sigma\tau})j(h_{\sigma,\tau})xy) = f(j(h'_{\sigma,\tau})j(\psi_\sigma)j({}^{\sigma}\psi_\tau)xy) \\ &\overset{(8)}{=} f(j(h'_{\sigma,\tau})j(\psi_\sigma)xj(\psi_\tau)y)) = j'(h'_{\sigma,\tau}){}^{\psi}f(x){}^{\psi}f(y) \\ &= {}^{\psi}f(x) \star {}^{\psi}f(y), \end{aligned}$$

so that ${}^{\psi}f : ({}^hS,\ j) \to ({}^{h'}S',\ j')$ is actually an isomorphism of Clifford system extensions of R, that is, $[{}^hS,\ j] = [{}^{h'}S',\ j']$.

Therefore, $([h], [S,\ j]) \mapsto [{}^hS,\ j]$ is a well-defined action of the abelian group $H^2_\Phi(G, C(R)^*)$ on $\mathrm{Cliff}_k(G, R; \Phi)$, which furthermore is a principal one. In fact, if we suppose that $[{}^hS,\ j] = [S,\ j]$, there must exist a grade preserving k-algebra isomorphism $f : {}^hS \to S$ such that $fj = j'$. For each $\sigma \in G$, the restriction $f_{/S_\sigma} : S_\sigma \to S_\sigma$ is a $R \otimes_k R^\circ$-isomorphism, and, by Lemma 3.1, there exists a unique $\psi_\sigma \in C(R)^*$ such that $f(x) = j(\psi_\sigma)x$ for all $x \in S_\sigma$. Thus $\psi : G \to C(R)^*$ is a 1-cochain. Since $f(x \star y) = f(x)f(y)$, for any $x \in S_\sigma$, $y \in S_\tau$, $\sigma, \tau \in G$, we have $j(\psi_{\sigma\tau}h_{\sigma,\tau})xy = j(\psi_\sigma)xj(\psi_\tau)y \overset{(8)}{=} j(\psi_\sigma{}^{\sigma}\psi_\tau)xy$. Therefore, since $S_\sigma S_\tau = S_{\sigma\tau}$, Lemma 3.1 implies that $\psi_{\sigma\tau}h_{\sigma,\tau} = \psi_\sigma{}^{\sigma}\psi_\tau$, that is, $h = \partial(\psi)$ represents the zero class in $H^2_\Phi(G, C(R)^*)$.

Finally, we observe that action (15) is transitive. Let $(S,\ j)$,$(S',\ j')$ be any two G-graded Clifford system extensions of R representing elements in $\mathrm{Cliff}_k(G,\ R;\ \Phi)$. Since S_σ,$S'_\sigma \in \Phi(x)$ for any $\sigma \in G$, there must exist $R \otimes_k R^\circ$-isomorphisms $f_\sigma : S_\sigma \to S'_\sigma$, $\sigma \in G$, with $f_1 = j'j^{-1}$. For each pair $\sigma, \tau \in G$, the square

$$\begin{tikzcd}
S_\sigma \otimes_R S_\tau \arrow[r] \arrow[d, "f_\sigma \otimes f_\tau"'] & S_{\sigma\tau} \arrow[d, "f_{\sigma\tau}"] \\
S'_\sigma \otimes_R S'_\tau \arrow[r] & S'_{\sigma\tau}
\end{tikzcd} ,$$

where the horizontal arrows represent the canonical isomorphisms $x \otimes y \mapsto xy$, need not be commutative. But, by Lemma 3.1, there exists a unique $h_{\sigma,\tau} \in C(R)^*$ such that $f_{\sigma\tau}(xy) = j'(h_{\sigma,\tau})f_\sigma(x)f_\tau(y)$ for all $x \in S_\sigma, y \in S_\tau$. Thus $h : G \to C(R)^*$ is a normalized 2-cochain. For any $x \in S_\sigma$, $y \in S_\tau$ and $z \in S_\gamma$, we have

$$f_{\sigma\tau\gamma}(xyz) = j'(h_{\sigma\tau,\gamma})f_{\sigma\tau}(xy)f_\gamma(z) = j'(h_{\sigma\tau,\gamma}h_{\sigma,\tau})f_\sigma(x)f_\tau(y)f_\gamma(z),$$

and analogously,

$$\begin{aligned} f_{\sigma\tau\gamma}(xyz) &= j'(h_{\sigma,\tau\gamma})f_\sigma(x)f_{\tau\gamma}(z) \\ &= j'(h_{\sigma,\tau\gamma})f_\sigma(x)j'(h_{\tau,\gamma})f_\tau(y)f_\gamma(z) \\ &\overset{(8)}{=} j'(h_{\sigma,\tau\gamma}{}^\sigma h_{\tau,\gamma})f_\sigma(x)f_\tau(y)f_\gamma(z). \end{aligned}$$

Lemma 3.1 implies that $h_{\sigma\tau,\gamma}h_{\sigma,\tau} = h_{\sigma,\tau\gamma}{}^\sigma h_{\tau,\gamma}$, that is, h is a 2-cocycle of G on $C(R)^*$. Clearly $f = \bigoplus_{\sigma \in G} f_\sigma$ establishes a G-graded Clifford system extension isomorphism $(S,\ j) \to ({}^hS',\ j')$, and so action (15) is transitive. □

To end this section, we shall focus on that class of rings known as crossed-product group k-algebras. According to ([4], §5) an *extension of a k-algebra R by a group G* in the sense of Hacque [8], is the same as G-graded Clifford system extension of R satisfying the condition that in any component there is at least one unit. As in Hacque's paper [8], let $\mathrm{Ext}_k(G,\ R)$ denote the set of isomorphism classes of extensions of a k-algebra R by a group G. Then $\mathrm{Ext}_k(G,\ R) \subseteq \mathrm{Cliff}_k(G,\ R)$, and we shall characterize this subset of $\mathrm{Cliff}_k(G,\ R)$ by means of collective characters as in the following proposition, where we take into account the canonical group embedding $\mathrm{Out}_k(R) \overset{\delta}{\hookrightarrow} \mathrm{Pic}_k(R)$ induced by the group exact sequence (4), whose image is ([1], Chap. II, (5.3))

$$\mathrm{Img}(\delta) = \{[P] \in \mathrm{Pic}_k(R) | P \cong R \text{ as left } R\text{-module}\}.$$

PROPOSITION 3.6

For any k-algebra R and group G there is a cartesian square

$$\begin{tikzcd}
\mathrm{Ext}_k(G,R) \arrow[r, tail] \arrow[d] & \mathrm{Cliff}_k(G,R) \arrow[d, "\chi"] \\
\mathrm{Hom}_{Gp}(G,\mathrm{Out}_k(R)) \arrow[r, tail, "\delta_*"] & \mathrm{Hom}_{Gp}(G,\mathrm{Pic}_k(R))
\end{tikzcd} \tag{16}$$

that is, $\mathrm{Ext}_k(G,\ R) = \chi^{-1}(\mathrm{Hom}_{Gp})(G,\ \mathrm{Out}_k(R))$ *is the set of classes of those G-graded Clifford system extensions of R which realize collective characters (in the sense of* [7, 8]).

Proof. Let $(S = \bigoplus_{\sigma \in G} S_\sigma,\ j)$ be a G-graded Clifford system extension of R such that for any $\sigma \in G$, there exists $u_\sigma \in S^* \cap S_\sigma$, that is, an extension of R by G. Right multiplication by u_σ is an isomorphism of left R-modules $R \overset{\sim}{\to} Ru_\sigma = S_\sigma$, $\sigma \in G$, and therefore the generalized collective character realized by $[S, j]$, $\chi_{[S,j]} : G \to \mathrm{Pic}_k(R)$, $\chi_{[S,j]}(\sigma) = [S_\sigma]$ factors through $\mathrm{Out}_k(R)$. Conversely, suppose $(S = \bigoplus_{\sigma \in G} S_\sigma,\ j)$ is a G-graded Clifford system extension of R such that $\chi_{[S,j]} = \delta\Phi$ for some $\Phi : G \to \mathrm{Out}_k(R)$. Then, if we choose any k-automorphism $f(\sigma) \in \Phi(\sigma)$ for each $\sigma \in G$, there must exist an $R \otimes_k R^\circ$-isomorphism $\varphi_\sigma : R_{f(\sigma)} \cong S_\sigma$. If $u_\sigma = \varphi_\sigma(1)$, then $S_\sigma = Ru_\sigma = u_\sigma R$. From $S_\sigma S_{\sigma^{-1}} = R1_S = S_{\sigma^{-1}} S_\sigma$, it follows that $u_\sigma R u_{\sigma^{-1}} = R1_S = u_{\sigma^{-1}} R u_\sigma$. Then there exist $a, b \in R$ such that $1 = u_\sigma a u_{\sigma^{-1}} = u_{\sigma^{-1}} b u_\sigma$ so that $u_\sigma \in S^* \cap S_\sigma$ and therefore (S, j) represents an extension of R by G. □

From the general results about Clifford system extensions of algebras, we deduce the following group cohomology classification of extensions of an algebra by a group, which was proved by Hacque in [8].

COROLLARY 3.7

Let G be a group and R be a k-algebra.

(i) *Each collective character of G in R, $\Phi : G \to \mathrm{Out}_k(R)$ determines in an invariant fashion a three-dimensional cohomology class $T(\Phi) \in H^3_\Phi(G, C(R)^*)$ of G with coefficients in the G-module (via Φ) of all units in the center of R.*

(ii) *There is a canonical partition of the set of equivalence classes of extensions of R by G,*

$$\mathrm{Ext}_k(G, R) = \coprod_{\Phi} \mathrm{Ext}_k(G, R; \Phi),$$

where, for any collective character $\Phi : G \to \mathrm{Out}_k(R)$, $\mathrm{Ext}_k(G, R; \Phi)$ is the set of equivalence classes of those extensions realizing Φ.

(iii) *A collective character $\Phi : G \to \mathrm{Out}_k(R)$ is realizable, that is, $\mathrm{Ext}_k(G, R; \Phi) \neq \emptyset$ if and only if its obstruction vanishes.*

(iv) *If the obstruction of a collective character $\Phi : G \to \mathrm{Out}_k(R)$ vanishes, then $\mathrm{Ext}_k(G, R; \Phi)$ is a principal homogeneous space under $H^2_\Phi(G, C(R)^*)$. In particular, there is a bijection*

$$\mathrm{Ext}_k(G, R; \Phi) \cong H^2(G, C(R)^*). \tag{17}$$

4. An obstructed collective character

It is very easy to find unobstructed generalized collective characters. Of course any Clifford system yields one of them. In this example we shall exhibit a non-realizable collective character, that is, a group homomorphism $\Phi : G \to \mathrm{Pic}_k(R)$, for particular group G and k-algebra R, such that there is no G-graded Clifford system extension of R, $(S = \bigoplus_{\sigma \in G} S_\sigma,\ j : R \cong S_1)$ such that $\Phi(\sigma) = [S_\sigma]$, $\sigma \in G$.

For example consider $G = C_2 = \langle t; t^2 = 1 \rangle$, the cyclic group of order two, $k = \mathbb{F}_5$, the Galois field with five elements and $R = \mathbb{F}_5[D_{10}]$, the group $\mathbb{F}_5$-algebra of the dihedral group $D_{10} = \langle r, s;\ r^{10} = 1 = s^2,\ srs = r^{-1} \rangle$.

Let $\beta : \mathbb{F}_5[D_{10}] \xrightarrow{\sim} \mathbb{F}_5[D_{10}]$ be the algebra automorphism defined by $\beta(r) = r^7$ and $\beta(s) = r^5 s$. Since $\beta^2(r) = r^{-1} = srs$ and $\beta^2(s) = sr^5(r^5)^7 = s$, the automorphism β^2

is simply conjugation by s. Therefore, the equations $\Phi(1) = 1$ and $\Phi(t) = [\beta]$ determine a homomorphism

$$\Phi : C_2 \longrightarrow \mathrm{Out}\,(\mathbb{F}_5[D_{10}]) \overset{\delta}{\subseteq} \mathrm{Pic}\,(\mathbb{F}_5[D_{10}]) \tag{18}$$

of the cyclic group C_2 into the Picard group of $\mathbb{F}_5[D_{10}]$, that is, a collective character of C_2 in $\mathbb{F}_5[D_{10}]$.

PROPOSITION 4.1

The Teichmüller obstruction $T(\Phi) \in H^3_\Phi(C_2, C(\mathbb{F}_5[D_{10}])^*)$ *is non-zero.*

Proof. First, let us observe that in this case, a (normalized) n-cochain $h : C_2 \times \cdots \times C_2 \longrightarrow C(\mathbb{F}_5[D_{10}])^*$ is determined by a single constant $h(t, \ldots, t) = h \in C(\mathbb{F}_5[D_{10}])^*$, whose coboundary is given by $\delta h = \beta(h)h^{-1}$ if n is even or $\delta h = \beta(h)h$ if n is odd. Since we easily see that the Teichmüller 3-cocycle is $T^\Phi = r^5$, the proof of the proposition amounts to checking that there is no unit h in the center of $\mathbb{F}_5[D_{10}]$ such that $\beta(h) = r^5h$.

The center of $\mathbb{F}_5[D_{10}]$ can be described as the 8-dimensional space over $\mathbb{F}_5$ generated by the elements

$$c_1 = 1, \quad c_2 = r + r^9, \quad c_3 = r^2 + r^8, \quad c_4 = r^3 + r^7, \quad c_5 = r^4 + r^6,$$
$$c_6 = r^5, \quad c_7 = (c_1 + c_3 + c_5)s, \quad c_8 = (c_2 + c_4 + c_6)s,$$

with multiplication given by

$$\begin{array}{ccccc}
c_2^2 = c_3 + 2 & c_2c_3 = c_2 + c_4 & c_2c_4 = c_3 + c_5 & c_2c_5 = c_4 + 2c_6 & c_2c_6 = c_5 \\
c_2c_7 = 2c_8 & c_2c_8 = 2c_7 & c_3^2 = c_5 + 2 & c_3c_4 = c_2 + 2c_6 & c_3c_5 = c_3 + c_5 \\
c_3c_6 = c_4 & c_3c_7 = 2c_7 & c_3c_8 = 2c_8 & c_4^2 = c_5 + 2 & c_4c_5 = c_2 + c_4 \\
c_4c_6 = c_3 & c_4c_7 = 2c_8 & c_4c_8 = 2c_7 & c_5^2 = c_3 + 2 & c_5c_6 = c_2 \\
c_5c_7 = 2c_7 & c_5c_8 = 2c_8 & c_6^2 = 1 & c_6c_7 = c_8 & c_6c_8 = c_7 \\
c_7^2 = 0 & c_7c_8 = 0 & c_8^2 = 0. & &
\end{array}$$

Let C_0 be the $\mathbb{F}_5$-subalgebra generated by c_2; that is, the span of $c_1, \ldots, c_6$ and note that the minimal polynomial of c_2 is $(t+2)^3(t-2)^3$. Then $C(\mathbb{F}_5[D_{10}]) = C_0 \oplus \mathbb{F}_5c_7 \oplus \mathbb{F}_5c_8$ with multiplication given by $c_7^2 = c_7c_8 = c_8^2 = 0$ and $c_2c_7 = 2c_8$, $c_2c_8 = 2c_7$ and we see that there is a homomorphism $\varphi : C(\mathbb{F}_5[D_{10}]) \to \mathbb{F}_5$ mapping c_2 to -2 and c_7, c_8 to 0. Hence, $\varphi(c_3) = 2$, $\varphi(c_4) = -2$, $\varphi(c_5) = 2$ and $\varphi(c_6) = -1$. Since $\beta(c_2) = c_4$, $\beta(c_7) = c_8$ and $\beta(c_8) = c_7$, this homomorphism φ satisfies that $\varphi(\beta(h)) = \beta(h)$ for all $h \in C\,(\mathbb{F}_5[D_{10}])$. Therefore, if $h \in C(\mathbb{F}_5[D_{10}])$ is such that $\beta(h) = r^5h$, then comparing the image of each side under φ yields $\varphi(h) = -\varphi(h)$, whence $\varphi(h) = 0$, and it follows that h is not invertible. □

Remark 4.2. Proposition 4.1 is an effect of inseparability. If we consider the Galois field $\mathbb{F}_3$ instead of $\mathbb{F}_5$, the resulting collective character (18), $\Phi : C_2 \to \mathrm{Pic}(\mathbb{F}_3[D_{10}])$ defined similarly by $\Phi(t) = [\beta]$, where β is the corresponding algebra automorphism determined by $\beta(r) = r^7$ and $\beta(s) = r^5s$, is unobstructed. In this case the Teichmüller cocycle $T^\Phi = r^5 = \delta(h)$, is the coboundary of the order 4 element $h = c_4 + c_5 + c_7 + 2c_8 \in C\,(\mathbb{F}_3[D_{10}])^*$, and therefore $T(\Phi) = 0$.

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The multiplication map for global sections of line bundles and rank 1 torsion free sheaves on curves

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Abstract. Let X be an integral projective curve and $L \in \mathrm{Pic}^a(X)$, $M \in \mathrm{Pic}^b(X)$ with $h^1(X, L) = h^1(X, M) = 0$ and L, M general. Here we study the rank of the multiplication map $\mu_{L,M} : H^0(X, L) \otimes H^0(X, M) \to H^0(X, L \otimes M)$. We also study the same problem when L and M are rank 1 torsion free sheaves on X. Most of our results are for X with only nodes as singularities.

Keywords. Singular projective curve; rank 1 torsion free sheaf; nodal curve; cuspidal curve; line bundle; special divisor.

1. Introduction

Let X be a smooth projective curve of genus $g \geq 0$ and $L, M \in \mathrm{Pic}(X)$ with L, M spanned. Call $h_L : X \to \mathbf{P}(H^0(X, L))$ and $h_M : X \to \mathbf{P}(H^0(X, M))$ the associated morphisms. Denote with $\mu_{L,M} : H^0(X, L) \otimes H^0(X, M) \to H^0(X, L \otimes M)$ the multiplication map and $\mathbf{i}_{L,M} : \mathbf{P}(H^0(X, L)) \times \mathbf{P}(H^0(X, M)) \to \mathbf{P}(H^0(X, L) \otimes H^0(X, M))$ the Segre embedding. Let $h_{L,M} : X \to \mathbf{P}(H^0(X, L)) \times \mathbf{P}(H^0(X, M))$ be the morphism induced by h_L and h_M on the two factors. Call $f_{L,M} : X \to \mathbf{P}(H^0(X, L \otimes M))$ the morphism obtained from $h_{L,M}$ and the multiplication map $\mu_{L,M}$. The surjectivity of $\mu_{L,M}$ means that $f_{L,M}(X)$ is linearly normal in its linear span and $\dim(\mathrm{Ker}(\mu_{L,M}))$ is the codimension of its linear span. For any L, M the surjectivity of $\mu_{L,M}$ has several important geometric consequences (see e.g. [7]) and very good criteria for the surjectivity of $\mu_{L,M}$ are known (see [10], Th. 4.a.1, and [7], p. 514).

In §2 we will give a proof the following result, proved also in [3].

Theorem 1.1. *Fix integers m, n and g with* $m \geq 1, n \geq 1$ *and* $g \geq 0$. *Let X be a general smooth projective curve of genus g. Take a general pair* $(L, M) \in \mathrm{Pic}^{g+m}(X) \times \mathrm{Pic}^{g+n}(X)$. *Then the multiplication map* $\mu_{L,M} : H^0(X, L) \otimes H^0(X, M) \to H^0(X, L \otimes M)$ *has maximal rank, i.e. it is injective if* $g \geq mn$ *and it is surjective if* $g \leq mn$.

Remark 1.2. In the set-up of 1.1 since $\deg(L) \geq g$, $\deg(M) \geq g$ and both L and M are general, we have $h^1(X, L) = h^1(X, M) = 0$. Hence by Riemann–Roch we have $h^0(X, L) = m + 1$ and $h^0(X, M) = n + 1$. We explain the numerology in the statement of 1.1 with the following example. Fix positive integers m and n. Let C be a smooth projective curve of genus mn and $A \in \mathrm{Pic}^{m+mn}(C)$, $B \in \mathrm{Pic}^{n+mn}(C)$ with $h^1(C, A) = h^1(C, B) = 0$. We have $h^0(C, A) = m+1$, $h^0(C, B) = n+1$, $\deg(A \otimes B) = n+m+2mn$, $h^1(C, A \otimes B) = 0$ and $h^0(C, A).h^0(C, B) = (m + 1)(n + 1) = n + m + 1 + mn = h^0(C, A \otimes B)$.

At the end of §2 we will prove the following result.

Theorem 1.3. *Fix integers m, n, g and q with* $g \geq q \geq 0, m \geq 1, n \geq 1$ *and* $g \geq 3$. *Let* $\pi : Y \to C$ *be a birational morphism with Y general curve of genus q and C general nodal curve with* $g - q$ *nodes and Y as normalization, i.e. assume that* $\pi^1(\mathrm{Sing}(C))$ *is formed by* $2g - 2q$ *general points of Y. Take a general pair* $(L, M) \in \mathrm{Pic}^{g+m}(C) \times \mathrm{Pic}^{g+n}(C)$. *Then the multiplication map* $\mu_{L,M} : H^0(C, L) \otimes H^0(C, M) \to H^0(C, L \otimes M)$ *has maximal rank, i.e. it is injective if* $g \geq mn$ *and it is surjective if* $g \leq mn$.

In §3 we will use the classical Brill–Noether theory of special divisors to study the multiplication map for line bundles on nodal or cuspidal curves. In §4 we will use 1.3 to study some problems related to the multiplication map for rank 1 torsion free sheaves on nodal curves.

2. Proofs of 1.1 and 1.3

We work over an algebraically closed field $\mathbf{K}$ with $\mathrm{char}(\mathbf{K}) = 0$; for the case $\mathrm{char}(\mathbf{K}) > 0$, see Remark 3.4. For all positive integers m and n set $\prod(m, n) := \mathbf{P}^m \times \mathbf{P}^n$. Call $\pi_1(m, n) : \prod(m, n) \to \mathbf{P}^m$ and $\pi_2(m, n) : \prod(m, n) \to \mathbf{P}^n$ (or just π_1 and π_2) the projections. We have $\mathrm{Pic}(\prod(m, n)) \cong \mathbf{Z}^{\oplus 2}$ and we will take $\pi_1{}^*(\boldsymbol{O}_{\mathbf{P}^m}(1))$ and $\pi_2{}^*(\boldsymbol{O}_{\mathbf{P}^n}(1))$ as generators of $\mathrm{Pic}(\prod(m, n))$. Sometimes we will write $\prod$ instead of $\prod(m, n)$. Set $\boldsymbol{O} := \boldsymbol{O}_{\prod}$ and call $\boldsymbol{O}(1, 0)$ and $\boldsymbol{O}(0, 1)$ the two choosen generators of $\mathrm{Pic}(\prod)$. Every one-dimensional cycle T of $\prod$ has a bidegree (a, b) with $a := T.\boldsymbol{O}(1, 0)$ and $b := T.\boldsymbol{O}(0, 1)$. If T is effective and irreducible we have $a = \deg(\pi_1|T)\deg(\pi_1(T))$ and $b = \deg(\pi_2|T)\deg(\pi_2(T))$. The tangent bundle, $T\prod(m, n)$, of $\prod(m, n)$ is isomorphic to $\pi_1{}^*(T\mathbf{P}^m) \oplus \pi_2{}^*(T\mathbf{P}^n)$. Notice that $T\mathbf{P}^m(-1)$ and $T\mathbf{P}^n(-1)$ are spanned (e.g. by the Euler sequence of $T\mathbf{P}^s$, $s = m$ or n). Hence for every integral curve $X \subset \prod$ of type (a, b), the vector bundle $T\prod(m, n)|X$ is the direct sum of a rank m vector bundle which is the quotient of $m + 1$ copies of $\boldsymbol{O}_X(1, 0)$ (and hence the quotient of line bundles of degree a) and a rank n vector bundle which is a quotient of $n + 1$ copies of $\boldsymbol{O}_X(0, 1)$ (and hence a quotient of $n + 1$ line bundles of degree b). For any locally complete intersection curve $X \subset \prod(m, n)$, let $N_{X]/\prod(m,n)}$ be its normal bundle. If X is smooth, then the normal bundle $N_{X/\prod(m,n)}$ of X in $\prod(m, n)$ is a quotient of $T\prod(m, n)|X$. To prove Theorem 1.1 we introduce the following statement:

$H(m, n), m \geq 1, n \geq 1$: There exists a smooth connected curve $X[m, n] \subset \prod(m, n)$ such that $p_a(X[m, n]) = mn$, $X[m, n]$ has bidegree $(mn + m, mn + n)$, the embedding of $X[m, n]$ in $\prod(m, n)$ is induced by a pair of line bundles (L, M) with $h^1(X[m, n], L) = h^1(X[m, n], M) = 0$, $X[m, n]$ spans $\mathbf{P}^{mn+m+n}$ and $h^1(X[m, n], N_{X[m,n]/\prod(m,n)}) = 0$.

Since $h^1(X[m, n], L) = h^1(X[m, n], M) = h^1(X[m, n], L \otimes M) = 0$, the condition that $X[m, n]$ spans $\mathbf{P}^{mn+m+n}$ in the statement of $H(m, n)$ is equivalent to the condition that the two maps $X[m, n] \to \mathbf{P}^m$ and $X[m, n] \to \mathbf{P}^n$ induced the inclusion of $X[m, n]$ into $\prod(m, n)$ are given by a complete linear system (i.e. by Riemann–Roch, that they are non-degenerate) and that the multiplication map $\mu_{L,M} : H^0(X[m, n], L) \otimes H^0(X[m, n], M) \to H^0(X[m, n], L \otimes M)$ is bijective.

Remark 2.2. $H(1, 1)$ is true because a smooth quadric surface $\prod(1, 1) \subset \mathbf{P}^3$ contains a smooth non-degenerate elliptic curve of bidegree $(2, 2)$ and such curve has as normal bundle a degree 4 line bundle.

PROPOSITION 2.2

Fix an integer $m \geq 1$. *If* $H(m, m)$ *is true, then* $H(m + 1, m + 1)$ *is true.*

Proof. See $\mathbf{P}^{m^2+2m}$ as a codimension $2m+3$ linear subspace, A, of $\mathbf{P}^{m^2+4m+3}$. Take a solution $X[m,m] \subset \prod(m,m)$ for $H(m,m)$ and see $\prod(m,m)$ as a linear section of $\prod(m+1,m+1) \subset \mathbf{P}^{m^2+4m+3}$. Fix $S \subset X[m,m]$ with $\mathrm{card}(S) = 2m+2$ and S spanning a linear subspace $\langle S\rangle$ of $\mathbf{P}^{m^2+2m}$ with $\dim(\langle S\rangle) = 2m+1$. Let C be a smooth rational curve and consider the pair $(R,R) \in \mathrm{Pic}(C) \times \mathrm{Pic}(C)$ with $\deg(R) = 2m+2$. The multiplication map $\mu_{R,R} : H^0(C,R) \otimes H^0(C,R) \to H^0(C,R^{\otimes 2})$ is surjective and $R^{\otimes 2}$ embeds C into a $(4m+4)$-dimensional projective space W as a rational normal curve; call $D \subset W$ its image. Hence D may be seen both as a smooth rational curve of degree $4m+4$ in W and a curve of bidegree $(2m+2, 2m+2)$ in $\prod(t,t)$ for any $t \geq 2m+1$. We may take $W \subset \mathbf{P}^{m^2+4m+3}$ in such a way that $W \cap A$ contains S; here we use that A has codimension $2m+3 = \dim(W) - (2m+1)$ in $\mathbf{P}^{m^2+4m+3}$ and that $\mathrm{card}(S) \leq \dim(W)$. The group $\mathrm{Aut}(\langle S\rangle)$ acts transitively on the set of ordered $(2m+2)$-ples of points in linear general in $\langle S\rangle$. Any such $(2m+2)$-ple is contained in a codimension $2m+3$ linear section of a rational normal curve of W. Hence we may assume that $D \cap A = S$. Set $Y := X[m,m] \cup D$. Y has bidegree $((m+1)(m+2), (m+1)(m+2))$, the same bidegree of $X[m+1,m+1]$.

Claim. We may find such D with $D \subset \prod(m+1,m+1)$, i.e. with $Y \subset \prod(m+1,m+1)$ and $Y \cap A = X[m,m]$.

Proof of the Claim. First we will check that $\mathrm{Pic}(Y)$ is an extension of $\mathrm{Pic}(X[m,m]) \times \mathrm{Pic}(D) \cong \mathrm{Pic}(X[m,m]) \times \mathbf{Z}$ by a multiplicative group isomorphic to $(\mathbf{K}^*)^{\oplus(2m+1)}$. More precisely, every $E \in \mathrm{Pic}(Y)$ is uniquely determined by $E|X[m,m]$, $E|D$ and by the gluing data at each of the $2m+2$ points of S; since $D \cong \mathbf{P}^1$, $E|D$ is uniquely determined by the integer $\deg(E|D)$; each of these gluing data is uniquely determined by a non-zero scalar (and vice versa, each non-zero scalar induces a gluing datum at one point of S); however, since for any $E' \in \mathrm{Pic}(X[m,m])$ and $E'' \in \mathrm{Pic}(D)$ we have $\mathrm{Aut}(E') \cong \mathrm{Aut}(E'') \cong \mathrm{Aut}(E) \cong \mathbf{K}^*$, we may multiply all these gluing data by a common non-zero scalar and obtain an isomorphic line bundle on Y. Hence $\mathrm{Pic}(Y)$ is an extension of $\mathrm{Pic}(X[m,m]) \times \mathrm{Pic}(D)$ by $(\mathbf{K}^*)^{\oplus(2m+1)}$. Take any $L' \in \mathrm{Pic}(Y)$, $M' \in \mathrm{Pic}(Y)$ with $L'|X[m;m] \cong L$, $M'|X[m,m] \cong M$ and $\deg(L'|D) = \deg(M'|D) = 2m+1$. Consider the Mayer–Vietoris exact sequence for L',

$$0 \to L' \to L'|X[m,m] \oplus L'|D \to L'|S \to 0 \tag{1}$$

and the corresponding Mayer–Vietoris exact sequence for M'. Since $\mathrm{card}(S) = 2m+2$ and $\deg(L'|D) = \deg(M'|D) = 2m+1$, the restriction maps $H^0(D, L'|D) \to H^0(S, L'|S)$ and $H^0(D, M'|D) \to H^0(S, M'|S)$ are surjective. Hence by the Mayer–Vietoris exact sequences we obtain $h^0(Y,L') = m+1$, $h^0(Y,M') = m+1$, $h^1(Y,L') = 0$ and $h^1(Y,M') = 0$. Similarly, we obtain that L' and M' are spanned and (for general gluing data) induce an embedding of Y into $\prod(m+1,m+1)$, proving the Claim.

The variety $\prod(m,m)$ is the complete intersection of two Cartier divisors of $\prod(m+1,m+1)$, one of type $(1,0)$ and one of type $(0,1)$. Hence $N_{X[m,m]/\prod(m+1,m+1)} \cong N_{X[m,m]/\prod(m,m)} \oplus L \oplus M$. Thus $h^1(X[m,m], N_{X[m,m]/\prod(m+1,m+1)}) = 0$. By construction $D \cap X[m,m] = S$ and D intersects quasi-transversally $X[m,m]$. Hence Y is a connected nodal curve with $p_a(Y) = m^2+2m+1$. Since $A \cap W = \langle S\rangle$ and $\dim(\langle S\rangle) + \dim(W) = \mathrm{codim}(A)$, Y spans $\mathbf{P}^{m^2+4m+3}$. Hence by semicontinuity it is sufficient to prove that Y is smoothable and that $h^1(Y, N_{Y/\prod(m+1,m+1)}) = 0$. Since D has bidegree $(2m+1, 2m+1)$ in $\prod(m+1,m+1)$, its normal bundle is a quotient of a direct sum of line bundles of degree $2m+1$. Since every vector bundle on $D \cong \mathbf{P}^1$ is a direct sum of line bundles, we obtain that every line bundle appearing in a decomposition of $N_{D/\prod(m+1,\ m+1)}$ has degree at least $2m+1$. By [11], Cor. 3.2 and Prop. 3.3, or [13], $N_{Y/\prod(m+1,\ m+1)}|X[m,m]$ (resp. $N_{Y/\prod(m+1,\ m+1)}|D$) is obtained from $N_{X[m,m]/\prod(m+1,\ m+1)}$ (resp. $N_{D/\prod(m+1,\ m+1)}$) making $2m+2$ positive elementary transformations. Hence $h^1(X[m,m], N_{Y/\prod(m+1,\ m+1)}|X[m,m]) = 0$ and every line bundle appearing in a decomposition of $N_{Y/\prod(m+1,\ m+1)}|D$ has degree at least $2m+1$. The last remark implies the surjectivity of the restriction map $\rho : H^0(D, N_{Y/\prod(m+1,\ m+1)}|D) \to H^0(S, N_{Y/\prod(m+1,\ m+1)}|S)$. By the Mayer–Vietoris exact sequence

$$\begin{aligned}0 \to N_{Y/\prod(m+1,\ m+1)} \to N_{Y/\prod(m+1,\ m+1)}|X[m,\ m] \oplus N_{Y/\prod(m+1,\ m+1)}|D\\ \to N_{Y/\prod(m+1,\ m+1)}|S \to 0,\end{aligned} \qquad (2)$$

we obtain $h^1(Y, N_{Y/\prod(m+1,\ m+1)}) = 0$. Furthermore, as in [11], Th. 4.1, or [13] we obtain also that Y is smoothable. Notice that we may apply the semicontinuity theorem for the dimension of the kernel of the multiplication map for a flat family of pairs of non-special line bundles on a flat family of curves, because the non-speciality condition implies that the corresponding cohomology groups have constant dimension. By semicontinuity we obtain the result for a general triple (Z, L'', M'') with Z of genus $(m+1)^2$ and (L'', M'') a general pair of line bundles on Z with degree $(m+1)^2 + m + 1$.

PROPOSITION 2.3

Fix integers m, n with $n \geq m \geq 1$. Assume that $H(m,n)$ is true. Then $H(m,n+1)$ is true.

Proof. We will show how to modify the proof of 2.2. Notice that $p_a(X[m,n+1]) = p_a(X[m,n]) + m$. We start with $(X[m,n], L, M)$ satisfying $H(m,n)$. Hence $L, M \in \mathrm{Pic}(X[m,n])$, $\deg(L) = p_a(X[m,n]) + m = mn + n$ and $\deg(M) = mn + n$. We take $S \subset X[m,n] \subset A := \langle\prod(m,n)\rangle$ with $\mathrm{card}(S) = m+1$ and $\dim(\langle S\rangle) = m$. Now D is a smooth rational curve and it is embedded into $\prod(x,y), x \geq m, y \geq m+1$, by a pair (R_1, R_2) with $\deg(R_1) = m$ and $\deg(R_2) = m+1$, i.e. of bidegree $(m,\ m+1)$. Hence $\deg(R_1 \otimes R_2) = 2m+1$. Set $Y := X[m,n] \cup D$. Since $h^0(D, R_i) = \deg(R_i)+1 \geq \mathrm{card}(S)$ for $i = 1, 2$, every part of the proof of 2.2 works in our new set-up, proving 2.3.

Proof of 1.1. (i) Here we will cover the case $0 \leq g \leq mn$, i.e. when we need to prove that for a general triple (X, L, M) the multiplication map $\mu_{L,M}$ is surjective. Since the case $g \leq 1$ is well-known and trivial, we assume $g \geq 2$ and hence $n \geq 2$. Since $H(m,n)$ is true, we know the case $g = mn$. Hence we may assume $2 \leq g < mn$. We start with $X[1,1]$ satisfying $H(1,1)$ and then we follow the proofs of 2.2 and 2.3 made to obtain a proof of $H(m,n)$. However, at each step of the proof we take D intersecting the other curve in a subset, S', of S. For instance if $n > m$ and $mn - m - 1 \leq g < mn$, we take $\mathrm{card}(S') = g - mn + n$. Call Y' the curve $X[m,n-1] \cup D$ with $D \cap X[m,n-1] = S'$. The proofs of 2.2 and 2.3 and semicontinuity proves 1.1 for this triple (m,n,g).

(ii) Now we assume $g > mn$. By induction on g for a fixed pair (m,n) and the case $g' = mn$ (the bijective case) proved in part (i) we may assume the result for the triple $(m,n,g-1)$. Let (C, A, B) a general triple satisfying the statement of 1.1 for the triple $(m,n,g-1)$. Fix two general points $\{P, Q\}$ of C and let Y be the nodal curve $C \cup D$ with $D \cong \mathbf{P}^1$ and $C \cap D = \{P, Q\}$. By semistable reduction Y is the flat limit of a flat family of smooth connected curves of genus g. Take any $L, M \in \mathrm{Pic}(Y)$ with $L|C \cong A, M|C \cong B$ and $\deg(L|D) = \deg(M|D) = 1$. We saw in the proof of 2.2 that the set of all such L (resp. M) is not empty and parametrized by an extension of $\mathrm{Pic}^0(C)$ by $\mathbf{K}^*$. Since the restriction maps $H^0(D, L|D) \to \mathcal{O}_{\{P,Q\}}$ and $H^0(D, M|D) \to \mathcal{O}_{\{P,Q\}}$

are surjective, as in the proof of 2.2 a Mayer–Vietoris exact sequence similar to (1) shows that $h^0(Y, L) = m + 1$, $h^0(Y, M) = m + 1$ and $h^1(Y, L) = h^1(Y, M) = 0$. Furthermore, the same exact sequence induces an isomorphism of $H^0(C, A)$ (resp. $H^0(C, B)$) with $H^0(Y, L)$ (resp. $H^0(Y, M)$) and a surjection of $H^0(C, A \otimes B)$ onto $H^0(Y, L \otimes M)$. Hence the injectivity of $\mu_{A,B}$ implies the injectivity of $\mu_{L,M}$. By semicontinuity we conclude as in the last part of the proof of 2.2.

Proof of 1.3. Look again to the proof of 1.1 and in particular to the proof of 2.2. Now we take as $X[1, 1]$ a rational curve with an ordinary node as only singularity. As in the proof of 2.2 we obtain the result in the case $m = n = 1$. Now we consider the inductive step in the proofs of 2.2 and 2.3. Just to fix the notation we assume the case (m, m) and prove the case $(m+1, m+1)$. Now $X[m, m]$ is the general rational curve with mn ordinary nodes as only singularities. Set $Y := X[m, m] \cup D$. We need to deform Y inside $\prod(m + 1, m + 1)$ to an irreducible rational curve with only nodes as singularities. Hence it is sufficient to prove that we may smooth exactly one node (any node we chose) in $Y \cap D$ keeping singular the other singular points of $Y \cap D$ and without smoothing the other points and keeping singular the singular points of $X[m, n]$. If instead of $X[m, n]$ we would have a smooth curve, this would be the notion of strong smoothability considered in [11], §1. The part concerning the nodes in $X[m, m] \cap D$ is easy because $\mathrm{card}(X[m, m] \cap D) = 2m + 2$ and every line bundle appearing in a decomposition of $N_{Y/\prod(m+1,\ m+1)}|D$ has degree at least $2m + 1$. Hence $h^1(D, (N_{Y/\prod(m+1,\ m+1)}|D)(-S)) = 0$ and we may apply the proof of [11], Th. 4.1. We know that $h^1(Y, N_{Y/\prod(m+1,m+1)}) = 0$ and hence that Y is a smooth point of $\mathrm{Hilb}(\prod(m + 1, m + 1))$. Furthermore, by induction on m we may assume that each subset of the set of all nodes of $X[m, m]$ may be smoothing independently, i.e. that for every subset Γ of $\mathrm{Sing}(X[m, m])$ the set of curves in $\prod(m + 1, m + 1)$ near $X[m, m]$ in which we smooth exactly the nodes in $\mathrm{Sing}(X[m, m])\backslash\Gamma$ has, near Y, codimension $\mathrm{card}(\Gamma)$ in $\mathrm{Hilb}(\prod(m + 1, m + 1))$. The same assertion for Y follows from this, $\mathrm{card}(S) = 2m + 2$, that every line bundle appearing in a decomposition of $N_{Y/\prod(m+1,m+1)}|D$ has degree at least $2m + 1$ and a Mayer–Vietoris exact sequence as in the proof of [11], Th. 4.1. Hence we obtain the case $q = 0$ of 1.3. If $q > 0$ we just smooth q nodes and apply semicontinuity.

3. Line bundles on singular curves

For any triple g, r, d of integers, let $\rho(g, r, d) := g - (r + 1)(g + r - d)$ be the so-called Brill–Noether number associated to g, r and d. For any smooth projective curve X, set $W^r_d(X) := \{L \in \mathrm{Pic}(X) : h^0(X, L) \geq r + 1\}$. On a general smooth curve X of genus $g \geq 2$ we have $W^r_d(X)) \neq \emptyset$ if and only if $\rho(g, r, d) \geq 0$; if $\rho(g, r, d) \geq 0$, then $W^r_d(X)$ is non-empty, smooth outside $W^{r+1}_d(X)$ and of pure dimension $\rho(g, r, d)$; $W^r_d(X)$ is irreducible if $\rho(g, r, d) > 0$ ([1], chs V and VII, and in particular the references [9] and for the smoothness and irreducibility in arbitrary characteristic). If $\rho(g, r, d) \geq 0$ this implies that a general $L \in W^r_d(X)$ has no base points and $h^0(X, L) = r + 1$; here and in the statements of 3.1, 3.2 and 3.5 if $\rho(g, r, d) = 0$ (i.e. if $W^r_d(X)$ is finite) the word 'general $L \in W^r_d(X)$' means 'every $L \in W^r_d(X)$'; if C is singular (i.e. $q \neq g$) in the statement of 3.1, 3.2 and 3.3 the word 'general' means only 'general in a smooth component with the expected dimension $\rho(g, x - 1, a)$ and $\rho(g, y - 1, b)$' because we do not claim any irreducibility result for the schemes $W^r_d(C)$ when C is a singular curve. In the smooth case ($q = g$) when $\rho(g, x - 1, b) = 0$ to have 'for all $L \in W^r_d(X)$' we need to use [6] and hence we need to assume $\mathrm{char}(\mathbf{K}) = 0$.

Theorem 3.1. *Fix integers g and q with $g \geq q \geq 0$ and $g \geq 3$. Let $\pi : Y \to C$ be a birational morphism with Y general curve of genus q and C general nodal curve with $g - q$ nodes and Y as normalization, i.e. assume that $\pi^{-1}(\mathrm{Sing}(C))$ is formed by $2g - 2q$ general points of Y. Fix integers a, b, x and y with $2 \leq x \leq g - 2, 2 \leq y \leq g + x - a - 1, \rho(g, x - 1, a) \geq 0, 0 \leq a \leq 2g - 2$ and $g + y - x - 1 \leq b \leq g + y - 1$. Let $L \in W_a^{x-1}(C)$ and $M \in W_b^{y-1}(X)$ be general elements. Then the multiplication map $\mu_{L,M} : H^0(C, L) \otimes H^0(C, M) \to H^0(C, L \otimes M)$ is injective.*

Proof. By [9], Prop. 1.2, there is a nodal curve D with $p_a(D) = g$ and exactly g ordinary nodes such that for every $L \in W_a^{x-1}(D)$ with $h^0(D, L) = x$ the multiplication map $\mu_{L,\omega_D \otimes L^*} : H^0(D, L) \otimes H^0(D, \omega_D \otimes L^*) \to H^0(D, \omega_D)$ is injective; we will only use that this is true just for one $L \in W_a^{x-1}(D)$ with $h^0(D, L) = x$. By semicontinuity for a general nodal curve, C, with $p_a(C) = g$ and with exactly $g - q$ nodes as only singularities there is $L \in \mathrm{Pic}(C)$ with $\deg(C) = a$, $h^0(C, L) = x$ and such that the multiplication map $\mu_{L,\omega_C \otimes L^*} : H^0(C, L) \otimes H^0(C, \omega_C \otimes L^*) \to H^0(C, \omega_C)$ is injective. By Riemann–Roch we have $h^1(C, L) = g + x - a - 1$. By assumption we have $g - a - 1 \leq y \leq g + x - a - 1$ and $b \geq g + y - x - 1$. Let D (resp. D') be the union of $g + x - a - 1 - y$ (resp. $b - g - y + x + 1$) general points of C. Set $R := \omega_C \otimes L^*(-D)$ and $A := R(D')$. Hence $\deg(R) = g + y - x + 1 \leq b = \deg(A)$. Since D is general, we have $h^0(C, R) = y$. Adding D as a base locus we may see the vector space $H^0(C, R)$ as a subspace of $H^0(C, \omega_C \otimes L^*)$. Thus the multiplication map $\mu_{L,R} : H^0(C, L) \otimes H^0(C, R) \to H^0(C, L \otimes R)$ is injective. By Rieman–Roch we have $h^1(C, R) = x$. Thus $h^1(C, R) \geq \deg(D')$. Hence $h^0(C, A) = h^0(C, R)$ by the generality of D', i.e. $A \in W_b^{y-1}(C)$ and the complete linear system associated to A has D' in its base locus. Thus the multiplication map $\mu_{L,A} : H^0(C, L) \otimes H^0(C, A) \to H^0(C, L \otimes A)$ is injective. Hence by semicontinuity for general $M \in W_b^{y-1}(C)$ the multiplication map $\mu_{L,M} : H^0(C, L) \otimes H^0(C, M) \to H^0(C, L \otimes M)$ is injective. Quoting [5] instead of [9] we have the following result.

Theorem 3.2. *Fix integers g and q with $g \geq q \geq 0$ and $g \geq 3$. Let $\pi : Y \to C$ be a birational morphism with Y general curve of genus q and C general cuspidal curve with $g - q$ nodes and Y as normalization, i.e. assume that $\pi^1(\mathrm{Sing}(C))$ is formed by $g - q$ general points of Y. Fix integers a, b, x and y with $2 \leq x \leq g - 2, 2 \leq y \leq g + x - a - 1, \rho(g, x - 1, a) \geq 0, 0 \leq a \leq 2g - 2$ and $g + y - x - 1 \leq b \leq g + y - 1$. Let $L \in W_a^{x-1}(C)$ and $M \in W_b^{y-1}(X)$ be general elements. Then the multiplication map $\mu_{L,M} : H^0(C, L) \otimes H^0(C, M) \to H^0(C, L \otimes M)$ is injective.*

Remark 3.3. Theorem 3.2 is true with the same proof for every rational cuspidal curve, not just the general one ([5]).

4. Rank 1 torsion free sheaves

Let C be an integral projective curve and F and G rank 1 torsion free sheaves on C. The sheaf $F \otimes G$ may have torsion, but the sheaf $F \otimes G/\mathrm{Tors}(F \otimes G)$ is a rank 1 torsion free sheaf. Call $\beta_{F,G} : H^0(C, F) \otimes H^0(C, G) \to H^0(C, F \otimes G/\mathrm{Tors}(F \otimes G))$ the composition of the multiplication map $\mu_{F,G} : H^0(C, F) \otimes H^0(C, G) \to H^0(C, F \otimes G)$ with the map $H^0(C, F \otimes G) \to H^0(C, F \otimes G/\mathrm{Tors}(F \otimes G))$ induced by the quotient map $F \otimes G \to F \otimes G/\mathrm{Tors}(F \otimes G)$. We believe that the linear map $\beta_{F,G}$ is more significant and has better behaviour than the plain multiplication map $\mu_{F,G}$. In this section we study $\beta_{F,G}$ and $\mu_{F,G}$ in the case of nodal curves. The general set-up works for curves with only ordinary nodes and ordinary cusps as singularities (see (4.1)). The restriction to nodal curves come from the use of 1.3. In many interesting cases the map $\beta_{F,G}$ is induced from a multiplication map for line bundles on a partial normalization of C (see (4.2)). Here is the general set-up. Let $f : Y \to C$ be a birational morphism between integral projective curves. Set $\delta := p_a(Y) - p_a(C)$. We have $\delta \geq 0$ and $\delta = 0$ if and only if f is an isomorphism. For every rank 1 torsion free sheaf A on Y the coherent sheaf $f_*(A)$ is a rank 1 torsion free sheaf on C. If $A \cong f^*(B)$ for some rank 1 torsion free sheaf B on C, then $f_*(A) \cong B \otimes f^*(O_Y)$ (projection formula) and hence $\deg(f_*(A)) = \deg(A) + \delta$. By the very definition of the direct image functor we have $h^0(C, f_*(A)) = h^0(Y, A)$. Since f is finite, we have $h^1(C, f_*(A)) = h^1(Y, A)$. It is easy to check that for every rank 1 torsion free sheaf B on C the natural map $f_B{}^* : H^0(C, B) \to H^0(Y, f^*(B)/\mathrm{Tors}(f^*(B)))$ is injective. Let L, M be rank 1 torsion free sheaves on Y. Since $H^0(Y, L) \cong H^0(C, f_*(L))$, $H^0(Y, M) \cong H^0(C, f_*(M))$, the multiplication map $\mu_{L,M} : H^0(Y, L) \otimes H^0(Y, M) \to H^0(Y, L \otimes M)$ induces a morphism $\beta_{L,M} : H^0(Y, L) \otimes H^0(Y, M) \to H^0(Y, L \otimes M/\mathrm{Tors}(L \otimes M))$, a morphism $\beta_{L,M,f} : H^0(C, f_*(L)) \otimes H^0(C, f_*(M)) \to H^0(C, f_*(L \otimes M))$ and a morphism $\alpha_{L,M,f} : H^0(C, f_*(L)) \otimes H^0(C, f_*(M)) \to H^0(C, f_*(L \otimes M)/\mathrm{Tors}(f_*(L \otimes M)))$. A section of a torsion free sheaf on a reduced curve is uniquely determined by its restriction to a Zariski open dense subset of the curve. Hence if $\mu_{L,M}$ is injective, then $\alpha_{L,M,f}$ is injective.

(4.1) Let R be the local ring either of an ordinary node (i.e. of an A_1 singularity), P, of an irreducible curve or of an ordinary cusp (i.e. of an A_2 singularity). Let $\mathbf{m}$ be the maximal ideal of R. If R is an ordinary node will say that a coherent sheaf on Spec(R) is torsion free near P if its completion has no nonzero element killed by an element of R which is not a zero-divisor of R; this is the definition used in [4]. With this convention every finitely generated torsion free R-module M (up to a completion) is of the form $R^{\oplus a} \oplus \mathbf{m}^{\oplus b}$ for some integer $a \geq 0, b \geq 0, a+b > 0$, with $a+b = \mathrm{rank}(M)$ ([4], Th. 2.4.2 and Remark 1 after that, or [14], Prop. 2 at p. 162). The same is true if R is an ordinary cusp. We will need only the case $\mathrm{rank}(M) = 1$; hence either $M \cong R$ or $M \cong \mathbf{m}$. It is easy to check that $\mathbf{m}$ contains a rank 1 submodule M with $M \cong R$ and $\mathbf{m}/M \cong \mathbf{K}$; obviously $\mathbf{m}$ is contained in the rank 1 free module R and $R/\mathbf{m} \cong \mathbf{K}$.

For any coherent sheaf F on an integral projective curve X with pure rank r the degree $\deg(F)$ of F is defined by the Riemann–Roch formula $\deg(F) := \chi(F) + r(g-1)$. If F is a torsion free sheaf on X, set $\mathrm{Sing}(F) := \{P \in X : F \text{ is not locally free at } P\} = \{P \in \mathrm{Sing}(X) : F \text{ is not locally free at } X\}$.

(4.2) Let C be an integral projective curve whose only singularities are ordinary nodes and ordinary cusps. Let F be a rank 1 torsion free sheaf on C. Set $S := \mathrm{Sing}(F) = \{P \in C : F \text{ is not locally free at } P\}$. Hence by 4.1 for every $P \in \mathrm{Sing}(F)$ near P the sheaf F is formally equivalent to the maximal ideal of $O_{C,P}$. Set $\delta := \mathrm{card}(S)$. Let $\pi : Y \to C$ be the partial normalization of C in which we normalize only the points of S. We have $p_a(C) = p_a(Y) + \delta$. Set $L := \pi^*(F)/\mathrm{Tors}(\pi^*(F))$. By 4.1 we have $L \in \mathrm{Pic}(C)$, $F \cong \pi_*(L)$ and $\deg(F) = \deg(L) + \delta$. Let $M(C; x, S)$ be the set of all rank 1 torsion free sheaves, G, on C with $\deg(G) = x$ and $\mathrm{Sing}(G) = S$. Now we will use the following observation.

Remark 4.3. Let C be an integral projective curve whose only singularities are ordinary nodes and ordinary cusps. Fix $S \subseteq \mathrm{Sing}(C)$ and let $\pi : Y \to C$ be the partial normalization of C in which we normalize only the points of S. $\mathrm{Pic}^0(Y)$ is a q-dimensional algebraic group, $q := p_a(C) - \mathrm{card}(S) = p_a(Y)$, which is an extension of an abelian variety of dimension $p_a(C)$ − card(Sing(C)) by a connected affine group G; G is the product of some copies of the additive group (the number of copies being the number of cusps of Y, i.e. of the cusps in Sing$(C)\setminus S$) and some copies of the multiplicative group (the number of copies being the number of nodes of Y). In particular $\mathrm{Pic}^0(Y)$ is an irreducible q-dimensional variety. Hence for every integer x the set $M(C; x, S)$ has a natural structure of q-dimensional irreducible algebraic variety. Hence we are allowed to consider the general element of $M(C; x, S)$.

Take another rank 1 torsion free sheaf G with $S = \mathrm{Sing}(G)$. Set $M := \pi^*(G)/\mathrm{Tors}(\pi^*(G))$. Hence $G \cong \pi_*(M)$ and $\deg(G) = \deg(M)+\delta$. By (4.1) we have $F\otimes G/\mathrm{Tors}(F\otimes G) \cong \pi_*(L\otimes M)$. Since $H^0(C, F) \cong H^0(Y, L)$, $H^0(C, G) \cong H^0(Y, M)$ and $H^0(Y, L\otimes M) \cong H^0(C, \pi_*(L \otimes M))$, the linear maps $\mu_{L,M}$ and $\alpha_{L,M,\pi}$ have kernel and cokernel with the same dimension. In particular $\mu_{L,M}$ is surjective (resp. injective) if and only if $\alpha_{L,M,f}$ is surjective (resp. injective). Hence by Theorem 1.3 for the integer $q := g - \delta$ we obtain the following result.

PROPOSITION 4.4

Let C be an integral projective curve whose only singularities are ordinary nodes. Fix a set $S \subseteq \mathrm{Sing}(C)$ and set $g := p_a(C)$ and $\delta := \mathrm{card}(S)$. Let $\pi : Y \to C$ be the partial normalization of C in which we normalize only the points of S. Fix integers a, b with $a \geq g$ and $b \geq g$. Then for general element $\pi_*(L) \in M(C; a, S)$ and $\pi_*(M) \in M(C; b, S)$ the map $\alpha_{L,M,\pi}$ has maximal rank.

Remark 4.5. Let C be an integral projective curve whose only singularities are ordinary nodes or ordinary cusps. Set $g := p_a(C)$. Fix $S \subseteq \mathrm{Sing}(C)$ and set $s := \mathrm{card}(S)$. Let $\pi : Y \to C$ be the partial normalization of C in which we normalize the set S. For every $L \in \mathrm{Pic}(C)$ we have $\pi_*(L) \in M(C; x, S)$ with $x = \deg(L)+s = \deg(L)+p_a(C)-p_a(Y)$ and $h^0(Y, L) = h^0(C, \pi_*(L)), h^1(Y, L) = h^1(C, \pi_*(L))$. Hence taking a general $L \in \mathrm{Pic}^{x-s}(Y)$ we obtain that for every integer $x \geq g - 1$ a general $F \in M(C; x, S)$ has $h^1(C, F) = 0$, i.e. $h^0(C, F) = \deg(F) + 1 - g$.

(4.6) Let C be an integral projective curve whose only singularities are ordinary nodes and ordinary cusps. Let F and G be rank 1 torsion free sheaves on C with $\mathrm{Sing}(F) \cap \mathrm{Sing}(G) = \emptyset$. This condition is equivalent to the torsion freeness of $F \otimes G$. We have $\deg(F \otimes G) = \deg(F) + \deg(G)$ and $\mathrm{Sing}(F \otimes G) = \mathrm{Sing}(F) \cup \mathrm{Sing}(G)$. Since $F \otimes G$ has no torsion, here we will consider the usual multiplication map $\mu_{F,G}$. For the injectivity of $\mu_{F,G}$ it is usually not restrictive to assume F spanned (otherwise we reduce to the study of the subsheaf F' of F spanned by $H^0(C, F)$, although $\mathrm{Sing}(F') \neq \mathrm{Sing}(F)$ in general). Usually we will consider a range in which $F \otimes G$ is spanned and hence to obtain the surjectivity of $\mu_{F,G}$ it is necessary to assume that F and G are spanned.

(4.7) Let C be an integral projective curve whose only singularities are ordinary nodes and ordinary cusps. Let F be a rank 1 spanned torsion free sheaf on C and $\pi : Y \to C$ the partial normalization of C in which we normalize exactly the points of $\mathrm{Sing}(F)$. Set $L := \pi^*(F)/\mathrm{Tors}(\pi^*(F))$. By 4.1 we have $L \in \mathrm{Pic}(C)$, $F \cong \pi_*(L)$ and $\deg(F) = \deg(L) + \delta$ and $h^0(Y, L) = h^0(C, F)$. Since F is spanned, $\pi^*(L)$ is spanned and hence L is spanned.

Remark 4.8. Let U be a quasi-projective one-dimensional scheme with a unique singular point, P, which is either an ordinary node or an ordinary cusp. Let F and G be rank 1 torsion free sheaves on U such that F is not locally free at P, while G is locally free at

P, i.e. with $G \in \mathrm{Pic}(U)$. Let $\mathbf{K}_P$ be the skyscraper sheaf on U supported by P and with length l. By the last part of (4.1) there exist rank 1 torsion free sheaves F', F'', G', G'' on U with $F' \subset F \subset F''$, $G' \subset G \subset G''$, $F/F' \cong F''/F \cong G/G' \cong G''/G' \cong \mathbf{K}_P$ and such that F' and F'' are locally free, while G' and G'' are not locally free at P.

Remark 4.9. Let C be an integral projective curve whose only singularities are ordinary nodes and ordinary cusps. Take $S \subseteq \mathrm{Sing}(C)$ and any spanned $R \in \mathrm{Pic}(C)$ with $h^1(C, R) = 0$. By 4.8 we obtain the existence of $F \in M(C; x, S)$, $x = \deg(R) + \mathrm{card}(S)$, such that R is a subsheaf of F and $F/R \cong \boldsymbol{O}_S$. Since $\boldsymbol{O}_S$ is a skyscraper sheaf and $h^1(C, R) = 0$, we obtain $h^1(C, F) = 0$. Hence $h^0(C, F) = h^0(C, R) + \mathrm{card}(S)$. Since R is spanned, this implies the spannedness of F.

PROPOSITION 4.10

Fix non-negative integers g, q, s, s', a, b *with* $g \geq s + s' + q$, $a \geq g + s$, $b \geq g + s$ *and* $(a+l-g-s)(b+l-g-s') \geq a+b+l-g-s-s'$. *Let* C *be a general integral nodal curve with* $p_a(C) = g$ *and normalization of genus* q. *Fix* $S \subseteq \mathrm{Sing}(C)$ *and* $S' \subseteq \mathrm{Sing}(C)$ *with* $\mathrm{card}(S) = s$, $\mathrm{card}(S') = s'$ *and* $S \cap S' = \emptyset$. *Then for a general* $F \in M(C; a, S)$ *and a general* $G \in M(C; b, S')$ *the multiplication map* $\mu_{F,G}$ *is surjective.*

Proof. By Remark 4.9 for general F and G we have $h^1(C, F) = h^1(C, G) = h^1(C, F \otimes G) = 0$. Take general $L \in \mathrm{Pic}(C)$ and $M \in \mathrm{Pic}(C)$. By 1.3 and the assumptions on g, a, b, s and s' the linear map $\mu_{L,M}$ is surjective. Take as F' (resp. G') any element of $M(C; a, S)$ (resp. $M(C; b, S')$ containing L (resp. M) and with $F'/L \cong \boldsymbol{O}_S$ (resp. $G'/M \cong \boldsymbol{O}_S$) (Remark 4.8). By Remark 4.9 we have $h^1(C, F') = h^1(C, G') = 0$, $h^0(C, F') = h^0(C, L) + s$, $h^0(C, G') = h^0(C, M) + s'$ and both F' and G' are spanned. See $L \otimes M$ as a subsheaf of $F' \otimes G'$ with $F' \otimes G'/L \otimes M \cong \boldsymbol{O}_{S \cup S'}$. Since both F' and G' are spanned $\mathrm{Im}(\mu_{F',G'})$ spans $F' \otimes G'$. Hence $\dim(\mathrm{Im}(\mu_{F',G'})) \geq \dim(\mathrm{Im}(\mu_{L,M})) + s + s' = a + b - s - s' + l - g + s + s' = h^0(C, F' \otimes G')$. Hence $\mu_{F',G'}$ is surjective and we conclude by semicontinuity.

PROPOSITION 4.11

Fix non-negative integers g, q, s, s', a, b *with* $g \geq s + s' + q$, $a \geq g$, $b \geq g + s$ *and* $(a+l-g+s)(b+l-g+s') \leq a+b+l-g+s+s'$. *Let* C *be a general integral nodal curve with* $p_a(C) = g$ *and normalization of genus* q. *Fix* $S \subseteq \mathrm{Sing}(C)$ *and* $S' \subseteq \mathrm{Sing}(C)$ *with* $\mathrm{card}(S) = s$, $\mathrm{card}(S') = s'$ *and* $S \cap S' = \emptyset$. *Then for a general* $F \in M(C; a, S)$ *and a general* $G \in M(C; b, S')$ *the multiplication map* $\mu_{F,G}$ *is injective.*

Proof. Since $(a + l - g + s)(b + l - g + s'') \leq a + b + l - g + s + s'$, Theorem 1.3 shows that for a general $L \in \mathrm{Pic}^{a+s}(C)$ and a general $M \in \mathrm{Pic}^{b+s'}(C)$ the multiplication map $\mu_{L,M}$ is injective. By Remark 4.9 there is $F' \in M(C; a, S)$ and $G' \in M(C; b, S')$ with $F' \subset L$, $G' \subset M$, $L/F' \cong \boldsymbol{O}_S$ and $M/G' \cong \boldsymbol{O}_{S'}$. Since F' is a subsheaf of L and G' is a subsheaf of M the map $\mu_{F',G'}$ is injective. Since $h^1(C, F' \otimes G') = 0$, we conclude using semicontinuity.

There is a geometrically important case in which iterations of the multiplication maps do occur. Let X be a smooth projective curve of genus q and $L \in \mathrm{Pic}^k(X)$ with $h^0(X, L) = 2$ and L spanned. The ordered sequence of integers $\{h^0(X, L^{\otimes t})\}_{t \geq 0}$ uniquely determines the so-called scrollar invariants of the pencil L (see e.g. [12], §2). If $2k \leq q$ and X is a general k-gonal curve of genus q we have $h^0(X, L^{\otimes t}) = t + 1$ if $0 \leq t \leq [q/(k-1)]$, while $h^0(X, L^{\otimes t}) = kd + 1 - q$ (i.e. $h^1(X, L^{\otimes t}) = 0$) if $t > [q/(k-1)]$ ([2]). Fix an integer with $2 \leq a \leq g$. The equalities $h^0(X, L^{\otimes t}) = t + 1$ if $0 \leq t \leq a$ are equivalent to the surjectivity of all multiplication maps $\mu_{L^{\otimes b}, L}$ with $1 \leq b < a$. On singular curve when L is not locally free the sheaf $L \otimes L$ has always torsion and hence it is more interesting to consider the associated map $\alpha_{L,L,f}$ and its iterations.

PROPOSITION 4.12

Fix integers g, q and k with $g > q \geq 2k \geq 4$. Let C be an integral projective curve with $p_a(C) = g$ and whose only singularities are ordinary nodes and ordinary cusps and $f : X \to C$ its normalization. Assume that X is a general k-gonal curve of genus q and call L its degree k pencil. For every integer $t \geq 1$ set $F_t := f(L^{\otimes t})$. For every integer $t \leq [q/(k-1)]$ we have $h^0(C, F_t) = t + 1$ and for every integer $a < [q/(k-1)]$ the map $\alpha_{F_a, F_1, f}$ is surjective.

Proposition 4.12 follows at once from the next observation which also explain the meaning of the sheaves involved in the statement of 4.12.

Remark 4.13. By 4.2 each F_t is a rank 1 torsion free sheaf on X with $\deg(F_t) = tk + g - q$ and $\mathrm{Sing}(F_t) = \mathrm{Sing}(C)$. Since $h^0(C, F_t) = h^0(X, L^{\otimes t})$, the first assertion of 4.12 follows from [2]. If $t \geq 2$ we have $F_t \cong F_{t-1} \otimes F_1/\mathrm{Tors}(F_{t-1} \otimes F_1) \cong F_1^{\otimes t}/\mathrm{Tors}(F_1^{\otimes t})$ (4.1 and induction on t). Hence we obtain the last assertion of 4.12 from the first assertion of 4.12.

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Boundedness results for periodic points on algebraic varieties

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Abstract. We give some conditions under which the periods of a self map of an algebraic variety are bounded.

Keyword. Periodic points.

Let X be an algebraic variety over a field K and let $f : X \to X$ be a morphism. A point P in $X(K)$ is *f-periodic* if $f^n(P) = P$ for some $n > 0$, and the smallest such n is called the *period* of P. We shall prove that if X and f satisfy certain hypotheses, then the set of possible periods is finite.

Our results may be viewed as an analogue of the finiteness of the torsion of abelian varieties over finitely generated fields. It is then natural to ask for an analogue of the full Mordell–Weil theorem. We believe that the following conjecture is the appropriate generalization.

Conjecture 1. Let X be a proper algebraic variety over a finitely generated field K of characteristic zero and $f : X \to X$ a morphism. Suppose there exists a subset S of $X(K)$ which is Zariski dense in X and such that f induces a bijection of S onto itself. Then f is an automorphism.

This can be easily checked for $X = \mathbf{P}^n$ or X an abelian variety using heights and the Mordell–Weil theorem respectively.

1. Finitely generated fields

Theorem 1. *Let X be a proper variety over a field K which is finitely generated over the prime field and let $f : X \to X$ be a morphism.*

(i) *If* $\mathrm{char}(K) = 0$ *then the set of periods of all f-periodic points in $X(K)$ is finite.*

(ii) *If* $\mathrm{char}(K) = p \neq 0$ *then the prime to p parts of the set of periods is finite i.e. there exists $n > 0$ such that all the f^n-periodic points in $X(K)$ have periods which are powers of p.*

Many special cases of this result have been known for a long time, the first such being the theorem of Northcott ([5], Theorem 3), proving the finiteness of the number of periodic points in certain cases. We refer the reader to [4] for a more detailed list of references.

Remark. We do not know whether the periods can really be unbounded if $\mathrm{char}(K) > 0$.

The theorem is obvious if K is a finite field and we will reduce the general case to this one by a specialization argument. A little thought shows that the following proposition suffices to prove the theorem.

PROPOSITION 1

Let R be a discrete valuation ring with quotient field K and residue field k. Let $\mathcal{X}$ be a proper scheme of finite type over $\mathrm{Spec}(R)$ *and $f : \mathcal{X} \to \mathcal{X}$ an R-morphism. Assume that the conclusions of the theorem hold for f restricted to the special fibre, and that for each $n > 0$ there are only finitely many roots of unity contained in all extensions of k of degree $\leq n$. Then the same holds for f restricted to the generic fibre, except possibly in the case* $\mathrm{char}(K) = 0$ *and* $\mathrm{char}(k) = p > 0$, *when the result holds modulo powers of p.*

Proof. Let $p = 1$ if $\mathrm{char}(k) = 0$. The hypotheses imply that by replacing f with a suitable power we may assume that all the f-periodic points in $\mathcal{X}(k)$ have period a power of p. Let P be a f-periodic point in $\mathcal{X}(K)$. By replacing f by f^{p^n}, for some n which may depend on P, we may assume that the specialization of P in $\mathcal{X}(k)$ (which exists since $\mathcal{X}$ is proper) is a fixed point of f restricted to the special fibre. Let $\mathcal{Z}$ be the Zariski closure (with reduced scheme structure) of the f-orbit of P. $\mathcal{Z}$ is finite over $\mathrm{Spec}(R)$ with a unique closed point, hence is equal to $\mathrm{Spec}(A)$ where A is a finite, local R-algebra (with rank equal to the period of P) which, since A is reduced, is torsion free as an R-module.

The key observation of the proof is that f restricted to $\mathcal{Z}$ induces an *automorphism* of finite order of A (which we also denote by f): Since f preserves the orbit of P and $\mathcal{Z}$ is reduced, it follows that f induces a map from $\mathcal{Z}$ to itself, hence an endomorphism of A. f^n is the identity on the orbit of P for some $n > 0$, hence f^n is the identity on $A \otimes_R K$. Since A is torsion free, it follows that f^n is the identity on A as well.

Let m be the maximal ideal of A. Since $\mathcal{Z}$ is a closed subscheme of $\mathcal{X}$, it follows that the dimension of m/m^2 is bounded independently of P. By the hypothesis on roots of unity, we may replace f by some power, independently of P, so that the endomorphism of m/m^2 induced by f is the identity. Thus, f is a unipotent map with respect to the (exhaustive) filtration of A induced by powers of m. This implies that the order of f, hence the period of P, is a power of p. □

Remarks. (1) For any explicitly given example, the proof furnishes an effective method for computing a bound for the periods. (2) In the non-proper case one can prove the following result by the methods of this paper: Let S be a flat, separated, integral scheme of finite type over $\mathbf{Z}$, let $\mathcal{X}$ be a separated scheme of finite type over S and let $f : \mathcal{X} \to \mathcal{X}$ be an S-morphism. If one defines the notion of f-periodic points and periods for elements of $\mathcal{X}(S)$ in the obvious way, then the set of periods is again bounded. (3) One may also ask whether Theorem 1 itself holds without the assumption of properness, for example when X is arbitrary but f is finite. The results of Flynn–Poonen–Schaefer [1] may be viewed as some positive evidence, however, aside from this we do not have many other examples. If true, this would imply the uniform boundedness of torsion of abelian varieties and other similar conjectures.

In general the set of periodic points is of course not finite. However, one can often use some geometric arguments to deduce finiteness of the number of periodic points from Theorem 1 as in the following:

Lemma 1. *Let X be a proper variety over a finitely generated field K of characteristic zero and $f : X \to X$ a morphism. Suppose that there does not exist any positive dimensional* *subvariety Y of X such that f induces an automorphism of finite order of Y. Then the number of f-periodic points in $X(K)$ is finite.*

Proof. Theorem 1 implies that f induces an automorphism of finite order on the closure of the set of f-periodic points in $X(K)$. □

The following gives a useful method for checking the hypothesis of the previous lemma.

Lemma 2. Let X be a projective variety over a field K and $f : X \to X$ a morphism. Suppose there exists a line bundle $\mathcal{L}$ on X such that $f^*(L) \otimes L^{-1}$ is ample. Then there is no positive dimensional subvariety Y of X such that f induces an automorphism of finite order of Y.

Proof. By replacing X by $f^n(X)$ for some large n, we may assume that f is a finite morphism. Suppose there exists a Y as above and assume that $f^m|Y$ is the identity of Y. Then

$$f^m(L) \otimes L^{-1} = \bigotimes_{i=1}^{m-1} f^{i*}(f^*(L) \otimes L^{-1}).$$

By assumption $f^*(L) \otimes L^{-1}$ is ample so $f^m(L) \otimes L^{-1}$, being a tensor product of ample bundles, is also ample. But $f^m(L) \otimes L^{-1}|_Y$ is trivial, so it follows that Y must be 0-dimensional. □

In case $\mathcal{L}$ is also ample, finiteness can also be proved using heights, see for example [2]. One advantage of our method is that it applies also when f is an automorphism, in which case an ample $\mathcal{L}$ as above can never exist. Using this, one can for example extend the finiteness results of Silverman [6] to apply to all automorphisms of infinite order of projective algebraic surfaces X with $H^1(X, \mathcal{O}_X) = 0 = H^0(X, T_X)$ and the Picard rank $\rho = 2$.

The following proposition gives a simple class of examples for which boundedness of the periods holds for non-proper varieties.

PROPOSITION 2

Let K be a finitely generated extension of $\mathbf{Q}$ and let G be a linear algebraic group over K. Then there exists an integer $M(G)$ such that for all varieties X over K with a G action and for all g in $G(K)$, the set of periods of g-periodic points in X is bounded above by $M(G)$.

Proof. The G orbit of any $x \in X(K)$ is isomorphic to G/H, where H is a closed subgroup of G, hence we may restrict ourselves to the case where $X = G/H$. By Weil restriction of scalars we may assume that $K = \mathbf{Q}(t_1, t_2, \ldots, t_r)$ for some $r \geq 0$, and since any linear algebraic group can be embedded in GL_n, we may also assume that $G = GL_n$ for some integer n.

Assume $K = \mathbf{Q}$. Let $G_{\mathbf{Z}} = GL_{n,\mathbf{Z}}$ and let $H_{\mathbf{Z}}$ be the Zariski closure of H in $G_{\mathbf{Z}}$. We may also form the quotient $G_{\mathbf{Z}}/H_{\mathbf{Z}}$ on which there is a natural action of $G_{\mathbf{Z}}$ extending the action of G on G/H. Let x be a g-periodic point of G/H with period equal to l. There exists a finite set of primes S such that g extends to an element of $G_{\mathbf{Z}}(R)$ and x extends to an element of $G_{\mathbf{Z}}/H_{\mathbf{Z}}(R)$, where R is the ring of S-integers. For a prime number p not in S, let G_p, H_p, G_p/H_p, g_p, x_p denote the reductions mod p of the corresponding objects defined above and let l_p denote the g_p-period of x_p. It is clear that l_p divides the order of

$G_p(\mathbf{F}_p) = GL_n(\mathbf{F}_p)$ for all $p \notin S$ and for $p >> 0$, $l_p = l$. By Lemma 1 below, it follows that $l < M$ for some constant M independent of H, g and x.

Now let $K = \mathbf{Q}(t_1, t_2, \ldots, t_r)$ with $r > 0$. We repeat the arguments of the above paragraph, replacing $\mathbf{Z}$ with $\mathbf{Q}[t_1, t_2, \ldots, t_r]$. Since the rational points are dense in Spec($\mathbf{Q}$ $[t_1, t_2, \ldots, t_r]$), it follows that the same constant M bounds the periods. □

Lemma 3. *For each positive integer n, there exists an integer M_n such that if l is any integer which divides $|GL_n(\mathbf{F}_p)|$ for all $p >> 0$, then $l < M_n$.*

Proof. Let N be an integer such that for all $a > N$, $(\mathbf{Z}/a\mathbf{Z})^*$ contains an element of order greater than n. Let q be a prime and assume that q^b divides $|GL_n(\mathbf{F}_p)| = p^{n(n-1)/2}(p^n - 1)(p^{n-1} - 1)\cdots(p - 1)$ for all $p >> 0$. If $q^c > N$, then by Dirichlet's theorem on primes in arithmetic progressions there exist infinitely many primes p such that the order of p mod q^c is greater than n. This bounds the powers of q that can divide each of $p^n - 1, p^{n-1} - 1, \ldots, p - 1$ and hence bounds b. It is clear that if $q > N$ then $b = 0$, so we obtain a bound for l by multiplying together the bounds for each prime $q \leq N$. □

Remark. Note that Proposition 2 applies to all automorphisms of affine space, where in many cases finiteness of the number of periodic points is also known; see for example the paper [3] of Marcello.

2. *p*-Adic fields

Proposition 1 shows that one also has boundedness of periods for p-adic fields, up to powers of p, as long as the variety and the morphism extend to the ring of integers. We now show that in fact we can bound the extra powers of p.

Theorem 2. *Let $\mathcal{O}$ be the ring of integers in K, a finite extension of $\mathbf{Q}_p$, and let $\mathcal{X}$ be a proper scheme of finite type over* Spec($\mathcal{O}$). *Then there exist a constant $M > 0$ such that for any $\mathcal{O}$-morphism $f : \mathcal{X} \to \mathcal{X}$, the periods of the f-periodic points of $\mathcal{X}(K)$ are all less than M.*

If X is any variety over a finite field k then it is clear that a statement analogous to the theorem holds for $X(k)$: since this is a finite set the periods are bounded above by $|X(k)|$, and hence are bounded independently of the morphism. To bound the powers of p that occur, one sees from the proof of Proposition 1 that it is enough to prove the following:

PROPOSITION 3

Let $\mathcal{O}$ be the ring of integers in K, a discrete valuation ring of characteristic zero with residue field k of characteristic p. Let (A, m) be a local sub-$\mathcal{O}$-algebra of $\mathcal{O}^{p^n}$ of rank p^n which is preserved by the automorphism σ given by cyclic permutation of the coordinates. Furthermore, assume that σ acts trivially on m/m^2. Then $n < r = \nu(p)$ if $p > 2$ and $n \leq r$ if $p = 2$, where ν is the normalized valuation on K.

Proof. Assume that $n \geq r$ if $p > 2$ and $n > r$ if $p = 2$. Since σ acts trivially on m/m^2, it follows that σ^{p^t} acts trivially on m/m^{t+2} for all $t \geq 0$. Thus, by replacing A by a quotient algebra corresponding to the Zariski closure in Spec(A) of the $\sigma^{p^{n-1}}$ orbit of any $\mathcal{O}$ valued point, we obtain a local rank p subalgebra of $\mathcal{O}^p$ which is stable under (the new) σ and such that σ acts trivially on m/m^{r+1} (m/m^{r+2} if $p = 2$).

For a in A we denote by $\nu(a)$ the *minimum* of the valuations of the coordinates. Let

$$U(m) = \{a \in m \mid \nu(a_i) \neq \nu(a_j) \text{ for some } i, j\}$$

and let

$$P(m) = \{a \in U(m) \mid \nu(a) \leq \nu(b) \text{ for all } b \in U(m)\}.$$

Suppose $\nu(a) = 1$ for $a \in P(m)$. Since $P(m) \subset U(m)$, it follows that $\nu(\sigma^s(a) - a) = 1$ for some s, which in turn implies that $\sigma^s(a) - a \notin m^2$. This is a contradiction, hence $\nu(a) > 1$ for all $a \in P(m)$. Also, one easily sees that any element of m can be written as $a = x + b$ with $x \in \pi \cdot \mathcal{O}$ and $b \in U(m) \cup \{0\}$, where π in $\mathcal{O}$ is a uniformizing parameter.

Now let $a \in P(m)$ and consider $\sigma(a) - a$. Letting $t = r + 1$ if $p > 2$ and $t = r + 2$ if $p = 2$, we see that

$$\sigma(a) - a = \sum_i \prod_{j=1}^{t} (x_{i,j} + b_{i,j}),$$

with $x_{i,j} \in \pi \cdot \mathcal{O}$ and $b_{i,j} \in U(m) \cup \{0\}$. Expanding the products and using the fact that $\nu(b_{i,j}) > 1$, we see that

$$\sigma(a) - a = x + \sum_k z_k d_k \bmod \pi^{\nu(a)+t},$$

with $x, z_k \in \pi^{t-1} \cdot \mathcal{O}$ and $d_k \in P(m)$. Further, using the fact that a is in $U(m)$, one sees that $\nu(x) = \nu(\sigma(a) - a) = \nu(a)$. Thus, we get

$$0 = \sigma^p(a) - a = \sum_{i=0}^{p-1} \sigma^i(\sigma(a) - a) = p \cdot x + \sum_k z_k \left(\sum_{i=0}^{p-1} \sigma^i(d_k) \right) \bmod \pi^{\nu(a)+t}. \quad (1)$$

Now the d_k's are also in $P(m)$, so we have

$$\sigma(d_k) - d_k = w_k \bmod \pi^{\nu(a)+t-1},$$

with w_k in $\pi^t \cdot \mathcal{O}$. This implies that

$$\sum_{i=0}^{p-1} \sigma^i(d_k) = p \cdot d_k + \frac{(p-1)p}{2} w_k \bmod \pi^{\nu(a)+t-1}.$$

Substituting this in eq. (1) (using that the z_k's are in $\pi \cdot \mathcal{O}$) we get

$$p \cdot x + \sum_k z_k \left(p \cdot d_k + \frac{(p-1)p}{2} w_k \right) = 0 \bmod \pi^{\nu(a)+t}.$$

We have $\nu(x) = \nu(a) = \nu(d_k) = \nu(w_k)$, $\nu(z_k) \geq t - 1$ and $\nu(p) = r$. By the choice of t it follows that the only term in the above equation with valuation less than or equal to $\nu(a) + r$ is $p \cdot x$. This is a contradiction since $t > r$. □

Remark. The assumption of properness is used only to guarantee the existence of specializations. If we consider an arbitrary separated scheme $\mathcal{X}$ of finite type over Spec$(\mathcal{O})$, then we obtain boundedness of the periods for the set of periodic points in $\mathcal{X}(\mathcal{O})$. One can also construct examples for which the set of periods of the periodic points in $\mathcal{X}(K)$ is unbounded.

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Spectra of Anderson type models with decaying randomness

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Abstract. In this paper we consider some Anderson type models, with free parts having long range tails and with the random perturbations decaying at different rates in different directions and prove that there is a.c. spectrum in the model which is pure. In addition, we show that there is pure point spectrum outside some interval. Our models include potentials decaying in all directions in which case absence of singular continuous spectrum is also shown.

Keywords. Anderson model; absolutely continuous spectrum; mobility edge; decaying randomness.

1. Introduction

There have been but few models in higher dimensional random operators of the Anderson model type in which presence of absolutely continuous spectrum is exhibited. We present here one family of models with such behaviour.

The results here extend those of Krishna [10] and part of those in Kirsch–Krishna–Obermeit [9], Krishna–Obermeit [12] while making use of wave operators to show the existence of absolutely continuous spectrum, the results of Jaksic–Last [14] to show its purity and those of Aizenman [1] for exhibiting pure point spectrum.

The new results in this paper allow for long range free parts, have models with compact spectrum (in dimensions 2 and more) which contains both absolutely continuous and dense pure point spectrum. Our models include the independent randomness on a surface considered by Jaksic–Molchanov [15, 16] and Jaksic–Last [14, 13], while allowing for the randomness to extend into the bulk of the material.

The literature on the scattering theoretic and commutator methods for discrete Laplacian includes those of Boutet de Monvel–Sahbani [4, 5] who study deterministic operators on the lattice.

The scattering theoretic method that we use is applicable even when the free operator is not the discrete Laplacian but has long range off diagonal parts. We impose conditions on the free part in terms of the structure it has in its spectral representation.

2. Main results

The models we consider in this paper are related to the discrete Laplacian $(\Delta u)(n) = \sum_{|i|=1} u(n+i)$ on $\ell^2(\mathbb{Z}^\nu)$. We denote by $\mathbb{T}^\nu$ the ν dimensional torus $\mathbb{R}^\nu/2\pi\mathbb{Z}^\nu$ and σ the invariant probability measure on it. We use the coordinate chart $\{\vartheta : \vartheta = (\theta_1, \ldots, \theta_\nu), 0 < \theta_i < 2\pi\}$ and the representation $\sigma = \prod_{i=1}^{\nu}(\mathrm{d}\theta_i/2\pi)$ on the torus for calculations below without further explanation. Then Δ is unitarily equivalent to multiplication by $2\sum_{i=1}^{\nu}\cos(\theta_i)$ acting on $L^2(\mathbb{T}^\nu, \sigma)$, written in the above coordinates. We consider a bounded self adjoint operator H_0 which commutes with Δ and which is given by, on $L^2(\mathbb{T}^\nu, \mathrm{d}\sigma)$, an operator of multiplication by a function $h(\vartheta)$ there with h satisfying the assumptions below.

Hypothesis 2.1. Let h be a real valued $C^{3\nu+3}(\mathbb{T}^\nu)$ function satisfying

1. h is separable, i.e. $h(\vartheta) = \sum_{j=1}^{\nu} h_j(\theta_j)$.
2. The sets

$$C(h_j) = \left\{x : \frac{\mathrm{d}h_j}{\mathrm{d}\theta}(x) = 0\right\}$$

are finite for each $j = 1, \ldots, \nu$. Let

$$\tilde{C}(h_j) = \mathbb{T} \times \ldots \times \mathbb{T} \times C(h_j) \times \mathbb{T} \ldots \times \mathbb{T},$$

where the set $C(h_j)$ occurs in the jth position. We denote by

$$C = \cup_{j=1}^{\nu} \tilde{C}(h_j)$$

and note that this is a closed set of measure zero in $\mathbb{T}^\nu$.

We consider random perturbations of bounded self adjoint operators coming from functions as in the above hypothesis. We assume the following on the distribution of the randomness.

Hypothesis 2.2. Let μ be a positive probability measure on $\mathbb{R}$ satisfying:

1. μ has finite variance $\sigma^2 = \int x^2 \mathrm{d}\mu(x)$.
2. μ is absolutely continuous.

Finally we consider some sequences of numbers a_n indexed by the lattice $\mathbb{Z}^\nu$ or $\mathbb{Z}_+^{\nu+1} = \mathbb{Z}^+ \times \mathbb{Z}^\nu$ and assume the following on them.

Hypothesis 2.3. (1) a_n is a bounded sequence of non-negative numbers indexed by $\mathbb{Z}^\nu$ which is non-zero on an infinite subset of $\mathbb{Z}^\nu$.
(2) Let $g(R) = a_n \chi_{\{n\in\mathbb{Z}^\nu : |n_i|>R,\ \forall 1\le i\le\nu\}}$. Then $g \in L^1((1,\infty))$.
(1′) a_n is a bounded sequence of non-negative numbers which are non-zero on an infinite subset of $\mathbb{Z}_+^{\nu+1}$.
(2′) Let $g(R) = a_n \chi_{\{n\in\mathbb{Z}_+^{\nu+1} : |n_i|>R,\ \forall 1\le i\le\nu\}}$. Then $g \in L^1((1,\infty))$.

Remark 1. In the case of $\mathbb{Z}^\nu$ our hypothesis on the sequence a_n allows for the following type of sequences

- $a_n = (1+|n|)^\alpha,\ \ \alpha < -1$.
- $a_n = (1+|n_i|)^\alpha$, for some $i,\ \ \alpha < -1$.
- $a_n = \prod_{i=1}^{\nu}(1+|n_i|)^{\alpha_i},\ \ \alpha_i \le 0$ with $\sum_{i=1}^{\nu}\alpha_i < -1$.

Therefore in the theorems, on the existence of absolutely continuous spectrum, we can allow the potentials to be stationary along all but one direction in dimensions $\nu \geq 2$.

2. In the case of $\mathbb{Z}_+^{\nu+1}$, we can allow the sequence to be of the type

- $a_n = 0, \; n_1 > N$ and $a_n = 1, \;$ for $n_1 \leq N$, for some $0 < N < \infty$.
- $a_n = (1 + |n_1|)^\alpha, \; \alpha < -1$.
- $a_n = \prod_{i=1}^{\nu}(1 + |n_i|)^{\alpha_i}, \; \alpha_i \leq 0$ with $\sum_{i=1}^{\nu} \alpha_i < -1$.

Thus allowing for models with randomness on a the boundary of a half space.

For the purposes of determining the spectra of the models we are going to consider here in this paper we recall a definition given in Kirsch–Krishna–Obermeit [9], namely,

DEFINITION 2.4

Let a_n be a non-negative sequence, indexed by $\mathbb{Z}^\nu$ or $\mathbb{Z}_+^{\nu+1}$. Let μ be a positive probability measure on $\mathbb{R}$. Then the a-supp(μ) is defined as

1. In the case of $\mathbb{Z}^\nu$,

$$\text{a-supp}(\mu) = \bigcap_{\substack{k \in \mathbb{Z}+ \\ k \neq 0}} \left\{ x : \sum_{n \in k\mathbb{Z}^\nu} \mu(a_n^{-1}(x - \epsilon, x + \epsilon)) = \infty, \quad \forall \; \epsilon > 0 \right\}.$$

2. In the case of $\mathbb{Z}_+^{\nu+1}$,

$$\text{a-supp}(\mu) = \bigcap_{\substack{k \in \mathbb{Z}+ \\ k \neq 0}} \left\{ x : \sum_{n \in k\mathbb{Z}_+^{\nu+1}} \mu(a_n^{-1}(x - \epsilon, x + \epsilon)) = \infty, \quad \forall \; \epsilon > 0 \right\}.$$

Remark. 1. In the sums occurring in the above definition we set $\mu(a_n^{-1}(x - \epsilon, x + \epsilon)) \equiv 0$, for those n for which $a_n = 0$. This notation is to allow for sequences a_n that are everywhere zero except on an axis for example.

2. We note that when a_n is a constant sequence $a_n = \lambda \neq 0$,

$$\text{a-supp}(\mu) = \lambda \cdot \text{supp}(\mu).$$

3. When a_n converge to zero as $|n|$ goes to ∞, the a-supp(μ) is trivial if μ has compact support. It could be trivial even for some class of μ of infinite support depending upon the sequence a_n.

4. If a_n is bounded below by a positive number on an infinite subset along the directions of the axes in $\mathbb{Z}^\nu$ (respectively $\mathbb{Z}_+^{\nu+1}$), then the a-supp(μ) could be non-trivial even for compactly supported μ.

We consider the operator (for $u \in \ell^2(\mathbb{Z}^+)$),

$$(\Delta_+ u)(n) = \begin{cases} u(n+1) + u(n-1), & n > 0, \\ u(1), & n = 0. \end{cases}$$

Below we use either Δ_+ or its extension by $\Delta_+ \otimes I$ to $\ell^2(\mathbb{Z}_+^{\nu+1})$ by the same symbol, the correct operator is understood from the context. Given a real valued continuous function on the torus $\mathbb{T}^\nu$, we consider the bounded self adjoint operators H_0 on $\ell^2(\mathbb{Z}^\nu)$ which is unitarily equivalent to multiplication by h on $\ell^2(\mathbb{T}^\nu, \sigma)$. We also denote the extension $I \otimes H_0$ of H_0 to $\ell^2(\mathbb{Z}_+^{\nu+1})$ by the symbol H_0 and $L^2(\mathbb{T}^\nu, \sigma)$ as simply $L^2(\mathbb{T}^\nu)$ in the sequel.

We then consider the random operators

$$H^\omega = H_0 + V^\omega, \quad V^\omega = \sum_{n \in I} a_n q^\omega(n) P_n, \quad \text{on } \ell^2(\mathbb{Z}^\nu),$$

$$H_+^\omega = H_{0+} + V^\omega, \quad V^\omega = \sum_{n \in I} a_n q^\omega(n) P_n, \, H_{0+} = \Delta_+ + H_0, \quad \text{on } \ell^2(\mathbb{Z}_+^{\nu+1}), \tag{1}$$

where P_n is the orthogonal projection onto the one dimensional subspace generated by δ_n when $\{\delta_n\}$ is the standard basis for $\ell^2(I)$ ($I = \mathbb{Z}^\nu$ or $\mathbb{Z}_+^{\nu+1}$). $\{q^\omega(n)\}$ are independent and identically distributed real valued random variables with distribution μ. The operator H_0 is some bounded self adjoint operator to be specified in the theorems later.

Then our main theorems are the following. First we state a general theorem on the spectrum of H_0 in such models. For this we consider the operator H_0 to denote a bounded self adjoint operator on $\ell^2(\mathbb{Z}^\nu)$ coming from a function h satisfying the Hypothesis 2.1 and Δ_+ defined as before.

Theorem 2.5. *Let H_0 and H_{0+} be the operators defined as in eq.* (1), *coming from functions h satisfying the hypothesis* 2.1(1)(2). *Let*

$$E_+ = \sum_{j=1}^{\nu} \sup_{\theta \in [0, 2\pi]} h_i(\theta), \quad E_- = \sum_{j=1}^{\nu} \inf_{\theta \in [0, 2\pi]} h_i(\theta).$$

Then, the spectra of both H_0 and H_{0+} are purely absolutely continuous and

$$\sigma(H_0) = [E_-, E_+], \quad \textit{and} \quad \sigma(H_{0+}) = [-2 + E_-, 2 + E_+].$$

Part of the essential spectra of the operators h^ω and H_+^ω are determined via Weyl sequences constructed from rank one perturbations of the free operators H_0 and H_{0+} respectively. The proof of this theorem is done essentially on the line of the proof of Theorem 2.4 in [9].

Theorem 2.6. *Let the indexing set I be $\mathbb{Z}^\nu$ or $\mathbb{Z}_+^{\nu+1}$ and consider the operator H_0 coming from a function h satisfying the conditions of hypothesis* 2.1(1) *in the case of $I = \mathbb{Z}^\nu$ and consider the associated H_{0+} in the case of $I = \mathbb{Z}_+^{\nu+1}$. Suppose $q^\omega(n)$, $n \in I$ are i.i.d random variables with the distribution μ satisfying the hypothesis* 2.2(1). *Let a_n be a sequence indexed by I satisfying the hypothesis* 2.3(1) (*or* (1′) *as the case may be*). *Assume also that* $0 \in$ a-supp(μ), *then*

$$\bigcup_{\lambda \in \text{a-supp}(\mu)} \sigma(H_0 + \lambda P_0) \subset \sigma_{\text{ess}}(H^\omega) \textit{ almost every } \omega$$

and

$$\bigcup_{\lambda \in \text{a-supp}(\mu)} \sigma(H_{0+} + \lambda P_0) \subset \sigma_{\text{ess}}(H_+^\omega) \textit{ almost every } \omega.$$

Remark 1. When μ has compact support and a_n goes to zero at infinity, or when μ ha infinite support but a_n has appropriate decay at infinity, there is no essential spectrum outside that of H_0 for H^ω almost every ω. So the point of this theorem is to show that ther is essential spectrum outside that of H_0 based on the properties of the pairs $(\{a_n\}, \mu)$.

2. In Kirsch–Krishna–Obermeit [9] some examples of random potentials which have essential spectrum outside $\sigma(H_0)$ even when a_n goes to zero at ∞ were given. The examples presented there had a-supp(μ) as a half axis or the whole axis, this is because of the decay of the sequences a_n. Here however, since we allow for a_n to be constant along some directions, our examples include cases where the spectra of H^ω are compact with some essential spectrum outside $\sigma(H_0)$.

We let $E_\pm$ be as in Theorem 2.5.. We also set $\mathcal{H}_{\omega,n}$ to be the cyclic subspace generated by δ_n and H^ω.

Theorem 2.7. *Consider a bounded self adjoint operator H_0 coming from a function h satisfying the conditions of hypothesis* 2.1(1), (2). *Suppose q^ω are i.i.d random variables with the distribution μ satisfying the hypothesis* 2.2(1).

1. *Let $I = \mathbb{Z}^\nu$ and a_n be a sequence satisfying the hypothesis* 2.3(1), (2). *Then,*

$$\sigma_{ac}(H^\omega) \supset [E_-,\ E_+] \text{ almost every } \omega.$$

Further when μ satisfies the hypothesis 2.3(2), *$a_n \neq 0$ on $\mathbb{Z}^\nu$, $\mathcal{H}_{\omega,n}$, $\mathcal{H}_{\omega,m}$ not mutually orthogonal for any n, m in $\mathbb{Z}^\nu$ for almost all ω and $E_\pm$ as in theorem* 2.5., *we also have*

$$\sigma_s(H^\omega) \subset \mathbb{R} \setminus (E_-,\ E_+) \text{ almost every } \omega.$$

2. *Let $I = \mathbb{Z}_+^{\nu+1}$ and a_n be a sequence satisfying the hypothesis* 2.3(1′), (2′). *Then,*

$$\sigma_{ac}(H_+^\omega) \supset [-2 + E_-,\ 2 + E_+] \text{ almost every } \omega.$$

Further when μ satisfies the hypothesis 2.3(2), *$a_n \neq 0$ on a subset of $\mathbb{Z}_+^{\nu+1}$ that contains the surface $\{(0,n) : n \in \mathbb{Z}^\nu\}$, the subspaces $\mathcal{H}_{\omega,n}$, $\mathcal{H}_{\omega,m}$ are not mutually orthogonal almost every ω for m, n in $\{(0,k) : k \in \mathbb{Z}^\nu\}$, we also have*

$$\sigma_s(H^\omega) \subset \mathbb{R} \setminus (-2, +E_-,\ 2 + E_+) \text{ almost every } \omega.$$

Remark 1. When μ is absolutely continuous the theorem says that the spectrum of H^ω in (E_-, E_+) (respectively in $(-2 + E_-, 2 + E_+)$ for the $\mathbb{Z}_+^{\nu+1}$ case) is purely absolutely continuous, this is a consequence of a remarkable theorem of Jaksic–Last [14] who showed that in such models with independent randomness, with the randomness non-zero a.e. on a sufficiently big set (H_0 can be any bounded self adjoint operator in their theorem, provided the set of points where the randomness lives gives a cyclic family for the operators H^ω), whenever there is an interval of a.c. spectrum it is pure almost every ω. Their proof is based on considering spectral measures associated with rank one perturbations and comparing the spectral measures of different vectors (which give rise to the rank one perturbations).

2. Our theorem extends the models of surface randomness considered by Jaksic–Last [13], to allow for thick surfaces where the randomness is located in a strip beyond the surface into the bulk of the material. Such models (which are obtained by taking $a_n = 0,\ n_1 > N,\ a_n = 1,\ n_1 \leq N$ for some finite N) have purely absolutely continuous spectrum in $(-2\nu - 2, 2\nu + 2)$. The purity of the a.c. spectrum is again a consequence of a theorem of Jaksic–Last [14].

Finally we have the following theorem on the purity of a part of the pure point spectrum. We denote

$$e_+ = \sup \sigma(H_{0+}),\quad e_- = \inf \sigma(H_{0+}) \text{ and } e_0 = \max(|e_-|, |e_+|). \tag{2}$$

Theorem 2.8. *Consider a bounded self adjoint operator H_0 coming from a function h satisfying the conditions of hypothesis* 2.1. *Let I be the indexing set and suppose $q^{\omega}(n)$, $n \in I$ are i.i.d random variables with the distribution μ satisfying the hypothesis* 2.2(1), (2). *Assume further that the density $f(x) = \mathrm{d}\mu(x)/\mathrm{d}x$ is bounded. Set $\sigma_1 = \int \mathrm{d}\mu(x)|x|$. Then,*

1. *Let $I = \mathbb{Z}^{\nu}$ and let a_n be a sequence satisfying the hypothesis* 2.3(1), (2). *Then there is a critical energy $E(\mu) > E_0$ depending upon the measure μ such that*

$$\sigma_c(H^{\omega}) \subset (-E(\mu), E(\mu)) \ \textit{almost every} \ \omega.$$

2. *Let $I = \mathbb{Z}_+^{\nu+1}$ and let a_n be a sequence satisfying the hypothesis* 2.3(1′), (2′). *Then there is a critical energy $e(\mu) > e_0$ such that*

$$\sigma_c(H_+^{\omega}) \subset (-e(\mu), e(\mu)) \ \textit{almost every} \ \omega.$$

Remark 1. The $E(\mu)$ and $e(\mu)$, while finite may fall outside the spectra of the operators H^{ω} and H_+^{ω}, for some pairs (a_n, μ) when μ is of compact support, so for such pairs this theorem is vacuous. However since the numbers $E(\mu)$ (respectively $e(\mu)$) depend only on the operators H_0 (respectively H_{0+}) and the measure μ we can still choose sequences a_n and μ of large support such that the theorem is non-trivial for such cases. Of course for μ of infinite support, the theorem says that there is always a region where pure point spectrum is present.

2. Since we allow for potentials with a_n not vanishing at ∞ in all directions, we could not make use of the technique of Aizenman–Molchanov [3], for exhibiting pure point spectrum.

3. When μ has compact support, comparing the smallness of a moment near the edges of support one exhibits pure point spectrum there by using the Lemma 5.1 proved by Aizenman [1], comparing the decay rate in energy of the sums of low powers of the integral kernels of the free operators with some uniform bounds of low moments of the measure μ weighted with singular but integrable factors occurring to the same power.

As in Kirsch–Krishna–Obermeit [9], Jaksic–Last [14] we also have examples of cases when there is pure a.c. spectrum in an interval and pure point spectrum outside. The part about a.c. spectrum follows as a corollary of theorem 2.6., while the pure point part is proven as in [9] (following the proof of their theorem 2.3, where Δ can be replaced by any bounded self adjoint operator on $\ell^2(\mathbb{Z}^d)$ and work through the details, as is done in Krishna–Obermeit [12], Lemma 2.1). Further when $H_0 = \Delta$, the Jaksic–Last condition on the mutual non-orthogonality of the subspaces $\mathcal{H}_{\omega,n}$, $\mathcal{H}_{\omega,m}$ is valid since given any n, m we can find a k so that $\langle \delta_n, \Delta^k \delta_m \rangle > 0$ (reason, take $k = |n - m| = \sum_{i=1}^{\nu} |n_i - m_i|$, then

$$\Delta^k = \left(\sum_{i=1}^{\nu} T_i + T_i^{-1}\right)^k = c\prod_{i=1}^{\nu} T_i^{|n_i - m_i|} + c\prod_{i=1}^{\nu} T_i^{-|n_i - m_i|} + \text{lower order}$$

with T_i denoting the bilateral shift in the ith direction and c a strictly positive constant coming from the multinomial expansion). We see that we can add any operator diagonal in the basis $\{\delta_n\}$ to Δ without altering the conclusion.

COROLLARY 2.9

Let a_n be a sequence as in Hypothesis 2.3 *and μ as in Hypothesis* 2.2. *Let $H_0 = \Delta$. Assume further that $a_n \neq 0$, $n \in \mathbb{Z}^{\nu}$ goes to zero at ∞ and* a-supp(μ) $= \mathbb{R}$. *Then we have, for almost all ω,*

1. $\sigma_{ac}(H^{\omega}) = [-2\nu, 2\nu]$.
2. $\sigma_{pp}(H^{\omega}) = \mathbb{R} \setminus (-2\nu, 2\nu)$.
3. $\sigma_{sc}(H^{\omega}) = \emptyset$.

The h given in the corollary below is a smooth $2\pi\mathbb{Z}^{\nu}$ periodic function, so it satisfies the conditions of the Hypothesis 2.1. It is also not hard to verify that, because of the term $\sum_{i=1}^{\nu}\cos(\theta_i)$ occurring in its expression, the cyclic subspaces generated by the associated H_0 on any pair of $\{\delta_n, \delta_m\}$ are mutually non-orthogonal.

COROLLARY 2.10

Let a_n be a sequence as in Hypothesis 2.3 and μ as in Hypothesis 2.2. Let H_0 be a bounded self adjoint operator coming from the function h given by $h(\vartheta) = \sum_{j=1}^{\nu}\sum_{k=1}^{N}\cos(k\theta_j)$. Assume that $a_n \neq 0$, $n \in \mathbb{Z}^{\nu}$ goes to zero at ∞ and a-supp$(\mu) = \mathbb{R}$. Then we have, for almost all ω,

1. $\sigma_{ac}(H^{\omega}) = [E_-, E_+]$.
2. $\sigma_{pp}(H^{\omega}) = \mathbb{R} \setminus (E_-, E_+)$.
3. $\sigma_{sc}(H^{\omega}) = \emptyset$.

3. Proofs

In this section we present the proofs of the theorems stated in the previous section.

Proof of Theorem 2.5. The statement about the spectrum of H_0 follows from the Hypothesis 2.1(1) on the function h. Each of the functions h_i is a real valued continuous 2π periodic function, hence has compact range. By the intermediate value theorem, we see that the range of $(0, 2\pi)$ under h_i is also an interval. Since the spectrum of H_0 is the algebraic sum of the intervals I_i, – if H_{0j} denotes the operator associated with h_j on $\ell^2(\mathbb{T})$, then $H_0 = H_{01} \otimes I + I \otimes H_{02} \otimes I + \cdots + I \otimes H_{0\nu}$ hence this fact – the statement follows.

We note that $\ell^2(\mathbb{Z}^+)$ is unitarily equivalent to the Hardy space $\mathbb{H}^2(\mathbb{T})$ of functions on $\mathbb{T}$ whose negative Fourier coefficients vanish. Under this unitary transformation, the operator Δ_+ is unitarily equivalent to the operator of multiplication by the function $2\cos(\theta)$ acting on $\mathbb{H}^2(\mathbb{T})$, which can be seen by the definitions of Δ_+, $\mathbb{H}^2(\mathbb{T})$ and the unitary isomorphism U that takes $\mathbb{H}^2(\mathbb{T})$ to $\ell^2(\mathbb{Z}^+)$ (explicitly this is $2\pi(Uf)(n) = \int_0^{2\pi} \mathrm{d}\theta\, \mathrm{e}^{-in\theta} f(\theta)$). Therefore the spectrum of Δ_+ is $[-2, 2]$ and is purely absolutely continuous (there are no eigenvalues). Therefore the spectrum of H_{0+} is also purely a.c. and equals $\sigma(\Delta_+) + [E_-, E_+]$, with $E_{\pm}$ as above. Hence the theorem follows.

Proof of Theorem 2.6. We prove the theorem for the case H^{ω} the proof for the case H_+^{ω} proceeds along essentially the same lines and we give a sketch of the proof for that case. We consider any $\lambda \in$ a-supp(μ), which means that we have

$$\sum_{n \in k\mathbb{Z}_+^{\nu+1}} \mu(a_n^{-1}(\lambda - \epsilon, \lambda + \epsilon)) = \infty, \quad \forall k \in \mathbb{Z}^+, k \neq 0, \quad \text{and all } \epsilon > 0.$$

We consider the distance function $|n| = \max|n_i|, i = 1, \ldots, \nu$ on $\mathbb{Z}^{\nu}$. We consider the events, with $\epsilon > 0$, $m \in k\mathbb{Z}^{\nu}$,

$$A_{k,m,\epsilon} = \{\omega : a_m q^{\omega}(m) \in (\lambda - \epsilon,\ \lambda + \epsilon),\ \ |a_n\, q^{\omega}(n)| < \epsilon,\ \forall 0 < |n - m| < k - 1\}$$

and

$$B_{k,m,\epsilon} = \{\omega : \ |a_n\, q^{\omega}(n)| < \epsilon, \ \forall 0 \leq |n-m| < k-1\},$$

where the index n in the definition of the above sets varies in $\mathbb{Z}^{\nu}$. Then each of the events $A_{k,m,\epsilon}$ are mutually independent for fixed k and ϵ as m varies in $k\mathbb{Z}^{\nu}$, since the random variable defining them live in disjoint regions in $\mathbb{Z}^{\nu}$. Similarly $B_{k,m,\epsilon}$ is a collection of mutually independent events for fixed k and ϵ as m varies in $k\mathbb{Z}^{\nu}$. Further these events have a positive probability of occurrence, the probability having a lower bound given by

$$\mathrm{Prob}(A_{k,m,\epsilon}) \geq \mu(a_m^{-1}(\lambda-\epsilon, \lambda+\epsilon))(\mu(-c\,\epsilon, c\,\epsilon))^{(k-1)^{\nu+1}}$$

and

$$\mathrm{Prob}(B_{k,m,\epsilon}) \geq (\mu(-c\,\epsilon, c\,\epsilon))^{(k-1)^{\nu+1}},$$

where we have taken $c = \inf_{n\in\mathbb{Z}^{\nu}} a_n^{-1} > 0$. The definition of c implies that

$$(-c\,\epsilon, c\,\epsilon) \subset a_m^{-1}(-\epsilon, \epsilon), \ \forall \ m \in \mathbb{Z}^{\nu}.$$

Therefore the assumption that $\lambda \in$ a-supp(μ) implies that $\forall k \in \mathbb{Z}^{+} \setminus \{0\}$,

$$\sum_{m\in k\mathbb{Z}^{\nu}} \mathrm{Prob}(A_{k,m,\epsilon}) \geq (\mu(-c\,\epsilon, c\,\epsilon))^{(k-1)^{\nu+1}} \sum_{m\in k\mathbb{Z}^{\nu}} \mu(a_m^{-1}(\lambda-\epsilon, \lambda+\epsilon)) = \infty$$

and similarly

$$\sum_{m\in k\mathbb{Z}^{\nu}} \mathrm{Prob}(B_{k,m,\epsilon}) = \infty, \ \forall k \in \mathbb{Z}^{+} \setminus \{0\}.$$

Then Borel–Cantelli lemma implies that for all $\epsilon > 0$, (setting $R_{\epsilon} = (\lambda-\epsilon, \lambda+\epsilon)$ and $S_{\epsilon} = (-\epsilon, \epsilon)$ and $\Lambda_k(m) = \{n \in \mathbb{Z}^{\nu} : 0 \leq |n-m| < k-1\}$), the events

$$\Omega(\epsilon, k) = \bigcap_{\substack{m\in I\subset\mathbb{Z}^{\nu}\\ \#I=\infty}} \{\omega : a_m q^{\omega}(m) \in R_{\epsilon}, \ a_n\, q^{\omega}(n) \in S_{\epsilon}, \ \forall n \in \Lambda_k(m) \setminus \{m\}\}$$

have full measure. Therefore the event

$$\Omega_1 = \bigcap_{l,k\in\mathbb{Z}^{+}\setminus\{0\}} \Omega\left(\frac{1}{l}, k\right)$$

has full measure, being a countable intersection of sets of full measure. Similarly the sets

$$\Omega_2(\epsilon, k) = \bigcap_{\substack{m\in I\subset\mathbb{Z}^{\nu}\\ \#I=\infty}} \{\omega : a_n\, q^{\omega}(n) \in S_{\epsilon}, \ \forall n \in \Lambda_k(m)\}$$

have full measure. Therefore the events

$$\Omega_2 = \bigcap_{l,k\in\mathbb{Z}^{+}\setminus\{0\}} \Omega_2\left(\frac{1}{l}, k\right)$$

have full measure.

We take

$$\Omega_0 = \Omega_1 \cap \Omega_2$$

and note that it has full measure. We use this set for further analysis. We denote $H(\lambda) = H_0 + \lambda P_0$. Then suppose $E \in \sigma(H(\lambda))$, then there is a Weyl sequence ψ_l of compact support, $\psi_l \in \ell^2(\mathbb{Z}^\nu)$ such that $\|\psi_l\| = 1$ and

$$\|(H(\lambda) - E)\psi_l\| < \frac{1}{l}.$$

Suppose the support of ψ_l is contained in a cube of side $r(l)$, centered at 0. Denote by $\Lambda_k(x)$ a cube of side k centered at x in $\mathbb{Z}^\nu$. We denote $V^\omega(n) = a_n q^\omega(n)$, for ease of writing. We then find cubes $\Lambda_{r(l)}(\alpha_l)$ centered at the points α_l such that

$$|V^\omega(\alpha_l) - \lambda| < \frac{1}{l}, \quad |V^\omega(x)| < \frac{1}{l}, \quad \forall x \in \Lambda_{r(l)}(\alpha_l) \setminus \{\alpha_l\}.$$

Now consider $\phi_l(x) = \psi_l(x - \alpha_l)$. Then by the translation invariance of H_0 we have for any $\omega \in \Omega_0$,

$$\begin{aligned} \|(H^\omega - E)\phi_l\| &\le \|(H_0 + V^\omega(\cdot + \alpha_l)) - E)\psi_l\| \\ &\le \|(H_0 + \lambda P_0 - E)\psi_l\| + \|V^\omega(\cdot + \alpha_l) - \lambda P_0)\phi_l\| \\ &\le \frac{1}{l} + \frac{1}{l}. \end{aligned} \tag{3}$$

Clearly since ϕ_l is just a translate of ψ_l, $\|\phi_l\| = 1$ for each l. We now have to show that the sequence ϕ_l goes to zero weakly. This is ensured by taking successively α_k large so that

$$\cup_{j=1}^{k-1} \mathrm{supp}(\phi_j) \cap \Lambda_{r(k)}(\alpha_k) = \emptyset, \quad \text{and} \quad \mathrm{supp}(\phi_k) \subset \Lambda_{r(k)}(\alpha_k).$$

This is always possible for each ω in Ω_0 by its definition, thus showing that the point E is in the spectrum of H^ω, concluding the proof of the theorem.

Proof of Theorem 2.7. We first consider the part (1) of the theorem and address the proof of (2) later. The set $\mathcal{C}$ below is as in Hypothesis 2.1. We consider the set

$$\mathcal{D} = \{\phi \in \ell^2(\mathbb{Z}^\nu) : \mathrm{supp}(\widehat{\phi}) \subset \mathbb{T}^\nu \setminus \mathcal{C} \text{ and } \widehat{\phi} \text{ smooth}\}, \tag{4}$$

where we denote by $\widehat{\phi}$ the function in $\ell^2(\mathbb{T}^\nu)$ obtained by taking the Fourier series of ϕ. Since the set $\mathcal{C}$ is of measure zero, such functions form a dense subset of $\ell^2(\mathbb{Z}^\nu)$. We also note that the set $\mathcal{C}$ is closed in $\mathbb{T}^\nu$, thus its complement is open (in fact it is a finite union of open rectangles) and each ϕ in $\mathcal{D}$ has compact support in $\mathbb{T}^\nu \setminus \mathcal{C}$.

We first consider the case when μ has compact support. The general case is addressed at the end of the proof.

If we show that the sequence $W(t, \omega) = e^{itH^\omega} e^{-itH_0}$ is strongly Cauchy for any ω, then standard scattering theory implies that $\sigma_{ac}(H^\omega) \supset \sigma_{ac}(H_0)$ for that ω. We will show below this Cauchy property for a set ω of full measure.

To this end we consider the quantity

$$\mathbb{E}\{\|(W(t, \omega) - W(r, \omega))\phi\|\}, \quad \phi \in \mathcal{D} \tag{5}$$

and show that this quantity goes to zero as t and r go to $+\infty$. Then the integrand being uniformly bounded by an integrable function $\|\phi\|$ and since ϕ comes from a dense set, Lebesgue dominated convergence theorem implies that $W(t, \omega)$ is strongly Cauchy for every ω in a set of full measure $\Omega(f)$ that depends on f in $\ell^2(\mathbb{Z}^\nu)$. Since $\ell^2(\mathbb{Z}^\nu)$ is separable, we take the countable dense set $\mathcal{D}_1$ and consider

$$\Omega_3 = \bigcap_{f \in \mathcal{D}_1} \Omega(f)$$

which also has full measure being a countable intersection of sets of full measure. For each $\omega \in \Omega_3$, $W(t, \omega)$ is a family of isometries such that $W(t, \omega)f$ is a strongly Cauchy sequence for each $f \in \mathcal{D}_1$, therefore this property also extends by density of $\mathcal{D}_1$ to all of $\ell^2(\mathbb{Z}^\nu)$ point wise in Ω_3. Thus it is enough to show that the quantity in (5) goes to zero as t and r go to $+\infty$.

We have the following inequality coming out of Cauchy–Schwarz and Fubini, for an arbitrary but fixed $\phi \in \mathcal{D}$. In the inequality below we denote, for convenience the operator of multiplication by the sequence a_n as A and in the first step we write the left hand side as the integral of the derivative to obtain the right hand side

$$\begin{aligned} \mathbb{E}\{\|W(t,\omega)\phi - W(r,\omega)\phi\|\} &\leq \mathbb{E}\left\{\left\| \int_r^t \mathrm{d}s \ \mathrm{e}^{isH^\omega} V^\omega \mathrm{e}^{-isH_0}\phi\right\|\right\} \\ &\leq \int_r^t \mathrm{d}s \ \mathbb{E}\{\|V^\omega \mathrm{e}^{-isH_0}\phi\|\} \\ &\leq \int_r^t \mathrm{d}s \ \|\sigma A \mathrm{e}^{-isH_0}\phi\|. \end{aligned} \tag{6}$$

The required statement on the limit follows if we now show that the quantity in the integrand of the last line is integrable in s. To do this we define the number

$$v_\phi = \inf_j \inf\{|h_j'(\theta_j)| : \vartheta \in \operatorname{supp}\widehat{\phi}\}, \quad \vartheta = (\theta_1, \ldots, \theta_\nu). \tag{7}$$

We note that since the support of $\widehat{\phi}$ is compact in $\mathbb{T}^\nu \setminus \mathcal{C}$, $h_j{}'$, $j = 1, \ldots, \nu$ (which are continuous by assumption), have non-zero infima there, so v_ϕ is strictly positive. Then consider the inequalities

$$\begin{aligned} \|\sigma A \mathrm{e}^{-isH_0}\phi\| &\leq \|\sigma A F(|n_j| > v_\phi\, s/4 \ \forall j\ )\mathrm{e}^{-isH_0}\phi\| \\ &\quad + \|\sigma A F(|n_j| \leq v_\phi\, s/4 \ \text{for some} \ \ j)\mathrm{e}^{-isH_0}\phi\| \\ &\leq \sigma|g(s)|\|\phi\| + \sigma\|A\|\|F(|n_j| \leq v_\phi\, s/4, \ \text{for some} \ \ j)\mathrm{e}^{-isH_0}\phi\|, \end{aligned} \tag{8}$$

where we have used the notation that $F(S)$ denotes the orthogonal projection (in $\ell^2(\mathbb{Z}^\nu)$) given by the indicator function of the set S and used the function g as in the Hypothesis 2.3(2) which is integrable in s, so the first term is integrable in s. We concentrate on the remaining term.

$$\|F(|n_j| \leq v_\phi\, s/4, \ \text{for some} \ \ j)\mathrm{e}^{-isH_0}\phi\|. \tag{9}$$

To estimate the term we go to the spectral representation of H_0 and do the computation there as follows. Since $|n_j| \leq v_\phi\, s/4$ for some j, we may without loss of generality set $j = 1$ and proceed with the calculation. Let us denote the set $S_1(s) = \{n : |n_1| \leq v_\phi s/4, \ \ n_j \in \mathbb{Z}, \ \ j \neq 1\}$. In the steps below we pass to $L^2(\mathbb{T}^\nu)$ via the Fourier series, (where the normalized measure on $\mathbb{T}^\nu$ is denoted by $\mathrm{d}\sigma(\vartheta)$).

$$\begin{aligned} T &= \|F(|n_1| \leq v_\phi\, s/4)\mathrm{e}^{-isH_0}\phi\| \\ &= \left\{\sum_{n \in S_1(s)} \left|\langle \delta_n, \mathrm{e}^{-isH_0}\phi\rangle\right|^2\right\}^{1/2} \\ &= \left\{\sum_{n \in S_1(s)} \left|\int_{\mathbb{T}^\nu} \mathrm{d}\vartheta \ \ \mathrm{e}^{-in\cdot\vartheta - is\sum_{j=1}^\nu h_j(\theta_j)}\widehat{\phi}(\vartheta)\right|^2\right\}^{1/2} \end{aligned}$$

$$= \left\{ \sum_{n\in\mathbb{Z}^{\nu-1}} \sum_{|n_1|\leq \frac{v_\phi s}{4}} \left| \int_{\mathbb{T}^{\nu-1}} \prod_{j=2}^{\nu} \mathrm{d}\sigma(\theta_j)\; \mathrm{e}^{-i\sum_{j=2}^{\nu}(n_j\theta_j+sh_j(\theta_j))} \int_{\mathbb{T}} \mathrm{d}\sigma(\theta_1) \right.\right.$$

$$\left.\left. \mathrm{e}^{-i(n_1\theta_1+sh_1(\theta_1))}\widehat{\phi}(\vartheta)\mathrm{d}\sigma(\theta_1) \right|^2 \right\}^{1/2}. \tag{10}$$

We define the function $J(\theta, s, n_1) = n_1\theta + sh_1(\theta)$. When ϑ is in the support of $\widehat{\phi}$, we have that $|h_1'(\theta_1)| \geq v_\phi$, by eq. (7). This in turn implies that when $\vartheta = (\theta_1, \dots, \theta_\nu) \in \mathrm{supp}\widehat{\phi}$,

$$\left| \frac{\partial}{\partial\theta} J(\theta_1, s, n_1) \right| = |n_1 + s\, h_1'(\theta_1)| \geq 3v_\phi\, s/4 \text{ when } n_1 \leq v_\phi\, s/4.$$

We use this fact and do integration by parts twice with respect to the variable θ_1 to obtain

$$T = \left\{ \sum_{|n_1|\leq \frac{v_\phi s}{4}} \sum_{n\in\mathbb{Z}^{\nu-1}} \left| \int_{\mathbb{T}^{\nu-1}} \prod_{j=2}^{\nu} \mathrm{d}\sigma(\theta_j)\; \mathrm{e}^{-i\sum_{j=2}^{\nu} n_j\theta_j+sh_j(\theta_j)} \int_{\mathbb{T}} \mathrm{d}\sigma(\theta_1) \right.\right.$$

$$\left.\left. \mathrm{e}^{-i(n_1\theta_1+sh_1(\theta_1))} \left\{ \left(\frac{\partial}{\partial\theta_1} \frac{1}{J'(\theta_1, n_1, s)} \right)^2 \widehat{\phi}(\vartheta) \right\} \mathrm{d}\sigma(\theta_1) \right|^2 \right\}^{1/2}. \tag{11}$$

We note that the quantity

$$I_1 = \left(\frac{\partial}{\partial\theta_1} \frac{1}{J'(\theta_1, n_1, s)} \right)^2 \widehat{\phi}(\vartheta)$$

$$= \left(\frac{-J^{(3)}}{(J')^3} + \frac{3J^{(2)}(J')^2}{(J')^6} \right) \widehat{\phi} + \frac{1}{(J')^2} \frac{\partial^2}{\partial\theta_1^2} \widehat{\phi} + \frac{-J^{(2)}}{(J')^3} \frac{\partial}{\partial\theta_1} \widehat{\phi} \tag{12}$$

is in $L^2(\mathbb{T}^\nu)$.

The assumptions on the lower bound on J' (when $|n_1| \leq v_\phi s/4$) and the boundedness of its higher derivatives by Cs (which is straightforward to verify by the assumption on h_j) together now yield the bound

$$T \leq \frac{C}{s^2} \left\{ \|\phi\| + \|\widehat{\frac{\partial}{\partial\theta_1}\phi}\|_{L^2(\mathbb{T}^\nu)} + \|\widehat{\frac{\partial^2}{\partial\theta_1^2}\phi}\|_{L^2(\mathbb{T}^\nu)} \right\}$$

which gives the required integrability.

We proved the case (1) of the theorem assuming that μ has compact support. The case when μ has infinite support requires only a comment on the function $\mathrm{e}^{-isH_0}\phi$ being in the domain on V^ω almost everywhere, when s is finite and for fixed $\phi \in \mathcal{D}$. Once this is ensured the remaining calculations are the same. To see the stated domain condition we first note that for each fixed s, the sequence $(\mathrm{e}^{-isH_0}\phi)(n)$ decays faster than any polynomial, (in $|n|$). The reason being that, by assumption, $\widehat{\phi}$ is smooth and of compact support in $\mathbb{T}^\nu \setminus \mathcal{C}$, $|\phi(n)| \leq |n|^{-N}$ for any $N > 0$, as $|n| \to \infty$. On the other hand for $|n - m| > s\|H_0\|$, we have

$$|\mathrm{e}^{-isH_0}(n, m)| \leq \frac{1}{|n-m|^N}, \text{ for any } N > 0.$$

These two estimates together imply that

$$\|(1 + |m|)^{2\nu+2}\mathrm{e}^{-isH_0}\phi\| < \infty, \ \forall\phi \in \mathcal{D}. \tag{13}$$

We now consider the events

$$A_n = \{\omega : |q^\omega(n)| > |n|^{2\nu+1}\}$$

and they satisfy the condition

$$\sum_{n \in \mathbb{Z}^\nu} \mathrm{Prob}(A_n) < \infty,$$

by a simple application of Cauchy–Schwarz and the finiteness of the second moment μ. Hence, by an application of Borel–Cantelli lemma, only finitely many events A_n occ with full measure. Therefore on a set of full measure all but finitely many $q^\omega(n)$ satisf $|q^\omega(n)| \leq |n|^{2\nu+1}$. Let the set of full measure be denoted by Ω_1. Then for each $\omega \in \Omega_1$ v have a finite set $S(\omega)$ such that $\mathrm{e}^{-isH_0}\phi$ is in the domain of the operator $V_1^\omega = V^\omega(I-P_{S(\omega)}$ where $P_{S(\omega)}$ is the orthogonal projection onto the subspace $\ell^2(S(\omega))$, in view of the e (13). Then the proof that the a.c. spectrum of the operator

$$H_1^\omega = H_0 + V_1^\omega, \forall \omega \in \Omega_1 \cap \Omega_0$$

goes through as before. Since for each $\omega \in \Omega_1 \cap \Omega_0$, H_1^ω differs from H^ω by a fin rank operator, its absolutely continuous spectrum is unaffected (by trace class theory scattering) and the theorem is proved.

The statement on the singular part of the spectrum of H^ω, is a direct corollary of t Theorem 5.2. We note firstly that since $\{\delta_n, n \in \mathbb{Z}^\nu\}$ is an orthonormal basis for $\ell^2(\mathbb{Z}^\nu$ is automatically a cyclic family for H^ω for every ω.

Secondly, by assumption, the subspaces $\mathcal{H}_{\omega,n}$ and $\mathcal{H}_{\omega,m}$ are not mutually orthogon so the conditions of Theorem 5.2 are satisfied. Therefore, since the a.c. spectrum of contains the interval (E_-, E_+) almost every ω the result follows.

(2) We now turn to the proof of part 2 of the theorem. The essential case to consic again as in (1) is when μ has compact support, the general case goes through as befo The proof is again similar to the one in (1), but we need to choose a dense set $\mathcal{D}_1$ in t place of $\mathcal{D}$ properly.

The operator Δ_+ is self adjoint on $\ell^2(\mathbb{Z}^+)$ and its restriction Δ_{+1} to $\ell^2(\mathbb{Z}^+ \setminus \{0\})$ unitarily equivalent to multiplication by $2\cos(\theta)$ acting on the image of $\ell^2(\mathbb{Z}^+ \setminus \{0\})$ unc the Fourier series map. We now consider the operator

$$H_{0+1} = \Delta_{+1} + H_0$$

in the place of H_{0+} and show the existence of the Wave operators

$$W_+ = \mathrm{slim}_{t\to\infty} \mathrm{e}^{itH_+^\omega} \mathrm{e}^{-itH_{0+1}}$$

almost every ω.

We take the set $\mathcal{D}$ as in eq. (4), $\mathcal{D}_2$ as in Lemma 3.1 and define

$$\mathcal{D}_+ = \left\{ \phi : \phi = \sum_{i,j \text{ finite}} \alpha_{ij} \phi_i \psi_j, \ \psi_j \in \mathcal{D}, \ \phi_i \in \mathcal{D}_2, \ \alpha_{ij} \in \mathbb{C} \right\}. \quad ($$

Then $\mathcal{D}_+$ is dense in

$$\mathcal{H}_0 = \{f \in \ell^2(\mathbb{Z}_+^{\nu+1}) : f(0,n) = 0\}.$$

We then define the minimal velocities for $\phi \in \mathcal{D}_+$ with w_{ϕ_1} defined as in Lemma 3.1 $\phi_1 \in \mathcal{D}_2$.

$$\begin{aligned} w_{1,\phi} &= \inf_{k} w_{\phi_k} \\ w_{2,\phi} &= \inf_{l} \inf_{j} \inf\{|h_j'(\theta_j)| : \vartheta \in \operatorname{supp}\widehat{\psi_l}\} \\ v_\phi &= \min\{w_{1,\phi}, w_{2,\phi}\}. \end{aligned} \tag{15}$$

Calculating the limits, as in eq. (5)

$$\begin{aligned} &\|(\mathrm{e}^{itH^{\omega+}}\mathrm{e}^{-itH_{0+1}} - \mathrm{e}^{irH^{\omega+}}\mathrm{e}^{-irH_{0+1}})\phi\| \\ &= \int_r^t \mathrm{d}s\; \|(\mathrm{e}^{isH^{\omega+}}(V^\omega - P_0\Delta_+ + -\Delta_+P_0 + P_0\Delta_+P_0)\mathrm{e}^{-itH_{0+1}}\phi\|, \end{aligned} \tag{16}$$

where P_0 is the operator $p_0 \otimes I$, with p_0 being the orthogonal projection onto the one dimensional subspace spanned by the vector δ_0 in $\ell^2(\mathbb{Z}^+)$. We note that by the definition of Δ_+, the term $P_0\Delta_+P_0$ is zero. The estimates proceed as in the proof of (1), after taking averages over the randomness and taking $\phi \in \mathcal{D}_+$. As in that proof it is sufficient to show the integrability in s of the functions

$$\|\sigma A\mathrm{e}^{-isH_{0+1}}\phi\|, \quad \||\delta_1\rangle\langle\delta_0| \otimes I\mathrm{e}^{-isH_{0+1}}\phi\|, \quad \||\delta_0\rangle\langle\delta_1| \otimes I\mathrm{e}^{-isH_{0+1}}\phi\|,$$

respectively. By the definition of $\mathcal{D}_+$, any ϕ there is a finite sum of terms of the form $\phi_j(\theta_1)\psi_j(\theta_2, \ldots, \theta_{\nu+1})$, so it is enough to show the integrability when ϕ is just one such product, say $\phi = \phi_1\psi_1$. Therefore we show the integrability in s of the functions

$$\|\sigma A\mathrm{e}^{-isH_{0+1}}\phi\|, \quad \||\delta_1\rangle\langle\delta_0| \otimes I\mathrm{e}^{-isH_{0+1}}\phi\|, \quad \||\delta_0\rangle\langle\delta_1| \otimes I\mathrm{e}^{-isH_{0+1}}\phi\|,$$

for s large we are done. We have

$$F(|n_1| > v_\phi s/4)\delta_i = 0, \quad i = 0, 1 \text{ and } \|\sigma A F(|n_j| > v_\phi s/4, \forall j)\| \in L^1(1, \infty),$$

by the Hypothesis 2.3(2′) on the sequence a_n. Therefore it is enough to show the integrability of the norms

$$\|F(|n_j| < v_\phi s/4)\mathrm{e}^{-is\Delta_{0+1}}\phi_1\psi_1\|, \quad \forall\phi_1 \in \mathcal{D}_2, \quad \psi_1 \in \mathcal{D},$$

for each $j = 1, \ldots, \nu + 1$. When $j = 2, \ldots, \nu + 1$, the proof is as in the previous theorem, while for $j = 1$, the proof is given in the Lemma 3.1 below.

The statement on the absence of singular part of the spectrum of $H^\omega +$ in $(E_- - 2, E_+ + 2)$, is as before a direct corollary of the Theorem 5.2, since the set of vectors $\{\delta_n, n = (0, m), m \in \mathbb{Z}^\nu\}$ is a cyclic family for H_+^ω, for almost all ω and $\mathcal{H}_{\omega,n}$ and $\mathcal{H}_{\omega,m}$ are not mutually orthogonal for almost all ω when m, n are in $\{(0, n) : n \in \mathbb{Z}^\nu\}$, and the fact that the a.c. spectrum of H^ω contains the interval $(-2 + E_-, 2 + E_+)$ almost every ω.

The lemma below is as in Jaksic–Last [13](Lemma 3.11) and the enlarging of the space in the proof is necessary since there are no non-trivial functions in $\ell^2(\mathbb{Z}^+)$ whose Fourier series has compact support in (0, 2π) (all of them being boundary values of functions analytic in the disk).

Lemma 3.1. *Consider the operator* Δ_{+1} *on* $\ell^2(\mathbb{Z}^+)$. *Then there is a set* $\mathcal{D}_2$ *dense in* $\ell^2(\mathbb{Z}^+)$ *and a number* w_ϕ *such that for* $s \geq 1$,

$$\|F(|n| < w_\phi s/4)\mathrm{e}^{-is\Delta_{1+}}\phi\| \leq C|s|^{-2}, \quad \forall\phi \in \mathcal{D}_2.$$

with the constant C independent of s.

Proof. We first consider the unitary map $\mathcal{W}$ from $\mathcal{H}_0$ to a subspace $\mathcal{S}$ of $\{f \in \ell^2(\mathbb{Z}) : f(0) = 0\}$, given by

$$(\mathcal{W}f)(n) = \begin{cases} \frac{1}{\sqrt{2}} f(n), & n > 0 \\ -\frac{1}{\sqrt{2}} f(-n), & n < 0. \end{cases} \tag{17}$$

Then the range of $\mathcal{W}$ is a closed subspace of $\ell^2(\mathbb{Z})$ and consists of functions

$$\mathcal{S} = \{f \in \ell^2(\mathbb{Z}) : f(n) = -f(-n)\}.$$

Under the Fourier series map this subspace goes to

$$\widehat{\mathcal{S}} = \{\phi \in L^2(\mathbb{T}) : \phi(\theta) = -\phi(-\theta)\}$$

so that the functions here have mean zero. Then under the map from $\ell^2(\mathbb{Z}^+ \setminus \{0\})$ to $\widehat{\mathcal{S}}$ obtained by composing $\mathcal{W}$ and the Fourier series map, the operator Δ_{1+} goes to multiplication by $2\cos(\theta)$. We now choose a set

$$\mathcal{D}_1 = \{\phi \in \widehat{\mathcal{S}} : \operatorname{supp}(\phi) \subset \mathbb{T} \setminus \{0, \pi\}\},$$

and define the number

$$w_\phi = \inf\{|2\sin(\theta)| : \theta \in \operatorname{supp}(\phi)\},$$

for each $\phi \in \mathcal{D}_1$. We denote by $\mathcal{D}_2$ all those functions whose images under the composition of $\mathcal{W}$ and the Fourier series lies in $\mathcal{D}_1$. The density of $\mathcal{D}_2$ in $\ell^2(\mathbb{Z}^+ \setminus \{0\})$ is then clear. We shall simply denote by f_ϕ elements in $\mathcal{D}_2$ whose images in $\mathcal{D}_1$ is ϕ. Given a $\phi \in \mathcal{D}_1$ and a w_ϕ we see that

$$\|F(|n| \leq w_\phi s/4) \mathrm{e}^{-is\Delta_{1+}} f_\phi\|^2 = \sum_{|n| < w_\phi s/4} \left| \int_{\mathbb{T}} \mathrm{d}\sigma(\theta) \mathrm{e}^{-in\theta - i2s\cos(\theta)} \phi(\theta) \right|^2 \leq C|s|^{-4},$$

by a simple integration by parts, done twice, using the condition that $||n| + 2s\sin(\theta)| > w_\phi s/4$ in the support of ϕ.

Proof of Theorem 2.8. The proof of this theorem is based on a technique of Aizenman [1]. We break up the proof into a few lemmas. First we show that the free operators H_0 and H_{0+} have resolvent kernels with some summability properties, for energies in their resolvent set.

Lemma 3.2. *Consider a function h satisfying the Hypothesis* 2.1 *and consider the associated operators H_0 or H_{0+}. Then for all $s \geq \nu/(3\nu + 3)$,*

$$\sup_{n \in \mathbb{Z}^\nu} \sum_{n \in \mathbb{Z}^\nu} |\langle \delta_n, (H_0 - E)^{-1} \delta_m \rangle|^s < C(E),$$

and $C(E) \to 0, |E| \to \infty$. Similarly we also have for all $s > \nu/(3\nu + 3)$,

$$\sup_{n \in \mathbb{Z}_+^{\nu+1}} \sum_{n \in \mathbb{Z}^\nu} |\langle \delta_n, (H_{0s} - E)^{-1} \delta_m \rangle|^s < C(E).$$

Proof. We will prove the statement for H_0, the proof for H_{0+} is similar. We write the expression for the resolvent kernel in the Fourier transformed representation (we write the Fourier series of an $\ell^2(\mathbb{Z}^\nu)$ function as $\widehat{u}(\vartheta) = \sum_{n\in\mathbb{Z}^\nu} \mathrm{e}^{in\cdot\vartheta} u(n))$, use the Hypothesis 2.1(1), and integrate by parts $3\nu + 3$ times with respect to the variable θ_j (recall that $\vartheta = (\theta_1, \dots, \theta_\nu)$), to get the inequalities

$$\begin{aligned}\langle\, \delta_n, (H_0 - E)^{-1}\delta_m \,\rangle &= \int_{\mathbb{T}^\nu} \mathrm{d}\sigma(\vartheta)\ \mathrm{e}^{i(m-n)\cdot\vartheta}\ (h(\vartheta) - E)^{-1} = \frac{(i)^{3\nu+3}}{((m-n)_j)^{3\nu+3}} \\ &\quad \times \int_{\mathbb{T}^\nu} \mathrm{d}\sigma(\vartheta)\ \mathrm{e}^{i(m-n)\cdot\vartheta}\ \frac{\partial^{3\nu+3}}{\partial\theta_j^{3\nu+3}}(h(\vartheta) - E)^{-1}, \qquad (18)\end{aligned}$$

where we have chosen the index j such that $|(m-n)_j| \geq |m-n|/\nu$ and assumed that $m \neq n$ (when $m = n$ the quantity is just bounded). Let us set

$$C_0(E) = \max\left\{ \sup_{\vartheta\in\mathbb{T}^\nu} \left| \frac{\partial^{3\nu+3}}{\partial\theta_j^{3\nu+3}}(h(\vartheta) - E)^{-1} \right|,\ |(h(\vartheta) - E)^{-1}| \right\}.$$

It is easy to see that since the function h is of compact range and all its $3\nu + 3$ partial derivatives are bounded, by hypothesis $C_0(E)$ goes to zero as $|E|$ goes to ∞. We then get the bound for any $s > \nu/(3\nu + 3)$,

$$|\,\langle\, \delta_n, (H_0 - E)^{-1}\delta_m \,\rangle\,| \leq \frac{\nu^{3\nu+3}}{|m-n|^{3\nu+3}} C_0(E).$$

Given this estimate we have

$$\begin{aligned}\sup_{n\in\mathbb{Z}^\nu} \sum_{n\in\mathbb{Z}^\nu} |\,\langle\, \delta_n, (H_0 - E)^{-1}\delta_m \,\rangle\,|^s &\leq C_0(E)^s \left(\sup_{n\in\mathbb{Z}^\nu} \left(1 + \sum_{\substack{n\in\mathbb{Z}^\nu \\ m\neq n}} \left| \frac{\nu^{s(3\nu+3)}}{|m-n|^{s(3\nu+3)}} \right| \right)\right) \\ &\leq C_0(E)^s \left(1 + \sum_{n\in\mathbb{Z}^\nu, m\neq 0} \left| \frac{\nu^{s(3\nu+3)}}{|m|^{s(3\nu+3)}} \right| \right), \\ &\leq C_0(E)^s C(s), \qquad (19)\end{aligned}$$

where $C(s)$ is finite since $|m|^{-s(3\nu+3|)}$, $m \neq 0$ is a summable function in $\mathbb{Z}^\nu$ when $s(3\nu + 3) > \nu$.

Proof of Corollary 2.9. We prove the theorem only for the case H^ω the proof of the other case is similar.

By the Hypothesis 2.3(2) on the finiteness of the second moment of μ we see that $\int \mathrm{d}\mu(x)\ |x| < \infty$, so that we can set $\tau = 1$ in the Lemma 5.1. Since the assumption in the theorem ensures the boundedness of the density of μ we can also set $q = \infty$ in the Lemma 5.1 with then $Q^{1/1+q} = \|\mathrm{d}\mu/\mathrm{d}x\|_\infty$. Then in the Lemma 5.1 the constant C is given by

$$C\left(Q, \frac{\kappa}{1-2\kappa}, \infty\right) = 1 + \frac{2\kappa Q}{1-\kappa}.$$

The condition on the constant κ becomes

$$\kappa < 1/3.$$

Below we choose a s satisfying $\frac{\nu}{(3\nu+3)} < s < 1/3$, and consider the expression

$$G(\omega, z, n, m) = \langle\, \delta_n, (H^\omega - z)^{-1}\delta_m \,\rangle,\quad G(0, z, n, m) = \langle\, \delta_n, (H_0 - z)^{-1}\delta_m \,\rangle,$$

where we take $z = E + i\epsilon$ with $\epsilon > 0$. Then by the resolvent equation we have

$$G(\omega, z, n, m) = G(0, z, n, m) - \sum_{l \in Znu} G(\omega, z, n, l)V^{\omega}(l)G(0, z, l, m). \qquad (2$$

We denote by

$$G_l(\omega, z, n, m) = \langle \delta_n, (H^{\omega} - V^{\omega}(l)P_l - z)^{-1}\delta_m \rangle,$$

where P_l is the orthogonal projection onto the subspace generated by δ_l. Then using t
rank one formula

$$G(\omega, z, n, l) = \frac{\frac{G_l(\omega,z,n,l)}{G_l(\omega,z,l,l)}}{V^{\omega}(l) + G_l(\omega, z, l, l)^{-1}}$$

whose proof is again by resolvent equation, we see that eq. (20) can be rewritten as

$$\begin{aligned} G(\omega, z, n, m) &= G(0, z, n, m) \\ &+ \sum_{l \in \mathbb{Z}^{\nu}} \left(\frac{\frac{G_l(\omega,z,n,l)}{G_l(\omega,z,l,l)}}{V^{\omega}(l) + G_l(\omega, z, l, l)^{-1}} \right) V^{\omega}(l)G(0, z, l, m). \end{aligned} \qquad (2$$

Raising both the sides to power s (noting that $s < 1$ so the inequalities are valid), we ge

$$\begin{aligned} |G(\omega, z, n, m)|^s &= |G(0, z, n, m)|^s \\ &+ \sum_{l \in \mathbb{Z}^{\nu}} \left| \left(\frac{\frac{G_l(\omega,z,n,l)}{G_l(\omega,z,l,l)}}{V^{\omega}(l) + (G_l(\omega, z, l, l)^{-1}} \right) \right|^s |V^{\omega}(l)|^s |G(0, z, l, m)|^s \end{aligned}$$

(2

Now observing that G_l is independent of the random variable $V^{\omega}(l)$, we see that

$$\begin{aligned} \mathbb{E}(|G(\omega, z, n, m)|^s) &= |G(0, z, n, m)|^s \\ &+ \sum_{l \in \mathbb{Z}^{\nu}} \mathbb{E} \left(\left| \left(\frac{\frac{G_l(\omega,z,n,l)}{G_l(\omega,z,l,l)}}{V^{\omega}(l) + (G_l(\omega, z, l, l)^{-1}} \right) \right|^s |V^{\omega}(l)|^s \right) |G(0, z, l, m)|^s. \end{aligned} \qquad (2$$

This then becomes, integrating with respect to the variable $q^{\omega}(l)$, remembering that $V^{\omega}(l)$
$a_l q^{\omega}(l)$,

$$\begin{aligned} \mathbb{E}(|G(\omega, z, n, m)|^s) &= |G(0, z, n, m)|^s \\ &+ \sum_{l \in \mathbb{Z}^{\nu}} \mathbb{E} \left(\left| \frac{G_l(\omega, z, n, l)}{G_l(\omega, z, l, l)} \right|^s \right) \\ &\times \int \left(\mathrm{d}\mu(x) \, \frac{|x|^s}{|x + a_l^{-1} G_l(\omega, z, l, l)^{-1}|^s} \right) |G(0, z, l, m)|^s \end{aligned}$$

(2

which when estimated using the Lemma 5.1 yields

$$\begin{aligned} \mathbb{E}(|G(\omega, z, n, m)|^s) &\le |G(0, z, n, m)|^s \\ &+ \sum_{l \in \mathbb{Z}^{\nu}} K_s \mathbb{E} \left(\left| \frac{G(\omega, z, n, l)}{G_l(\omega, z, l, l)} \right|^s \right) \\ &\times \int \left(\mathrm{d}\mu(x) \, \frac{1}{|x + a_l^{-1} G_l(\omega, z, l, l)^{-1}|^s} \right) |G(0, z, l, m)|^s \end{aligned}$$

(

where K_s is the constant appearing in Lemma 5.1 with κ set equal to s. We take $K = (\sup_n |a_n|^s)K_s$, and rewrite the above equation to obtain

$$\mathbb{E}(|G(\omega, z, n, m)|^s) = |G(0, z, n, m)|^s + \sum_{l \in \mathbb{Z}^\nu} K\mathbb{E}(|(G(\omega, z, n, l)|^s |G(0, z, l, m)|^s. \tag{26}$$

We now sum both the sides over m, set

$$I = \sum_{m \in \mathbb{Z}^\nu} \mathbb{E}(|G(\omega, z, n, m)|^s)$$

and obtain the inequality

$$I \leq \sum_{m \in \mathbb{Z}^\nu} |G(0, z, n, m)|^s + \sup_{l \in \mathbb{Z}^\nu} \sum_{m \in \mathbb{Z}^\nu} KI|G(0, z, l, m)|^s.$$

Therefore when there is an interval (a, b) in which

$$K \sup_{l \in \mathbb{Z}^\nu} \sum_{m \in Znu} |G(0, z, l, m)|^s < 1, \quad E \in (a, b), \tag{27}$$

we obtain that

$$\int_a^b \mathrm{d}E \sum_{m \in \mathbb{Z}^\nu} \mathbb{E}(|G(\omega, E + i0, n, m)|^s) < \infty,$$

by an application of Fatou's lemma implying that for almost all $E \in (a, b)$ and almost all ω, we have the finiteness of

$$\sum_{m \in \mathbb{Z}^\nu} |G(\omega, E + i0, n, m)|^2 < \infty,$$

satisfying the Simon–Wolff [19] criterion. This shows that (the proof follows as in Theorems II.5, II.6 [18]) the measures

$$\nu_n^\omega(\cdot) = \langle \delta_n, E_{H^\omega}(\cdot)\delta_n \rangle$$

are pure point in (a, b) almost every ω. This happens for all n, hence the total spectral measure of H^ω itself is pure point in (a, b) for almost all ω.

There are two different ways to fix the critical energy $E(\mu)$ now. Firstly if K is large, then in view of the Lemma 3.2 (by which $C_0(E) \to 0$, $|E| \to \infty$) and the fact that K is finite (by Lemma 5.1)

$$K \sup_{l \in \mathbb{Z}^\nu} \sum_{m \in Znu} |G(0, z, l, m)|^s \leq KC_0(E)^s C(s) < 1, \quad |E| \to \infty. \tag{28}$$

Therefore there is a large enough $E(\mu)$ such that for all intervals (a, b) in $(-\infty, -E(\mu)) \cup (E(\mu), \infty)$, the condition in eq. (25) is satisfied.

On the other hand if the moment $B = \int |x| \, \mathrm{d}\mu(x)$ is very small, then we can choose $E(\mu)$ by the condition,

$$KC_0(E)C_s < 1,$$

even when $C_0(E) > 1$, since it is finite for E in the resolvent set of H_0 by Lemma 3.2.

4. Examples

In this section we present some examples of the operators H_0 considered in the theorems. We only give the functions h stated in the Hypothesis 2.1.

- *Examples of operators* H_0

1. $h(\vartheta) = \sum_{i=1}^{\nu} 2\cos(\theta_i)$, corresponds to the usual discrete Schrödinger operator and it is obvious that the Hypothesis 2.1 are satisfied. The Jaksic–Last condition 5.2 on mutual non-orthogonality of the subspaces generated by H_0 and δ_n for different n in $\mathbb{Z}^\nu$ are also satisfied, by an elementary calculation taking powers of H_0 depending upon a pair of vectors δ_n and δ_m, since the operator H_0 is given by $T + T^{-1}$, with T being the bilateral shift on $\ell^2(\mathbb{Z})$.
2. $h(\vartheta) = \sum_{i=1}^{\nu} h_i(\theta_i), h_i(\theta_i) = \sum_{k=1}^{N(i)} \cos(k\theta_i), \quad N(i) < \infty$. Clearly each h_i is a smooth function in $\mathbb{R}^\nu$ and each h_i and all its derivatives are 2π periodic. Hence the Hypothesis 2.3 is satisfied. Further each of h_i is a trigonometric polynomial, and its derivative is also a trigonometric polynomial and hence has only finitely many zeros on the circle.

   The condition in Jaksic–Last condition Theorem 5.2 on mutual non-orthogonality is again elementary to verify in this case.
3. Consider the functions

$$h_i(\theta_i) = \theta_i^{3\nu+4}(2\pi - \theta_i)^{3\nu+4}, \quad 0 \le \theta_i \le 2\pi, \quad i = 1, \ldots, \nu$$

   and take $h = \sum_{i=1}^{\nu} h_i(\theta)$ extended to the whole of $\mathbb{R}^\nu$ periodically. Clearly these are in $C^{3\nu+3}(\mathbb{T}^\nu)$, by construction.

- *Examples of pairs* (a_n, μ)

We give next some examples of sequences a_n satisfying the Hypothesis 2.2 such that

$$\text{supp}(\mu) = \text{a-supp}(\mu).$$

We consider $\nu \ge 2$ and the sequence $a_n = (1 + |n_1|)^\alpha, \ \alpha < -1$. Then we have that

$$k\mathbb{Z}^\nu \cap \{(0, n) : n \in \mathbb{Z}^{\nu-1}\} = \{(0, n) : n \in k\mathbb{Z}^{\nu-1}\}$$

and $a_{(0,n)}^{-1}(a, b) =$ (a, b) for any interval (a, b) and any $n \in \mathbb{Z}^{\nu-1}$. Therefore for any positive integer k, we have

$$\sum_{m \in k\mathbb{Z}^\nu} \mu(a_m^{-1}(a, b)) \ge \sum_{m \in k\mathbb{Z}^{\nu-1}} \mu((a, b)) = \infty$$

whenever $\mu((a, b)) > 0$.

- *Examples of measures* μ *with small moment*

We next give an example of an absolutely continuous measure of compact support such that the Aizenman condition (in Lemma 5.1 is satisfied. We use the notation used in that lemma for the example.

We consider numbers $0 < \epsilon, \delta < 1$, R and let μ be given by

$$\mathrm{d}\mu(x)/\mathrm{d}x = \begin{cases} \frac{1-\epsilon}{\delta}, & 0 \leq x \leq \delta, \\ \frac{\epsilon}{R-\delta}, & \delta < x \leq R, \\ 0, & \text{otherwise.} \end{cases} \tag{29}$$

Then μ is an absolutely continuous probability measure and

$$Q \leq \frac{1}{\delta} + \frac{1}{R-\delta}.$$

We take $\tau = 1$, then the moment B is bounded by

$$B \leq (1-\epsilon)\delta + (R+\delta)\epsilon/2.$$

Now if we fix R large and choose $\epsilon = 1/R^3$ and $\delta = 1/R^2$, we obtain an estimate

$$B^{\kappa} \leq \frac{2^{\kappa}}{R^{2\kappa}} \text{ and } B^{\kappa} Q^{1-2\kappa} \leq 8R^{2-6\kappa}.$$

Taking $\kappa = s$ in the lemma and noting that $s < 1/3$ implies $2 - 6s < 0$ so that both the terms above go to zero as R goes to ∞. We see that by taking μ with large support but small moment, we can make the constant K in the Lemma 5.1 as small as we want. This in particular means that in the Theorem 2.8. given a energy E_0 outside the spectrum of H_0 we can find a measure μ which is absolutely continuous of small moment such that K is smaller than $C_0(E_0)^s C_s$ in the proof of Theorem 2.8. and hence $E(\mu) < |E_0|$. We can use such measures to give examples of operators with compact spectrum with both a.c. spectrum and pure point spectrum present but in disjoint regions.

- *Example when Jaksic–Last condition is violated*

We finally give examples where Jaksic–Last condition is violated and yet the conclusion of their theorem is valid.

Consider $\nu = 1$, for simplicity, and let $h(\theta) = 2\cos(2\theta)$. Then the associated H_0 has purely a.c. spectrum in [-2, 2] and we see that the operator $H_0 = T^2 + T^{-2}$ if T is the bilateral shift acting on $\ell^2(\mathbb{Z})$. Then if we consider the operators $H^{\omega} = H_0 + V^{\omega}$, and the cyclic subspaces $\mathcal{H}_{\omega,1}, \mathcal{H}_{\omega,2}$ generated by the H^{ω} and the vectors δ_1, δ_2 respectively, such an operator satisfies

$$\mathcal{H}_{\omega,1} \subset \ell^2(\{1\} + 2\mathbb{Z}), \ \mathcal{H}_{\omega,2} \subset \ell^2(\{1+1\} + 2\mathbb{Z}), \ \text{almost every } \omega.$$

We then have

$$\mathcal{H}_{\omega,1} \subset \ell^2(\{2n+1, \ n \in \mathbb{Z}\}), \ \mathcal{H}_{\omega,2} \subset \ell^2(2\mathbb{Z}), \ \text{almost every } \omega.$$

The subspaces $\ell^2(\{n : n \text{ odd}\})$ and $\ell^2(2\mathbb{Z})$ are generated by the families $\{\delta_k, \ k \text{ odd}\}$ and $\{\delta_k, \ k \text{ even}\}$ respectively. (We could have taken any odd integer k in the place of 1 to do the above)

These two are invariant subspaces of H^{ω} which are mutually orthogonal, a.e. ω. Therefore the Jaksic–Last theorem is not directly valid. However, by considering the restrictions of H^{ω} to these two subspaces, one can go through their proof in these subspaces to again obtain the purity of a.c. spectrum for such operators when they exist.

We consider two examples to illustrate the point, for which we let $q^{\omega}(n)$ denote a collection of i.i.d. random variables with an absolutely continuous distribution μ of compact support in $\mathbb{R}$, its support containing 0.

1. If $V^\omega(n) = a_n q^\omega(n)$, with $0 < a_n < (1+|n|)^{-\alpha}, \alpha > 0$, we see that there is pu a.c. spectrum in [–2, 2], a.e. ω by applying trace class theory of scattering.
2. On the other hand if, with $0 < a_n < (1+|n|)^{-\alpha}, \alpha > 1$,
$$V^\omega(n) = \begin{cases} a_n q^\omega(n), & n \text{ odd} \\ q^\omega(n), & n \text{ even}, \end{cases}$$
then there is dense pure point spectrum embedded in the a.c. spectrum in [–2, 2

We can give similar, but non trivial, examples in higher dimensions but we leave it to t reader.

5. Appendix

In this appendix we collect two theorems we use in this paper. One is a lemma of Aizenma [1] and another a theorem of Jaksic–Last [14].

The first lemma and its proof are those of Aizenman [1](Lemma A.1) which reprodu below (with some modifications in the form we need), with a slight change in notation (v in particular call the number s in Aizenman's lemma as κ),

Lemma 5.1 (*Aizenman*). *Let μ be an absolutely continuous probability measure who density f satisfies $\int_{\mathbb{R}} \mathrm{d}x |f(x)|^{1+q} = Q < \infty$ for some $q > 0$. Let $0 < \tau \leq 1$ and suppo $B \equiv \int_{\mathbb{R}} \mathrm{d}\mu(x)\ |x|^\tau < \infty$. Then for any*

$$\kappa < \left[1 + \frac{2}{\tau} + \frac{1}{q}\right]^{-1}$$

we have

$$\int_{\mathbb{R}} \mathrm{d}\mu(x)\ \frac{|x|^\kappa}{|x-\alpha|^\kappa} < K_\kappa \int_{\mathbb{R}} \mathrm{d}\mu(x)\ \frac{1}{|x-\alpha|^\kappa}, \quad \text{for all} \quad \alpha \in \mathbb{C},$$

with K_κ given by

$$K_\kappa = B^{\frac{\kappa}{\tau}}(2^{1+2\kappa} + 4)\left[B^{1-\frac{\kappa}{\tau}} + B^{\frac{\kappa}{\tau}}\ C(Q, \frac{\kappa}{1 - \frac{2\kappa}{\tau}}, q)^{\frac{\tau - 2\kappa}{\tau}}\right] < \infty.$$

Remark. We see from the explicit form of the constant K_κ that the moment B can made sufficiently small by the choice of μ even when its support is large. This will ensu that in some models of random operators, the region where the Simon–Wolff criterion valid extends to the region in the spectrum. This is the reason for our writing K_κ in t form.

Proof. The strategy employed in proving the lemma is to consider the ratio

$$\frac{\int_{\mathbb{R}} \mathrm{d}\mu(x)\ \frac{|x|^\kappa}{|x-\alpha|^\kappa}}{\int_{\mathbb{R}} \mathrm{d}\mu(x)\ \frac{1}{|x-\alpha|^\kappa}}$$

and obtain upper bounds for the numerator and lower bounds for the denominator.

Note first that B finite and $\kappa < \tau$ implies that $|x-\alpha|^\kappa$ is integrable even if α is pur real and we have

$$\int_a^b f(x)\mathrm{d}x \leq Q^{\frac{1}{1+q}} |b-a|^{\frac{q}{1+q}} \quad ($$

by Hölder inequality. Hence

$$\begin{aligned}\int \mathrm{d}\mu(x)\, \frac{1}{|x-\alpha|^{\kappa}} &\leq 1 + \int_1^{\infty} \mathrm{d}t\; \mu(\{x : \frac{1}{|x-\alpha|^{\kappa}} \geq t\}) \\ &\leq 1 + \frac{\kappa(2^q Q)^{\frac{1}{1+q}}}{\frac{q}{1+q} - \kappa} \\ &\equiv C(Q, \kappa, q), \end{aligned} \tag{31}$$

where the integral is estimated using the estimate in eq. (30).

Consider the region $|\alpha| > (2B)^{\frac{1}{\tau}}$: We then estimate for fixed α the contributions from the regions $|x| \leq |\alpha|/2$ and $|x| > |\alpha|/2$ to obtain

$$\begin{aligned}\int \mathrm{d}\mu(x)\, \frac{|x|^{\kappa}}{|x-\alpha|^{\kappa}} &\leq \frac{2^{\kappa}}{|\alpha|^{\kappa}} \left(\int \mathrm{d}\mu(x)\; |x|^{\kappa} + \int \mathrm{d}\mu(x)\, \frac{|x|^{2\kappa}}{|x-\alpha|^{\kappa}} \right) \\ &\leq \frac{2^{\kappa}}{|\alpha|^{\kappa}} (B + B^{\frac{2\kappa}{\tau}}\, C\left(Q, \frac{\kappa}{1-2\kappa/\tau}, q)^{\frac{\tau-2\kappa}{\tau}}\right), \end{aligned} \tag{32}$$

with κ chosen so that $\kappa/(1-2\kappa/\tau) < q/(1+q)$. (Here we have explicitly calculated the p occurring in the lemma of Aizenman in terms of κ and τ). For a fixed τ and q this condition is satisfied whenever κ satisfies the inequality stated in the lemma.

The lower bounds on $\int \mathrm{d}\mu(x)\; 1/|x-\alpha|^{\kappa}$ is obtained first by noting that $B < \infty$ implies

$$\mu(\{x : |x|^{\tau} > (2B)\}) \leq \frac{1}{2}.$$

Since $|\alpha| > (2B)^{\frac{1}{\tau}}$, we have the trivial estimate

$$\begin{aligned}\int \mathrm{d}\mu(x) \frac{1}{|x-\alpha|^{\kappa}} &\geq \int_{|x|>(2B)^{\frac{1}{\tau}}} \mathrm{d}\mu(x) \frac{1}{|x-\alpha|^{\kappa}} + \int_{|x|\leq(2B)^{\frac{1}{\tau}}} \mathrm{d}\mu(x) \frac{1}{|x-\alpha|^{\kappa}} \\ &\geq \int_{|x|\leq(2B)^{\frac{1}{\tau}}} \mathrm{d}\mu(x) \frac{1}{|x-\alpha|^{\kappa}} \\ &\geq \frac{1}{2(|\alpha| + (2B)^{\frac{1}{\tau}})^{\kappa}}. \end{aligned} \tag{33}$$

Putting the inequalities in (32) and (33) together we obtain, (remembering that $|\alpha| > (2B)^{\frac{1}{\tau}}$),

$$\frac{\int_{\mathbb{R}} \mathrm{d}\mu(x)\; \frac{|x|^{\kappa}}{|x-\alpha|^{\kappa}}}{\int_{\mathbb{R}} \mathrm{d}\mu(x)\; \frac{1}{|x-\alpha|^{\kappa}}} \leq 2^{1+2\kappa} B^{\frac{\kappa}{\tau}} \left[B^{1-\frac{\kappa}{\tau}} + B^{\frac{\kappa}{\tau}}\, C(Q, \frac{\kappa}{1-\frac{2\kappa}{\tau}}, q)^{\frac{\tau-2\kappa}{\tau}} \right]. \tag{34}$$

We now consider the region $|\alpha| < (2B)^{\frac{1}{\tau}}$: Estimating as in eq. (32) but now splitting the region as $|x| \leq (2B)^{\frac{1}{\tau}}$ and $|x| > (2B)^{\frac{1}{\tau}}$, we obtain the analogue of the estimate in eq. (32), in this region of α as

$$\begin{aligned}\int \mathrm{d}\mu(x)\, \frac{|x|^{\kappa}}{|x-\alpha|^{\kappa}} &\leq \frac{1}{(2B)^{\frac{1}{\tau}}} \left(\int \mathrm{d}\mu(x)\; |x|^{\kappa} + \int \mathrm{d}\mu(x)\, \frac{|x|^{2\kappa}}{|x-\alpha|^{\kappa}} \right) \\ &\leq \frac{1}{(2B)^{\frac{1}{\tau}}} \left(B + B^{\frac{2\kappa}{\tau}}\, C\left(Q, \frac{\kappa}{1-2\kappa/\tau}, q\right)^{\frac{\tau-2\kappa}{\tau}} \right). \end{aligned} \tag{35}$$

Similarly the estimate for the denominator term is done as in eq. (33),

$$\begin{aligned}\int \mathrm{d}\mu(x)\frac{1}{|x-\alpha|^{\kappa}} &\geq \int_{|x|>(2B)^{\frac{1}{\tau}}} \mathrm{d}\mu(x)\frac{1}{|x-\alpha|^{\kappa}} + \int_{|x|\leq(2B)^{\frac{1}{\tau}}} \mathrm{d}\mu(x)\frac{1}{|x-\alpha|}\\ &\geq \int_{|x|\leq(2B)^{\frac{1}{\tau}}} \mathrm{d}\mu(x)\frac{1}{|x-\alpha|^{\kappa}}\\ &\geq \frac{1}{2((2B)^{\frac{1}{\tau}}+(2B)^{\frac{1}{\tau}})}\\ &= \frac{1}{4(2B)^{\frac{1}{\tau}}}. \qquad (3\end{aligned}$$

Using the above two inequalities we obtain the estimate,

$$\frac{\int_{\mathbb{R}} \mathrm{d}\mu(x)\ \frac{|x|^{\kappa}}{|x-\alpha|^{\kappa}}}{\int_{\mathbb{R}} \mathrm{d}\mu(x)\ \frac{1}{|x-\alpha|^{\kappa}}} \leq 4\left[B^{1-\frac{\kappa}{\tau}} + B^{\frac{\kappa}{\tau}}\ C(Q, \frac{\kappa}{1-\frac{2\kappa}{\tau}}, q)^{\frac{\tau-2\kappa}{\tau}}\right], \qquad (3$$

when $|\alpha| \leq (2B)^{\frac{1}{\tau}}$. Using the inequalities (34) and (36) obtained for these two regions values of α we finally get

$$\frac{\int_{\mathbb{R}} \mathrm{d}\mu(x)\ \frac{|x|^{\kappa}}{|x-\alpha|^{\kappa}}}{\int_{\mathbb{R}} \mathrm{d}\mu(x)\ \frac{1}{|x-\alpha|^{\kappa}}} \leq B^{\frac{\kappa}{\tau}}(2^{1+2\kappa}+4)\left[B^{1-\frac{\kappa}{\tau}} + B^{\frac{\kappa}{\tau}}\ C(Q, \frac{\kappa}{1-\frac{2\kappa}{\tau}}, q)^{\frac{\tau-2\kappa}{\tau}}\right], \qquad ($$

for any $\alpha \in \mathbb{R}$.

We next state a theorem (Corollary 1.1.3) of Jaksic–Last [14] without proof, its proo as in Corollary 1.1.3 of Jaksic–Last [14]. We state it in the form we use in this paper.

Theorem 5.2 [*Jaksic–Last*]. *Suppose $\mathcal{H}$ is a separable Hilbert space and A a boun self adjoint operator. Suppose $\{\phi_n\}$ are normalized vectors and let P_n denote the orth onal projection on to the one dimensional subspace generated by each ϕ_n. Let $q^{\omega}(n$ independent random variables with absolutely continuous distributions μ_n. Consider*

$$A^{\omega} = A + \sum_n q^{\omega}(n)P_n, \ \textit{almost every}\ \omega.$$

Suppose that the following conditions are valid

1. *The family $\{\phi_n\}$ is a cyclic family for A^{ω} a.e. ω.*
2. *Let $\mathcal{H}_{\omega,n}$ denote the cyclic subspace generated by A^{ω} and ϕ_n. Then the cyclic subsp $\mathcal{H}_{\omega,n}$ and $\mathcal{H}_{\omega,m}$, are not orthogonal.*

Then whenever there is an interval (a, b) in the absolutely continuous spectrum of A $A + \sum_n q^{\omega}(n)P_n$, almost all ω, we have

$$\sigma_s(A^{\omega}) \cap (a, b) = \emptyset, \ \textit{almost every}\ \omega.$$

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Multipliers for the absolute Euler summability of Fourier series

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Abstract. In this paper, the author has investigated necessary and sufficient conditions for the absolute Euler summability of the Fourier series with miltipliers. These conditions are weaker than those obtained earlier by some workers. It is further shown that the multipliers are best possible in certain sense.

Keywords. Multipliers; absolute summability; summability of factored Fourier series; absolute Euler summability.

1. Definitions and notations

Let $\sum_{n=0}^{\infty} w_n$ be a given infinite series and let q be a real or complex number such that $q \neq -1$. Then we write

$$w_n^q = (1+q)^{-n-1} \sum_{m=0}^{n} \binom{n}{m} q^{n-m} w_m; \quad w_n^0 = w_n. \tag{1.1}$$

Following Chandra [2], $\sum w_n$ is said to be absolutely summable by (E, q) means (or Euler means) or simply $\sum_{n=0}^{\infty} a_n \in |E, q|$ if

$$\sum_{n=0}^{\infty} |w_n^q| \quad < \quad \infty. \tag{1.2}$$

For $q > 0$, a reference may be made to Hardy ([9]; p. 237). It may be observed that the method $|E, q|$ $(q > 0)$ is absolutely regular.

Let $L_{2\pi}$ be the space of all 2π-periodic and Lebesgue-integrable functions over $[-\pi, \pi]$. Then the Fourier series of $f \in L_{2\pi}$ at x is given by

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) = \sum_{n=0}^{\infty} A_n(x), \tag{1.3}$$

where a_n and b_n are the Fourier coefficients of f.

Throughout the paper, we assume that the constant term $a_0 = 0$. For real x, $q > 0$ and $\delta \geq 0$, we write

$$\phi(t) \quad = \quad \frac{1}{2}\{f(x+t) + f(x-t)\}, \tag{1.4}$$

$$\phi_1(t) \quad = \quad \frac{1}{t}\int_0^t \phi(u)\mathrm{d}u, \tag{1.5}$$

$$
\begin{aligned}
P(t) &= \phi(t) - \phi_1(t), && \text{(1.}\\
y(t) &= (1+q)^{-1}(1+q^2+2q\cos t)^{1/2}, && \text{(1.}\\
V_m^q(n) &= (1+q)^{-n-1}\binom{n+1}{m+1}q^{n-m} \quad (m \le n), && \text{(1.}\\
s_m(t) &= \sum_{r=1}^{m} V_r^q(n)\sin rt, && \text{(1.}\\
d_n &= \log^{-\delta}(n+1), && \text{(1.1}\\
\Delta d_n &= d_n - d_{n+1}, && \text{(1.1}\\
g(t) &= P(t)\log^{-\delta}\frac{k}{t}, && \text{(1.1}\\
b(t) &= t\log^{\delta}\frac{k}{t}, && \text{(1.1}\\
H_n(t) &= b(t)\frac{\sin nt}{nt} + \int_t^c \frac{\sin nu}{nu}\,\mathrm{d}\,b(u), && \text{(1.1}
\end{aligned}
$$

where $0 < c \le \pi$ and k is a suitable positive constant taken for the convenience in t analysis and possibly depending upon δ.

2. Introduction

In 1968, Mohanty and Mohapatra [12] began the study of absolute Euler summability Fourier series by proving the following:

Theorem A. *Let*

$$\phi(t)\log\frac{1}{t} \in BV(0,c), \quad 0 < c < 1. \qquad \text{(2}$$

Then

$$\sum A_n(x) \in |E,q| \quad (q > 0). \qquad \text{(2}$$

Among other results the above result was also proved by Kwee [10] independently. also proved that the condition (2.1) cannot be replaced by the weaker condition

$$\phi(t)\log^{\eta}\frac{1}{t} \in BV(0,c), \quad 0 < \eta < 1, \qquad \text{(}$$

in Theorem A. This result of Kwee [10] was further improved by the present author Dikshit [7].

In 1978, the present author [4] proved the following:

Theorem B. *Let*

$$\phi(t) \in BV(0,\pi). \qquad \text{(}$$

Then

$$\sum_{n=1}^{\infty}\frac{A_n(x)}{\log(n+1)} \in |E,q| \qquad (q > 0). \qquad \text{(}$$

Recently, Ray and Sahoo [15] have not only bridged the gap in between Theorems A and B but they have also improved Theorem B by proving the following:

Theorem C. *Let* $0 \le \delta \le 1$ *and let*

$$\phi(t) \log^{1-\delta} \frac{k}{t} \in BV(0, c), \quad 0 < c < 1. \tag{2.6}$$

Then

$$\sum_{n=1}^{\infty} \frac{A_n(x)}{\log^{\delta}(n+1)} \in |E, q| \qquad (q > 0). \tag{2.7}$$

It may be remarked that in Theorem C, δ has been restricted to be in [0,1] since for $\delta > 1$, (2.6) implies the absolute convergence of

$$\sum_{n=1}^{\infty} \frac{A_n(x)}{\log^{\delta}(n+1)}. \tag{2.8}$$

A reference may be made to Chandra [1]; Theorem 2 on page 6, and hence (2.8) is necessarily summable $|E, q|$ $(q > 0)$.

In a different setting, very recently, Dikshit [8] has obtained a few more results concerning the absolute Euler summability factors for Fourier series.

One of the main objects of the present paper is to improve Theorem C on replacing (2.6) by the following weaker condition:

$$\left.\begin{array}{lll} \text{(i)} & P(t) \log^{1-\delta} \frac{k}{t} & \in \; BV(0, c) \\ \text{(ii)} & t^{-1} P(t) \log^{-\delta} \frac{k}{t} & \in \; L(0, c) \end{array}\right\}, \tag{2.9}$$

where $0 \le \delta \le 1$ and $0 < c < 1$. The above claim that (2.9) is weaker than (2.6) has been settled in Lemma 1 of the present paper.

Secondly, we investigate necessary and sufficient conditions, imposed upon the generating functions of the Fourier series of f at x, for the truth of (2.7). Before we give the statement of the theorem to be proved, we give the following equivalent form of (2.9), which follows from Lemma 2 of the present paper:

$$\left.\begin{array}{lll} \text{(i)} & \int_0^c \log \frac{k}{t} \; |dg(t)| < \infty, & 0 < c < 1 \\ \text{(ii)} & g(0+) = 0 & \end{array}\right\}. \tag{2.10}$$

Precisely, we prove the following:

Theorem. *Let* $\delta \ge 0$ *and let* (2.10) (i) *hold. Then in order that* (2.7) *should hold, it is necessary and sufficient that* (2.10) (ii) *must hold. Further, the condition* (2.10) (i) *is best possible in the sense that it cannot be replaced by*

$$\int_0^{\pi} \log^{\eta} \frac{k}{t} \; |dg(t)| < \infty \quad (0 < \eta < 1). \tag{2.11}$$

3. Estimates

To prove the theorem, we shall require the following estimates for $\delta \geq 0$ but proved fo
real δ: uniformly in $0 < t < c$,

$$H_n(t) = O(1)\left(b(t) + b\left(\frac{1}{n+1}\right)\right), \qquad (3.1$$

$$H_n(t) = b(t)\frac{\sin nt}{nt} + O(1)n^{-2}t^{-2}b(t), \qquad (3.2$$

$$s_n(t) \leq y^n(t) + \left(\frac{q}{1+q}\right)^n. \qquad (3.3$$

Proof of (3.1). We have

$$H_n(t) = b(t)\frac{\sin nt}{nt} + \int_0^c \frac{\sin nu}{nu} \mathrm{d}b(u) - \int_0^t \frac{\sin nu}{nu} \mathrm{d}b(u).$$

Now, since $b(u)$ is monotonic increasing therefore

$$\left|\int_0^t \frac{\sin nu}{nu} \mathrm{d}b(u)\right| \leq \int_0^t \mathrm{d}b(u) = b(t)$$

and

$$\left|\int_0^c \frac{\sin nu}{nu} \mathrm{d}b(u)\right| \leq \int_0^{n^{-1}} \mathrm{d}b(u) + \left|\int_{n^{-1}}^c \frac{\sin nu}{nu} \mathrm{d}b(u)\right|.$$

Also $u^{-1}\frac{\mathrm{d}}{\mathrm{d}u}b(u)$ decreases therefore, we have, by the second mean value theorem

$$\begin{aligned}\int_{n^{-1}}^c \frac{\sin nu}{nu} \mathrm{d}\,b(u) &= \left[u^{-1}\frac{\mathrm{d}}{\mathrm{d}u}b(u)\right]_{u=n^{-1}} \int_{n^{-1}}^{\theta} \frac{\sin nu}{n} \mathrm{d}u \quad (n^{-1} < \theta < c \\ &= O(1)b\left(\frac{1}{n+1}\right).\end{aligned}$$

Collecting the results, we get (3.1).

Proof of (3.2). Since $u^{-1}\frac{\mathrm{d}}{\mathrm{d}u}b(u)$ decreases, therefore, by the second mean val
therorem

$$\begin{aligned}\int_t^c \frac{\sin nu}{nu} \mathrm{d}\,b(u) &= (nt)^{-1}\frac{\mathrm{d}}{\mathrm{d}t}b(t)\int_t^{c'} \sin nu\,\mathrm{d}u \qquad (t < c' < c) \\ &= O(n^{-2}t^{-2}b(t)).\end{aligned}$$

Using this estimate in the definition $H_n(t)$, we get (3.2).

Proof of (3.3). We have

$$s_n(t) = \text{imaginary part of} \sum_{k=0}^{n} V_k^q(n)\ \exp(ikt),$$

where

$$\begin{aligned}
&\sum_{k=0}^{n} V_k^q(n)\ \exp(ikt)\\
&= e^{-it} \sum_{k=0}^{n} V_k^q(n)\ \exp(i(k+1)t)\\
&= e^{-it} \sum_{m=1}^{n+1} V_{m-1}^q(n)\ \exp(imt)\\
&= e^{-it} \sum_{m=0}^{n+1} V_{m-1}^q(n)\ \exp(imt) - e^{-it} V_{-1}^q(n)\\
&= e^{-it} \frac{(q+\exp(it))^{n+1}}{(1+q)^n} - e^{-it} \frac{q^{n+1}}{(1+q)^{n+1}}\\
&= e^{-it} \frac{R^{n+1}}{(1+q)^{n+1}} (\cos\theta + i \sin\theta)^{n+1} - e^{-it} \left(\frac{q}{1+q}\right)^{n+1}\\
&= \left(\frac{R}{1+q}\right)^{n+1} e^{i(n+1)\theta - it} - e^{-it} \left(\frac{q}{1+q}\right)^{n+1},
\end{aligned}$$

where $R\cos\theta = q + \cos t$, $R\sin\theta = \sin t$ and

$$\theta = \tan^{-1}\left(\frac{\sin\theta}{q+\cos\theta}\right).$$

Hence imaginary part of

$$\sum_{k=0}^{n} V_k^q(n)\ \exp(ikt) = \left(\frac{R}{1+q}\right)^{n+1} \sin[(n+1)\theta - t] + \left(\frac{q}{1+q}\right)^{n+1} \sin t,$$

where

$$R = \sqrt{1+q^2+2q\cos t} = y(t)(1+q).$$

Hence

$$|s_n(t)| \leq y^n(t) + \left(\frac{q}{1+q}\right)^n.$$

This completes the proof.

4. Lemmas

We require the following lemmas for the proof of the theorem:

Lemma 1. *For* $0 \leq \delta \leq 1$,

$$(2.6) \Rightarrow (2.9) \tag{4.1}$$

but its converse is not true in general.

Proof. It has been observed (Chandra [5]; p. 19) that (2.6) with $\delta = 1$ holds if and only i

$$\text{(i)} \quad P(t) \in BV(0,c), \quad \text{(ii)} \quad t^{-t}Pt \in L(0,c), \tag{4.2}$$

which is stronger than (2.9) with $\delta = 1$.

We now consider the case $0 \leq \delta < 1$. In this case, we observe that

$$\begin{aligned} (2.6) \quad &\Rightarrow \quad \phi(t) \quad \in \quad BV(0,c) \\ &\Rightarrow \quad t^{-1}P(t) \quad \in \quad L(0,c) \quad \text{(see (4.2) (ii))} \\ &\Rightarrow \quad t^{-1}P(t)\ \log^{-\delta}\frac{k}{t} \quad \in \quad L(0,c). \end{aligned}$$

Hence (2.9) (ii) holds. Now for the truth of (2.9) (i), we write

$$h(t) = \phi(t)\ \log^{1-\delta}\frac{k}{t} \quad \text{and} \quad h_1(t) = \frac{1}{t}\int_0^t h(u)\,\mathrm{d}u.$$

Then $h_1(t) \in BV(0,c)$, where

$$\begin{aligned} th_1(t) &= \int_0^t \phi(u) \quad \log^{1-\delta}\frac{k}{u}\mathrm{d}u \\ &= t\phi_1(t) \quad \log^{1-\delta}\frac{k}{t} + (1-\delta)\int_0^t \phi_1(u) \quad \log^{-\delta}\frac{k}{u}\mathrm{d}u. \end{aligned}$$

Hence

$$h_1(t) = \phi_1(t) \quad \log^{1-\delta}\frac{k}{t} + (1-\delta)\frac{1}{t}\int_0^t \phi_1(u)\log^{-\delta}\frac{k}{u}\mathrm{d}u$$

from which one gets

$$\begin{aligned} P(t) \quad \log^{1-\delta}\frac{k}{t} \quad &= \quad h(t) - h_1(t) \\ &\quad + \frac{1-\delta}{t}\int_0^t \phi_1(u) \quad \log^{-\delta}\frac{k}{u}\mathrm{d}u. \end{aligned} \tag{4}$$

Observe that

$$\begin{aligned} h(t) \in BV(0,c) \quad &\Rightarrow \quad \phi_1(t) \quad \log^{-\delta}\frac{k}{t} \in BV(0,c) \\ \Rightarrow \quad &\left\{\frac{1}{t}\int_0^t \phi_1(u) \quad \log^{-\delta}\frac{k}{u}\mathrm{d}u\right\} \in BV(0,c). \end{aligned}$$

Hence using these results in (3.3), we get

$$P(t) \quad \log^{1-\delta}\frac{k}{t} \in BV(0,c).$$

To prove that converse is not true in general, let f be even function and $x = 0$. Then $\phi(t) = f(t)$ in $[0, \pi]$. We define

$$f(t) = \begin{cases} \left(\log\dfrac{k}{t}\right)^{-\frac{1}{2}(1-\delta)} & \text{in } (0,c) \\ 0 & \text{elsewhere.} \end{cases}$$

Then (2.6) does not hold.

On the other hand, since $\phi(t) \in BV(0,c)$, therefore

$$P(t) = \frac{1}{t}\int_0^t u\,\mathrm{d}\phi(u) = \frac{1}{2}(1-\delta)\frac{1}{t}\int_0^t \left(\log\frac{k}{u}\right)^{(\delta-3)/2} \mathrm{d}u \tag{4.4}$$

and hence

$$\begin{aligned}\int_0^c \left|\frac{P(t)}{t\log^\delta \frac{k}{t}}\right| \mathrm{d}t &< \frac{1}{2}\int_0^c \frac{\mathrm{d}t}{t^2\log^\delta\frac{k}{t}}\int_0^t \left(\log\frac{k}{u}\right)^{(\delta-3)/2}\mathrm{d}u \\ &= \frac{1}{2}\int_0^c \left(\log\frac{k}{u}\right)^{(\delta-3)/2}\mathrm{d}u\int_u^c \frac{t^{-2}}{\log^\delta\frac{k}{t}}\,\mathrm{d}t \\ &< \int_0^c \frac{\mathrm{d}u}{u\log^{(3+\delta)/2}\left(\frac{k}{u}\right)} < \infty,\end{aligned}$$

which proves (2.9) (ii). Also from (4.4)

$$P(t)\log^{1-\delta}\frac{k}{t} = \frac{1}{2}(1-\delta)t^{-1}\log^{1-\delta}\frac{k}{t}\int_0^t \log^{(\delta-3)/2}\left(\frac{k}{u}\right)\mathrm{d}u$$

$$= \frac{1}{2}(1-\delta)\log^{-\frac{1}{2}(1+\delta)}\left(\frac{k}{t}\right) + \frac{1}{2}(1-\delta)\frac{\delta-3}{2}t^{-1}\log^{1-\delta}\left(\frac{k}{t}\right)\int_0^t \log^{(\delta-3)/2}(ku)\mathrm{d}u.$$

Now it may be observed that each of the term on the right above is of bounded variation on $(0, c)$ and hence

$$P(t)\log^{1-\delta}\left(\frac{k}{t}\right) \in BV(0,c),$$

which proves (2.9) (i).

This completes the proof of the lemma.

Lemma 2 [11]. *If* $\eta > 0$, *then necessary and sufficient conditions that* (i) $h(t)\log\frac{k}{t} \in$ $BV(0,\eta)$ *and* (ii) $t^{-1}h(t) \in L(0,\eta)$ *are that*

$$h(0+) = 0 \quad \textit{and} \quad \int_0^{\eta} \log\left(\frac{k}{t}\right) |\,\mathrm{d}h(t)| < \infty.$$

Lemma 3 [15]. *Let, for* $0 < c < \pi$,

$$\alpha_n = \frac{2}{\pi}\int_c^{\pi} \phi(t)\cos nt\,\mathrm{d}t.$$

Then $\sum_{n=1}^{\infty} \alpha_n d_n \in |E,q|\ (q > 0)$.

This is really proved for $0 \le \delta \le 1$ *but the same arguments hold for* $\delta \ge 0$.

Lemma 4. *Let* $0 < \beta \le \pi$ *and* $\delta \ge 0$. *Then uniformly in* $0 < t < \beta$

$$\sum_{m=1}^{n} V_m^q(n)\, d_m\, \exp(imt) = O\left\{n^{-1/2}t^{-1}\log^{-\delta}\left(\frac{k}{t}\right)\right\}.$$

The case $\delta = 1$ *is dealt with in Lemma 2 of Chandra* [4]. *The general case may b* *obtained similarly.*

Lemma 5. *For* $0 < c < \pi$ *and for all real* β

$$\frac{2}{\pi}\int_0^{c} \frac{\sin nu}{u}\log^{\beta}\frac{k}{u}\,\mathrm{d}u \sim \log^{\beta} n.$$

The case $\beta = 1$ *with* $c = \pi$ *was dealt with by Mohanty and Ray* [13] *and for all real* *with* $c = \pi$, *references may be made to Ray* [14] *or Chandra* [6]. *Since the same argumen* *hold if we replace* π *by* c *in Ray* [14] *or Chandra* [6], *therefore one can get the above resu* *from either Ray* [14] *or Chandra* [6].

Lemma 6. *Uniformly in* $0 < t < \pi$,

$$\sum_{m=1}^{n} V_m^q(n)\, d_m \sin mt$$
$$= O(t^{-1})\Delta d_n + O\{d_n y^n(t)\} + O\left\{(d_n)\left(\frac{q}{1+q}\right)^n\right\}.$$

Proof. Let N denote the integral part of $\frac{n+1-q}{1+q}$ for $n > 2q$. Then we first observe th $V_m^q(n)$ increases monotonically with $m \le N$ and decreases with $m > N$. And, by Abel transformation

$$\begin{aligned}
&\sum_{m=1}^{n} \quad V_m^q(n) d_m \sin mt \\
&= \quad \sum_{m=1}^{n-1} s_m(t)\Delta d_m + s_n(t) d_n \\
&= \quad \sum_{m=1}^{N} s_m(t)\Delta d_m + \sum_{m=N+1}^{n-1} s_m(t)\Delta d_m + s_n(t) d_n \\
&= \quad \sum_{m=1}^{N} s_m(t)\Delta d_m + \sum_{m=N+1}^{n-1} \left[s_n(t) - \sum_{k=m+1}^{n} V_k^q(n)\sin kt \right] \Delta d_m + s_n(t) d_n \\
&= \quad \sum_{m=1}^{N} s_m(t)\Delta d_m + d_{N+1} s_n(t) - \sum_{m=N+1}^{n-1} \Delta d_m \sum_{k=m+1}^{n} V_k^q(n)\sin kt \\
&= \quad \sum\nolimits_1 + \sum\nolimits_2 + \sum\nolimits_3, \quad \text{say.} \qquad (4.5)
\end{aligned}$$

owever, by Abel's lemma

$$\begin{aligned}
|s_m(t)| \quad &\le \quad V_m^q(n) \max_{1 \le m' < m'' \le m} \left| \sum_{k=m'}^{m''} \sin kt \right| \qquad (\text{for } m < N) \\
&= \quad O(t^{-1}) V_m^q(n).
\end{aligned}$$

ence

$$\begin{aligned}
\sum_{1} \quad &= \quad O(t^{-1}) \sum_{m=1}^{N} \Delta d_m V_m^q(n) \\
&= \quad O(t^{-1}) \sum_{m=1}^{n} \Delta d_m V_m^q(n) \\
&= \quad O(t^{-1}) n^2 \Delta d_n \sum_{m=1}^{n} m^{-2} V_m^q(n) \\
&= \quad O(t^{-1}) \Delta d_n, \qquad (4.6)
\end{aligned}$$

ice $m^2 \Delta d_m$ is increasing and

$$\sum_{m=1}^{n} m^{-2} V_m^q(n) = O(n^{-2}).$$

id by (3.3)

$$\sum_{2} = O\left\{ d_n y^n(t) \right\} + O\left\{ d_n \left(\frac{q}{1+q} \right)^n \right\}. \qquad (4.7)$$

nally, once again by applying Abel's lemma in the inner sum of $\sum_3$, we get

$$\sum_{3} \quad = \quad \sum_{m=N+1}^{n-1} \Delta d_m O(t^{-1}) V_m^q(n)$$

$$
\begin{aligned}
&= O(t^{-1}) \sum_{m=1}^{n} V_m^q(n) \Delta d_m \\
&= O(t^{-1}) \Delta d_n, \qquad (4.8)
\end{aligned}
$$

as in $\sum_1$.

Combining (3.5) through (4.8), we get the required result.

Lemma 7. *There exists an* $f \in L_{2\pi}$ *for which* (2.10) (i) *and* (2.11) *hold but the series* (2.8) *at* $x = 0$ *diverges properly for every real* δ *and hence not summable by any regular summability method.*

Proof. Let f be even and let $x = 0$. Then $\phi(t) = f(t)$. Define f by periodicity. We first consider the case $\delta = 0$ for which we define

$$
f(t) = \begin{cases} \log\log\left(\frac{k}{t}\right), & 0 < t \leq \pi \\ 0, & t = 0 \end{cases} \qquad (4.9)
$$

where $k \geq \pi e^2$. Then

$$
\begin{aligned}
g(t) &= \phi(t) - \phi_1(t) \\
&= -\frac{1}{\log\left(\frac{k}{t}\right)} + \frac{1}{t} \int_0^t \log^{-2}\left(\frac{k}{u}\right) \mathrm{d}u,
\end{aligned}
$$

which is of bounded variation and $g(0+) = 0$. Hence

$$
\int_0^\pi \log^\eta \frac{k}{t} | \, \mathrm{d}g(t)| < \int_0^\pi t^{-1} \log^{\eta-2} \frac{k}{t} \, \mathrm{d}t,
$$

which converges whenever $0 < \eta < 1$. This proves that (2.10) (i) and (2.11) hold. However

$$
\begin{aligned}
A_n(x) &= \frac{2}{\pi} \int_0^\pi \log\log\frac{k}{t} \cos nt \, \mathrm{d}t \\
&= \frac{2}{\pi} \int_0^\pi \frac{\sin nt}{nt} \log^{-1} \frac{k}{t} \, \mathrm{d}t \\
&\sim \frac{1}{n \log n},
\end{aligned}
$$

by using Lemma 5. Thus $\sum_{n=1}^{\infty} A_n(x)$ diverges properly and hence it cannot be summable by any absolutely regular summability method and, *a fortiori*, (2.8) with $\delta = 0$ is not $|E, q|$ $(q > 0)$ summable.

In the case when δ is non-zero real number, we define

$$
f(t) = \begin{cases} \log^\delta \frac{k}{t}, & (0 < t \leq \pi) \\ 0, & t = 0 \end{cases} . \qquad (4.10)
$$

Then since $\phi(t) = f(t)$, we have

$$\begin{aligned} P(t) &= \delta \frac{1}{t} \int_0^t \log^{\delta-1}\left(\frac{k}{u}\right) \mathrm{d}u \\ &= \delta \log^{\delta-1}\left(\frac{k}{t}\right) + \frac{\delta(\delta-1)}{t} \int_0^t \log^{\delta-2}\left(\frac{k}{u}\right) \mathrm{d}u \end{aligned}$$

and hence

$$\begin{aligned} g(t) &= P(t) \log^{-\delta}\left(\frac{k}{t}\right) \\ &= \frac{\delta}{\log\left(\frac{k}{t}\right)} + \frac{\delta(\delta-1)}{t \log^{\delta}\left(\frac{k}{t}\right)} \int_0^t \log^{\delta-2}\left(\frac{k}{u}\right) \mathrm{d}u, \end{aligned}$$

which shows that $g(0+) = 0$ and

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} g(t) = \frac{\delta^2}{t \log^2\left(\frac{k}{t}\right)} \quad &+ \quad \delta(\delta-1) \left\{ \frac{\delta}{t^2 \log^{1+\delta}\left(\frac{k}{t}\right)} - \frac{1}{t^2 \log^{\delta}\left(\frac{k}{t}\right)} \right\} \\ &\times \int_0^t \log^{\delta-2}\left(\frac{k}{u}\right) \mathrm{d}u \end{aligned}$$

and for all real $\delta \neq 0$

$$\int_0^t \log^{\delta-2}\left(\frac{k}{u}\right) \mathrm{d}u \leq Mt \log^{\delta-2}\left(\frac{k}{t}\right),$$

where M is a positive constant not necessarily the same at each occurrence and possibly depending upon δ. Therefore

$$\int_0^\pi \log^\eta \frac{k}{t} |\mathrm{d}g(t)| \leq M \int_0^\pi t^{-1} \log^{\eta-2} \frac{k}{t} \, \mathrm{d}t,$$

which converges for $0 < \eta < 1$. This proves that (2.10) (i) and (2.11) hold for all real $\delta \neq 0$. But for the function defined by (3.10)

$$\begin{aligned} A_n(x) &= \frac{2}{\pi} \int_0^\pi \log^{\delta}\left(\frac{k}{t}\right) \cos nt \, \mathrm{d}t \quad (\delta \neq 0) \\ &= \frac{2\delta}{n\pi} \int_0^\pi \frac{\sin nt}{t} \log^{\delta-1}\left(\frac{k}{t}\right) \mathrm{d}t \\ &\sim \frac{\delta}{n} \log^{\delta-1} n, \end{aligned}$$

by Lemma 5 and hence,

$$\frac{A_n(x)}{\log^{\delta}(n+1)} \sim \frac{\delta}{n\log(n+1)}.$$

This shows that for every real $\delta \neq 0$, (2.8) is not $|E, q|$ $(q > 0)$ summable since

$$\sum \frac{1}{n\log(n+1)} = \infty.$$

This completes the proof of the lemma.

5. Proof of the theorem

In view of the inclusion: $|E, q| \subset |E, q'|$ $(q' > q > -1)$ (see Chandra [2]; Corollary 2 we assume $0 < q < 1$ for the proof of the theorem, without any loss of generality.

Let (2.10) (i) hold. Then proceeding as in Chandra ([3], p. 388–9), we have for $n \geq 1$

$$A_n(x) = \frac{2}{\pi}\int_c^{\pi} \phi(t) \quad \cos nt \,\mathrm{d}t + \frac{2}{\pi}\phi_1(c)\frac{\sin nc}{n} + \frac{2}{\pi}\int_0^c tP(t)\frac{\partial}{\partial t}\left(\frac{\sin nt}{nt}\right)\mathrm{d}t \qquad \text{(5.1}$$

and integrating by parts, we get

$$\begin{aligned}&\int_0^c tP(t)\frac{\partial}{\partial t}\left(\frac{\sin nt}{nt}\right)\mathrm{d}t\\ &= \int_0^c g(t)b(t)\frac{\partial}{\partial t}\left(\frac{\sin nt}{nt}\right)\mathrm{d}t\\ &= g(0+)\int_0^c b(u)\frac{\partial}{\partial u}\left(\frac{\sin nu}{nu}\right)\mathrm{d}u + \int_0^c \mathrm{d}g(t)\int_t^c b(u)\frac{\partial}{\partial u}\left(\frac{\sin nu}{nu}\right)\mathrm{d}u \qquad \text{(5.2}\end{aligned}$$

and for $0 \leq t < c$

$$\int_t^c b(u)\frac{\partial}{\partial u}\left(\frac{\sin nu}{nu}\right)\mathrm{d}u = b(c)\frac{\sin nc}{nc} - b(t)\frac{\sin nt}{nt} - \int_t^c \frac{\sin nu}{nu}\,\mathrm{d}b(u). \qquad \text{(5.3}$$

Using (5.3) in (5.2), we get

$$\begin{aligned}&\int_0^c tP(t)\frac{\partial}{\partial t}\left(\frac{\sin nt}{nt}\right)\mathrm{d}t\\ &= g(0+)\left[b(c)\frac{\sin nc}{nc} - \int_0^c \frac{\sin nu}{nu}\,\mathrm{d}b(u)\right] + \int_0^c b(c)\frac{\sin nc}{nc}\,\mathrm{d}g(t) - \int_0^c H_n(t)\,\mathrm{d}g(\\ &= g(c)b(c)\frac{\sin nc}{nc} - g(0+)\int_0^c \frac{\sin nu}{nu}\,\mathrm{d}b(u) - \int_0^c H_n(t)\,\mathrm{d}g(t). \qquad \text{(5.}\end{aligned}$$

And using (5.4) in (5.1), we get

$$\begin{aligned} A_n(x) &= \frac{2}{\pi}\int_c^\pi \phi(t)\cos nt\,\mathrm{d}t + \frac{2}{\pi}\phi(c)\frac{\sin nc}{n} \\ &\quad - \frac{2}{\pi}g(0+)\int_0^c \frac{\sin nu}{nu}\,\mathrm{d}b(u) - \frac{2}{\pi}\int_0^c H_n(t)\,\mathrm{d}g(t) \\ &= \alpha_n + \beta_n - \gamma_n - \delta_n, \text{ say.} \end{aligned} \tag{5.5}$$

Since $A_0 = \frac{1}{2}a_0 = 0$, therefore

$$\sum_{n=1}^{\infty} A_n(x)d_n \in |E, q| \quad (q > 0)$$

if and only if

$$\sum_{n=1}^{\infty} \frac{1}{(q+1)^{n+1}}\left|\sum_{m=1}^{n}\binom{n}{m}q^{n-m}d_m A_m(x)\right| < \infty. \tag{5.6}$$

However, it follows from Lemma 3 that

$$\sum_{n=1}^{\infty} \alpha_n d_n \in |E, q| \quad (q > 0) \tag{5.7}$$

and since

$$\beta_n = \frac{2}{\pi}\phi(c)\sin nc\left[\frac{1}{n+1} + \frac{1}{n(n+1)}\right]$$

and

$$\sum_{n=1}^{\infty}\left|\frac{2}{\pi}\phi(c)d_n\frac{\sin nc}{n(n+1)}\right| < \infty.$$

Therefore, in view of absolute regularity of $|E, q|$ $(q > 0)$ method,

$$\sum_{n=1}^{\infty} \beta_n d_n \in |E, q| \quad (q > 0) \tag{5.8}$$

if

$$\begin{aligned} &\sum_{n=1}^{\infty} \frac{1}{(q+1)^{n+1}}\left|\sum_{m=1}^{n}\binom{n}{m}q^{n-m}d_m\frac{\sin nc}{m+1}\right| \\ &\qquad = \sum_{n=1}^{\infty}\frac{1}{n+1}\left|\sum_{m=1}^{n} V_n^q(m)d_m \sin nc\right| < \infty, \end{aligned}$$

which holds by Lemma 4. Now

$$\sum_{n=1}^{\infty} \delta_n d_n \in |E, q| \quad (q > 0) \tag{5.9}$$

if and only if

$$Q = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{(q+1)^{n+1}}\left|\sum_{m=1}^{n}\binom{n}{m}q^{n-m}d_m\int_0^c H_m(t)\mathrm{d}g(t)\right| < \infty.$$

Clearly

$$Q \le \frac{2}{\pi}\int_0^c |\,\mathrm{d}g(t)|\sum_{n=1}^{\infty}\frac{1}{(q+1)^{n+1}}\left|\sum_{m=1}^{n}\binom{n}{m}q^{n-m}d_m H_m(t)\right|$$

and since by (2.10) (i),

$$\int_0^c \log\frac{k}{t}|\,\mathrm{d}g(t)| < \infty$$

therefore for the proof of (5.9) it is suffiecient to prove that

$$Z = \sum_{n=1}^{\infty}\frac{1}{(q+1)^{n+1}}\left|\sum_{m=1}^{n}\binom{n}{m}q^{n-m}d_m H_m(t)\right| = O\left(\log\frac{k}{t}\right), \tag{5.10}$$

uniformly in $0 < t < c$.

For $T = [k/t]$, the integral part of k/t, we write

$$Z = \sum_{n\le T} + \sum_{n>T} \qquad \text{say.} \tag{5.11}$$

By (3.1), we get

$$\begin{aligned}
\sum_{n\le T} &= b(t)\sum_{n=1}^{T}\frac{1}{(q+1)^{n+1}}\sum_{m=1}^{n}\binom{n}{m}q^{n-m}d_m \\
&\quad + O(1)\sum_{n=1}^{T}\frac{1}{(q+1)^{n+1}}\sum_{m=1}^{n}\binom{n}{m}q^{n-m}\frac{1}{m+1} \\
&= O(1)b(t)\sum_{n=1}^{T}d_n\sum_{m=1}^{n}V_m^q(n) + O(1)\sum_{n=1}^{T}\frac{1}{n+1}\sum_{m=1}^{n}V_m^q(n) \\
&= O(1)b(t)\sum_{n=1}^{T}d_n + O(1)\sum_{n=1}^{T}\frac{1}{n+1} \\
&= O\left(\log\frac{k}{t}\right),
\end{aligned} \tag{5.12}$$

uniformly in $0 < t < c$, since $\sum_{m=1}^{n} V_m^q(n) \le 1$. And by (3.2)

$$\begin{aligned}
\sum_{n\ge T} &= \log^{\delta}\left(\frac{k}{t}\right)\sum_{n=T}^{\infty}\frac{1}{(q+1)^{n+1}}\left|\sum_{m=1}^{n}\binom{n}{m}q^{n-m}d_m\frac{\sin mt}{m}\right| \\
&\quad + O\left(t^{-1}\log^{\delta}\frac{k}{t}\right)\sum_{n=T}^{\infty}\frac{1}{(q+1)^{n+1}}\sum_{m=1}^{n}\binom{n}{m}q^{n-m}\frac{d_m}{m(m+1)}
\end{aligned}$$

$$
\begin{aligned}
&= \log^{\delta}\frac{k}{t}\sum_{n=T}^{\infty}\frac{1}{(q+1)^{n+1}}\left|\sum_{m=1}^{n}\binom{n}{m}q^{n-m}d_m\frac{\sin mt}{m+1}\right| \\
&\quad + O\left(t^{-1}\log^{\delta}\frac{k}{t}\right)\sum_{n=T}^{\infty}\frac{1}{(q+1)^{n+1}}\sum_{m=1}^{n}\binom{n}{m}q^{n-m}\frac{d_m}{m(m+1)} \\
&= R(t)\log^{\delta}\frac{k}{t} + O\left(t^{-1}\log^{\delta}\frac{k}{t}\right)W(t), \quad \text{say,} \qquad (5.13)
\end{aligned}
$$

where

$$W(t) = \sum_{n=T}^{\infty}\frac{1}{(q+1)^{n+1}}\sum_{m=1}^{n}\binom{n}{m}q^{n-m}\frac{d_m}{m(m+1)}.$$

Now, by using repeatedly the relation:

$$\binom{r}{s} = \frac{s+1}{r+1}\binom{r+1}{s+1},$$

where r and s are integers such that $r \geq s \geq 0$, we get

$$
\begin{aligned}
&\sum_{m=1}^{n}\binom{n}{m}q^{n-m}\frac{d_m}{m(m+1)} \\
&\qquad = \frac{1}{n+1}\sum_{m=1}^{n}\binom{n+1}{m+1}q^{n-m}\frac{d_m}{m} \\
&\qquad = \frac{1}{n+1}\sum_{m=1}^{n}\binom{n+1}{m+1}q^{n-m}\left(\frac{2}{m}+1\right)\frac{d_m}{m+2} \\
&\qquad = \frac{1}{(n+1)(n+2)}\sum_{m=1}^{n}\binom{n+2}{m+2}q^{n-m}\left(\frac{2}{m}+1\right)d_m \\
&\qquad < \frac{3}{n^2}\sum_{m=1}^{n}\binom{n+2}{m+2}q^{n-m}d_m \\
&\qquad = \frac{3}{n^2(n+3)}\sum_{m=1}^{n}\binom{n+3}{m+3}q^{n-m}(m+3)d_m.
\end{aligned}
$$

However, the function $(x+2)\log^{-\delta}x$ increases with $x > \exp(3\delta)$, therefore

$$
\begin{aligned}
W(t) &= O(1)\sum_{n=T}^{\infty}(q+1)^{-n-1}n^{-2}d_n\sum_{m=1}^{n}\binom{n+3}{m+3}q^{n-m} \\
&= O(1)\sum_{n=T}^{\infty}(q+1)^{-n-1}n^{-2}d_n\sum_{m=0}^{m+3}\binom{n+3}{m}q^{n-m} \\
&= O(1)\sum_{n=T}^{\infty}(q+1)^{2}\frac{d_n}{n^2}
\end{aligned}
$$

$$= O(1)\sum_{n=T}^{\infty}\frac{d_n}{n^2}$$
$$= O\left(t\log^{-\delta}\left(\frac{k}{t}\right)\right), \tag{5.14}$$

uniformly in $0 < t < c$. And, by Lemma 6,

$$\begin{aligned}
R(t) &= \sum_{n=T}^{\infty}\frac{1}{n+1}\left|\sum_{m=1}^{n} V_m^q(n)d_m \sin mt\right| \\
&= O(t^{-1})\sum_{n=T}^{\infty}\frac{\Delta d_n}{n+1} + O(1)\sum_{n=T}^{\infty}\frac{d_n}{n+1}y^n(t) \\
&\quad + O(1)\sum_{n=T}^{\infty}\frac{d_n}{n+1}\left(\frac{q}{1+q}\right)^n \\
&= O(1)\log^{-\delta}\frac{k}{t} + O(1)\log^{-\delta}\left(\frac{k}{t}\right)\sum_{n=1}^{\infty}\frac{y^n(t)}{n} \\
&\quad + O(1)t\log^{-\delta}\left(\frac{k}{t}\right)\sum_{n=0}^{\infty}\left(\frac{q}{1+q}\right)^n \\
&= O(1)\log^{1-\delta}\left(\frac{k}{t}\right),
\end{aligned} \tag{5.15}$$

uniformly in $0 < t < c$, since

$$\sum_{n=1}^{\infty}\frac{y^n(t)}{n} = \log\frac{1}{1-y(t)}$$

and

$$\frac{1}{1-y(t)} = O\left(\frac{k}{t}\right)^2 \quad (t \to 0+).$$

Combining (5.11) through (5.15) we get (5.10). Also in view of (5.5) through (5.9), $\sum_{n=1}^{\infty} A_n(x)d_n \in |E, q|(q > 0)$ if and only if

$$\sum_{n=1}^{\infty}\gamma_n d_n \in |E, q| \quad (q > 0), \tag{5.16}$$

where

$$\gamma_n d_n = \frac{2}{\pi}g(0+)d_n\int_0^c \frac{\sin nu}{nu}\,\mathrm{d}b(u)$$

and, by Lemma 5,

$$\begin{aligned}
\frac{2}{\pi}\int_0^c \frac{\sin nu}{nu}\mathrm{d}b(u) &\sim \frac{1}{n}[\log^{\delta}(n+1) - \delta\log^{\delta-1}(n+1)] \\
&\sim \frac{1}{n}\log^{\delta}(n+1)
\end{aligned}$$

and hence

$$\frac{2}{\pi} d_n \int_0^c \frac{\sin nu}{nu} \, \mathrm{d}b(u) \sim \frac{1}{n}.$$

Thus in order that (5.16) should hold it is necessary and sufficient that

$$\sum_{n=1}^{\infty} \frac{g(0+)}{n} \in |E, q| \; (q > 0)$$

for which it is necessary and sufficient that (2.10)(ii) must hold, since $\sum_{n=1}^{\infty} 1/n$ diverges strictly.

The fact that the condition (2.10)(i) cannot be replaced by (2.11) follows by Lemma 7. This proves the theorem completely.

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On a Tauberian theorem of Hardy and Littlewood

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Abstract. In this paper, we give a simple alternative proof of a Tauberian theorem of Hardy and Littlewood (Theorem E stated below, [3]).

Keywords. Abel's theorem; Tauberian theorem; Hardy-Littlewood Tauberian theorem; divergent series.

1. Introduction

Let $\sum_{n=0}^{\infty} a_n$ be an infinite series of real terms. Let

$$0 \leq \lambda_0 < \lambda_1 <, \ldots, \lambda_n \to \infty$$

and let $\sum a_n e^{-\lambda_n x}$ be convergent for all $x > 0$. If

$$f(x) = \sum a_n e^{-\lambda_n x} \to s$$

as $x \to 0$, then we say that $\sum a_n$ is summable (A, λ_n) to s. When $\lambda_n = n$, the method (A, λ_n) reduces to the classical method summability (A), named after Abel.

It is a famous result due to Abel that if $\sum a_n$ is convergent to s, then $\sum a_n$ is summable (A) to s. That the converse is not necessarily true is evident from the example of the series

$$1 - 1 + 1 - 1 \cdots$$

which is summable (A) to $\frac{1}{2}$, but not convergent. The question naturally arises as to whether one can determine a suitable restriction or restrictions on the general term a_n so that $\sum a_n$ will be convergent to s whenever it is summable (A). The first answer to this question was given by Tauber in 1897 in the form of the following theorem.

Theorem A [7]. *If* $\sum a_n$ *is summable* (A) *to* s *and* $na_n = o(1)$, *then* $\sum a_n$ *is convergent to* s.

A generalization of Theorem A to the set-up of summability (A, λ_n) was proved by Landau [4].

Another significant generalization of Theorem A was obtained by Littlewood in 1910 in the form of

Theorem B. *If* $\sum a_n$ *is summable* (A) *to* s, *and* $na_n = O(1)$, *then* $\sum a_n$ *is convergent to* s.

In fact Littlewood proved the following more general theorem.

Theorem C [5]. *If* $\sum \mu_n$ *is a series of positive terms such that, as* $n \to \infty$,

$$\lambda_n = \mu_1 + \mu_2 + \cdots + \mu_n \to \infty, \quad \mu_n/\lambda_n \to 0,$$

$\sum a_n e^{-\lambda_n x} \to s$ *as* $x \to 0$, *and*

$$a_n = O\left(\frac{\lambda_n - \lambda_{n-1}}{\lambda_n}\right),$$

then $\sum a_n$ *is convergent to s.*

Littlewood had stated that Theorem C is true even without the restriction: $\mu_n/\lambda_n \to$
This result is stated below as Theorem C*. It was proved in 1928 by Ananda-Rau [1].
simple alternative proof was supplied by Bosanquet (see Hardy [2]).

Theorem C*. *If* $\sum \mu_n$ *is a series of positive terms such that* $\lambda_n = \mu_1 + \mu_2 + \cdots + \mu_n \to$
as $n \to \infty$, $\sum a_n e^{-\lambda_n x} \to s$ *as* $x \to 0$ *and*

$$a_n = O\left(\frac{\lambda_n - \lambda_{n-1}}{\lambda_n}\right), \qquad (1.$$

then $\sum a_n$ *is convergent to s.*

Littlewood also conjectured [5] that the following theorem is true.

Theorem D. *If* $\lambda_1 > 0$, $\lambda_{n+1}/\lambda_n \geq \theta > 1$ $(n = 1, 2, \ldots)$, *and*

$$f(x) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n x} \to s \quad as \quad x \to 0,$$

then $\sum a_n$ *converges to s.*

The truth of this conjecture was proved by Hardy and Littlewood [3][1]. Theorems of t
kind are called 'high indices' theorems, as distinguished from 'Tauberian' theorems, si
in such theorems no restriction is needed to be imposed upon the general term a_n of
series in question, excepting, of course, that $\sum a_n e^{-\lambda_n x}$ is convergent for every $x >$
Such a theorem shows that the method (A, λ_n) with the type of λ_n involved does not s
any series which is not convergent, and therefore shows the 'ineffectiveness' of the meth
(A, λ_n).

Hardy and Littlewood first established Theorem D in the special case in which

$$a_n = O(1)$$

and then, by further analysis, derived Theorem D itself. This is an instance of a Taube
theorem leading to a high indices theorem. Thus Hardy and Littlewood first establis
the following Tauberian theorem.

Theorem E. *If* $\lambda_1 > 0$, $\lambda_{n+1}/\lambda_n \geq \theta > 1$ $(n = 1, 2, \ldots)$,

$$f(x) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n x} \to s \quad as \quad x \to 0,$$

---

[1] For a proof of Theorem D due to A E Ingham, see [2], proof of Theorem 114, where too, the result has
obtained via a Tauberian theorem.

and $a_n = O(1)$, *then* $\sum a_n$ *is convergent to* s.

It should be observed that Theorem E is included in the theorem of Ananda-Rau, in which no extra restriction is imposed on λ_n, in view of the fact that whenever $\lambda_{n+1}/\lambda_n \geq \theta > 1$, and $a_n = O(1)$, $a_n = O\left(\frac{\lambda_n - \lambda_{n-1}}{\lambda_n}\right)$. On the other hand, under the hypotheses of Theorem C*, (1.1) implies: $s_n = O(1)$, and hence $a_n = O(1)$ (see Lemma 2 in the sequel).

The object of the present paper is to give an alternative proof of Theorem E which is quite straightforward, not requiring Lemmas 1 and 2 of Hardy and Littlewood [3].

2. Lemmas

We shall need the following lemmas.

Lemma 1 [5]. *If, as* $y \to 0$, $\psi(y) \to s$, *and for every positive integer* r,

$$y^r \psi^{(r)}(y) = O(1),$$

then for every positive integer r, $y^r \psi^{(r)}(y) = o(1)$.

Lemma 2 [4].[2] *If* $0 < \lambda_1 < \lambda_2 <, \ldots, \lambda_n \to \infty$, *as* $n \to \infty$, $f(x) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n x} = O(1)$ *as* $x \to 0$, *and*

$$a_n = O\left(\frac{\lambda_n - \lambda_{n-1}}{\lambda_n}\right),$$

then

$$s_n = a_1 + a_2 + \cdots + a_n = O(1).$$

Lemma 3 [3]. *If* $\lambda_1 > 0$ *and*

$$\frac{\lambda_{n+1}}{\lambda_n} \geq \theta > 1 \quad (n = 1, 2, \ldots)$$

then, for $r = 1, 2, \ldots,$

$$\sum_{n=1}^{\infty} \lambda_n^r e^{-\lambda_n x} = O(x^{-r}).$$

3. Proof of Theorem E

We may assume, without loss of generality, that $s = 0$. Thus $f(x) = o(1)$ as $x \to 0$. Also, since $a_n = O(1)$, for $r = 1, 2, \ldots,$

$$\begin{aligned} x^r f^{(r)}(x) &= (-1)^r x^r \sum_{n=1}^{\infty} a_n \lambda_n^r e^{-\lambda_n x} \\ &= O\left(x^r \sum_{n=1}^{\infty} \lambda_n^r e^{-\lambda_n x}\right) \\ &= O(1), \end{aligned}$$

---

[2] As remarked by Ananda-Rau in [1], the argument in Landau [4], pp. 13–14, has only to be slightly modified to yield the result of Lemma 2.

by Lemma 3. Hence, by Lemma 1,

$$x^r f^{(r)}(x) = o(1).$$

Since

$$f(x) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n x} = \sum_{n=1}^{\infty} s_n \left(e^{-\lambda_n x} - e^{-\lambda_{n+1} x}\right) = x \sum_{n=1}^{\infty} s_n \int_{\lambda_n}^{\lambda_{n+1}} e^{-xu} du,$$

we have

$$\begin{aligned}(-1)^r x^r f^{(r)}(x) &= x^{r+1} \sum_{n=1}^{\infty} s_n \int_{\lambda_n}^{\lambda_{n+1}} u^r e^{-xu} du - r x^r \sum_{n=1}^{\infty} s_n \int_{\lambda_n}^{\lambda_{n+1}} u^{r-1} e^{-xu} du \\ &= V_r - r V_{r-1}, \text{ say.}\end{aligned}$$

Hence, by Lemma 1,

$$\begin{aligned}V_r &= r V_{r-1} + o(1) \\ &= r(r-1) V_{r-2} + o(r) + o(1) \\ &= r(r-1)(r-2) V_{r-3} + o(r(r-1)) + o(r) + o(1) \\ &= \dots. \\ &= r! f(x) + o(r(r-1)\dots 2) + \dots + o(1),\end{aligned}$$

so that[3]

$$\begin{aligned}\frac{V_r}{r!} &= f(x) + o\left(1 + \frac{1}{2!} + \frac{1}{3!} + \dots + \frac{1}{r!}\right) \\ &= f(x) + o(1) \\ &= o(1). \end{aligned} \tag{3.1}$$

This can be explicitly written as

$$\frac{1}{r!}|F(x)| = \frac{1}{r!}\left| x^{r+1} \sum s_n \int_{\lambda_n}^{\lambda_{n+1}} t^r e^{-xt} dt \right| \to 0 \quad (r = 1, 2, \dots) \tag{3.2}$$

as $x \to 0$. If s_n does not converge to zero, there exists a positive constant h such that $|s_n| > h$ for an infinite number of values of n. Let m be any one of these values. We shall show that, when r exceeds a sufficiently large positive integer r_0,

$$\overline{\lim}_{x \to 0} \frac{1}{r!} |F(x)| \geq \delta > 0,$$

where δ is a positive constant. This will contradict (3.2), and hence we will conclude that $\sum a_n$ converges to zero, which is required to be proved.

---

[3] In Hardy and Littlewood [3], (2.41) should be replaced by our (3.1) $\frac{V_r}{r!} = o(1)$; line 4 from the top on p. 225 should be replaced by: $r! \sum_{n=0}^{\infty} s_n w_n = V_r$ so that $\sum_{n=0}^{\infty} s_n w_n = \frac{V_r}{r!} = o(1)$. For similar alterations needed in the papers [1], [5] and [8], see Pati [6].

Now, by the hypotheses of Theorem E and Lemma 2, $s_n = O(1)$ and hence

$$\begin{aligned} |F(x)| &\geq |s_m| x^{r+1} \sum_{n=1}^{\infty} \int_{\lambda_n}^{\lambda_{n+1}} t^r \mathrm{e}^{-xt} \mathrm{d}t - \left| x^{r+1} \sum_{n=1}^{\infty} (s_n - s_m) \int_{\lambda_n}^{\lambda_{n+1}} t^r \mathrm{e}^{-xt} \mathrm{d}t \right| \\ &> |s_m| x^{r+1} \sum_{n=1}^{\infty} \int_{\lambda_n}^{\lambda_{n+1}} t^r \mathrm{e}^{-xt} \mathrm{d}t - K x^{r+1} \sum_{m+1}^{\infty} \int_{\lambda_n}^{\lambda_{n+1}} t^r \mathrm{e}^{-xt} \mathrm{d}t \\ &\qquad - K x^{r+1} \sum_{n=1}^{m-1} \int_{\lambda_n}^{\lambda_{n+1}} t^r \mathrm{e}^{-xt} \mathrm{d}t, \end{aligned} \tag{3.3}$$

where K is a positive constant. We choose

$$x = \frac{2r}{\lambda_{m+1} + \lambda_m}.$$

Then, for fixed r, $x \to 0$ iff $m \to \infty$. Since ([5], p. 440)

$$\lim_{x \to 0} x^{r+1} \sum_{n=1}^{\infty} \int_{\lambda_n}^{\lambda_{n+1}} t^r \mathrm{e}^{-xt} \mathrm{d}t = r!,$$

$$\frac{1}{r!} \overline{\lim}_{x \to 0} |s_m| x^{r+1} \sum_{n=1}^{\infty} \int_{\lambda_n}^{\lambda_{n+1}} t^r \mathrm{e}^{-xt} \mathrm{d}t \geq h. \tag{3.4}$$

We now use the transformation $u = xt$, so that, for $t = \lambda_{m+1}$,

$$u = r \frac{2\lambda_{m+1}}{\lambda_{m+1} + \lambda_m} = r(1 + \eta),$$

where

$$\eta = \frac{\lambda_{m+1} - \lambda_m}{\lambda_{m+1} + \lambda_m}. \tag{3.5}$$

Thus the second term in (3.3) gives

$$\overline{\lim}_{x \to 0} x^{r+1} \sum_{m+1}^{\infty} \int_{\lambda_n}^{\lambda_{n+1}} t^r \mathrm{e}^{-xt} \mathrm{d}t \leq \int_{r(1+\eta)}^{\infty} u^r \mathrm{e}^{-u} \mathrm{d}u. \tag{3.6}$$

The third term in (3.3) gives

$$\overline{\lim}_{x \to 0} x^{r+1} \sum_{1}^{m-1} \int_{\lambda_n}^{\lambda_{n+1}} t^r \mathrm{e}^{-xt} \mathrm{d}t \leq \int_{0}^{r(1-\eta)} u^r \mathrm{e}^{-u} \mathrm{d}u, \tag{3.7}$$

where η is as defined in (3.5).

Combining (3.4), (3.6) and (3.7) we have

$$\frac{1}{r!} \overline{\lim}_{x \to 0} |F(x)| \geq h - \frac{K}{r!} \left[\int_0^{r(1-\eta)} u^r \mathrm{e}^{-u} \mathrm{d}u + \int_{r(1+\eta)}^{\infty} u^r \mathrm{e}^{-u} \mathrm{d}u \right]. \tag{3.8}$$

We show below that

$$I_1 \equiv \int_0^{r(1-\eta)} u^r \mathrm{e}^{-u} \mathrm{d}u < K_1 r^r \mathrm{e}^{-r} \tag{3.9}$$

and

$$I_2 \equiv \int_{r(1+\eta)}^{\infty} u^r \mathrm{e}^{-u} \mathrm{d}u < K_2 r^r \mathrm{e}^{-r},$$

where the K in each inequality denotes a positive constant, independent of r.

Proof of (3.9). We have

$$I_1 = -u^r \mathrm{e}^{-u}\big]_0^{r(1-\eta)} + r \int_0^{r(1-\eta)} u^{r-1} \mathrm{e}^{-u} \mathrm{d}u.$$

Hence

$$\int_0^{r(1-\eta)} \left(\frac{r}{u} - 1\right) u^r \mathrm{e}^{-u} \mathrm{d}u = r^r (1-\eta)^r \mathrm{e}^{-r(1-\eta)}.$$

Now, since $0 < \eta < 1$, $u \leq r(1-\eta)$ implies:

$$\frac{r}{u} - 1 \geq \frac{\eta}{1-\eta},$$

so that

$$\frac{\eta}{1-\eta} I_1 \leq r^r \mathrm{e}^{-r} \left[(1-\eta)\mathrm{e}^{\eta}\right]^r < r^r \mathrm{e}^{-r},$$

since

$$\mathrm{e}^{\eta} < 1 + \eta + \eta^2 + \cdots = \frac{1}{1-\eta}.$$

Thus

$$I_1 < K_1 r^r \mathrm{e}^{-r},$$

where

$$K_1 = \frac{1-\eta}{\eta} = \frac{2\lambda_m}{\lambda_{m+1} - \lambda_m} \leq \frac{2}{\theta - 1} \quad (\theta > 1).$$

Proof of (3.10). We have

$$I_2 = -u^r \mathrm{e}^{-u}\big]_{r(1+\eta)}^{\infty} + r \int_{r(1+\eta)}^{\infty} u^{r-1} \mathrm{e}^{-u} \mathrm{d}u.$$

Hence

$$\begin{aligned} \int_{r(1+\eta)}^{\infty} \left(1 - \frac{r}{u}\right) u^r \mathrm{e}^{-u} \mathrm{d}u &= r^r \mathrm{e}^{-r} \left(\frac{1+\eta}{\mathrm{e}^{\eta}}\right)^r \\ &< r^r \mathrm{e}^{-r}; \end{aligned}$$

since

$$u \geq r(1+\eta) \quad \text{implies:} \quad 1 - \frac{r}{u} \geq \frac{\eta}{1+\eta},$$

we have

so that

$$I_2 < K_2 r^r \mathrm{e}^{-r},$$

where

$$K_2 = \frac{1+\eta}{\eta} = \frac{2\lambda_{m+1}}{\lambda_{m+1}-\lambda_m} = \frac{2}{1-\frac{\lambda_m}{\lambda_{m+1}}} \leq \frac{2\theta}{\theta-1} \quad (\theta > 1).$$

Hence, from (3.8), (3.9) and (3.10), we have

$$\overline{\lim}_{x\to 0} \frac{1}{r!}|F(x)| \geq h - 2K\frac{\theta+1}{\theta-1}\frac{r^r \mathrm{e}^{-r}}{r!}.$$

Since by Stirling's theorem,

$$\frac{r^r \mathrm{e}^{-r}}{r!} \sim \frac{1}{\sqrt{2\pi}} r^{-\frac{1}{2}},$$

taking $r > r_0$, a sufficiently large positive integer, we have

$$\overline{\lim}_{x\to 0} \frac{1}{r!}|F(x)| \geq \delta > 0,$$

which contradicts (3.2). Hence our assumption that $\{s_n\}$ does not converge to 0 is false.

This completes the proof of Theorem E.

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Proximinal subspaces of finite codimension in direct sum spaces

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Abstract. We give a necessary and sufficient condition for proximinality of a closed subspace of finite codimension in c_0-direct sum of Banach spaces.

Keywords. Proximinality and strong proximinality.

0. Notation and preliminaries

Let X be a normed linear space and A be a closed subset of X. We say A is proximinal in X if for each $x \in X$ there exists an element $a \in A$ such that $\|x - a\| = d(x, A)$.

We say A is strongly proximinal in X if A is proximinal in X and given $\epsilon > 0$, there exists $\delta > 0$ such that

$$a \in A,\ \|x - a\| < d(x, A) + \delta \Rightarrow d(a, P_A(x)) < \epsilon,$$

where $P_A(x) = \{a \in A : \|x - a\| = d(x, A)\}$.

Proximinal subspaces of finite codimension have been studied by various authors (see [1–4, 7–10]). In this paper we obtain a necessary and sufficient condition for proximinality of subspaces of finite codimension in c_0-direct sum of Banach spaces in terms of the proximinality of the corresponding subspaces of finite codimension of the coordinate spaces. We also give an example to show that similar result does not hold in l_1-direct sum of Banach spaces.

Let X be a real normed linear space and X^* its dual. The closed unit ball of X is denoted by B_X and the unit sphere by S_X. Let Y be a closed, linear subspace of codimension n in X. For a set $f_1, f_2, \ldots, f_n$ of linear functionals in the annihilator space $Y^\perp$, we give the following definitions from [4]. We have modified the notation used in [4].

$$J_X(f_1) = \{x \in B_X : f_1(x) = \|f_1\|\}$$
$$J_X(f_1, f_2, \ldots, f_i) = \{J_X(f_1, \ldots, f_{i-1}) : f_i(x) = \sup_{x \in J_X(f_1, \ldots, f_{i-1})} f_i(x)\}$$

for $i = 2, 3, \ldots, n$.

Similarly we set

$$J_{(Y^\perp)^*}(f_1) = \{\Phi \in B_{(Y^\perp)^*} : \Phi(f_1) = \|f_1\|\}$$
$$J_{(Y^\perp)^*}(f_1, f_2, \ldots, f_i) = \{\Phi \in J_{(Y^\perp)^*}(f_1, f_2, \ldots, f_{i-1}) : \Phi(f_i) = \max_{\psi \in J_{(Y^\perp)^*}(f_1, f_2, \ldots, f_{i-1})} \psi(f_i)\}$$

for $i = 2, 3 \dots n$. Since $Y^{\perp}$ is finite dimensional, the above n sets are nonempty. We als set

$$M(f_1) = \|f_1\| = N(f_1)$$
$$M(f_1, \dots, f_i) = \sup\{f_i(x) : x \in J_X(f_1, \dots, f_{i-1})\}$$

and

$$N(f_1, \dots, f_i) = \max\{\Phi(f_i) : \Phi \in J_{(Y^{\perp})^*}(f_1, \dots, f_{i-1})\}.$$

We also need the following Theorem from [4].

Theorem A. *Let X be a normed linear space and Y be a subspace of codimension n in* *Then Y is proximinal in X if and only if for every basis $f_1, \dots, f_n$ of $Y^{\perp}$ we have*

1. $J_X(f_1, \dots, f_i) \neq \emptyset$ *for* $1 \leq i \leq n$.
2. $M(f_1, \dots, f_i) = N(f_1, \dots, f_i)$ *for* $1 \leq i \leq n$.

We shall first show that condition 2 of the above theorem can be reformulated wi conditions only involving the normed linear space X. This is easily done using the weal density of B_X in $B_{X^{**}}$ and in a manner similar to that of Vlasov [10]. For this purpose make the following definitions for $\epsilon > 0$ and any finite subset $f_1, \dots, f_n$ of $Y^{\perp}$.

$$\tilde{N}(f_1) = \|f_1\|,$$
$$J_X(f_1, \epsilon) = \{x \in B_X : f_1(x) > \|f_1\| - \epsilon\},$$
$$\tilde{N}(f_1, \dots, f_i, \epsilon) = \sup\{f_i(x) : x \in J_X(f_1, \dots, f_{i-1}, \epsilon)\},$$
$$\tilde{N}(f_1, \dots, f_i) = \inf\nolimits_{\epsilon > 0} \tilde{N}(f_1, \dots, f_i, \epsilon)$$

and

$$J_X(f_1, \dots, f_i, \epsilon) = \{x \in J_X(f_1, \dots, f_{i-1}, \epsilon) : f_i(x) > \tilde{N}(f_1, \dots, f_i) -$$

1. Proximinality of subspaces of finite codimension

We begin with the following proposition.

PROPOSITION 1.1

Let X be a normed linear space and Y be a closed subspace of codimension n. Then *every finite subset $f_1, \dots, f_n$ of $Y^{\perp}$ we have*

$$\tilde{N}(f_1, \dots, f_i) = N(f_1, \dots, f_i) \text{ for } 1 \leq i \leq n.$$

Proof. By induction. The case $i = 1$ is trivial. Assume

$$\tilde{N}(f_1, \dots, f_k) = N(f_1, \dots, f_k) \text{ for } 1 \leq k \leq i - 1.$$

Select any $\Phi \in J_{(Y^{\perp})^*}(f_1, f_2 \dots f_i)$. Since B_X is weak* dense in $B_{X^{**}}$, there exists a (x_α) in B_X that weak* converges to Φ. In particular,

$$\lim_{\alpha} f_k(x_\alpha) = \Phi(f_k) \text{ for } 1 \leq k \leq n.$$

Thus, given $\epsilon > 0$, $\exists\, \alpha_0$ such that

$$f_k(x_\alpha) > \tilde{N}(f_1, \ldots, f_k) - \epsilon \ \forall\, \alpha \geq \alpha_0 \text{ and } 1 \leq k \leq n.$$

This together with the induction hypothesis implies that

$$(x_\alpha) \in J_X(f_1, \ldots, f_{i-1}, \epsilon) \ \forall\ \alpha \geq \alpha_0.$$

and so

$$\tilde{N}(f_1, \ldots, f_i) \geq N(f_1, \ldots, f_i).$$

To prove the other inequality, for each positive integer n, select an element (x_n) in $J_X\ (f_1, \ldots, f_i, \frac{1}{n})$. Then

$$\tilde{N}(f_1, \ldots, f_k, \frac{1}{n}) > f_k(x_n) > \tilde{N}(f_1, \ldots, f_k) - \frac{1}{n} \text{ for } 1 \leq k \leq i.$$

Let $\psi_n = x_n | Y^\perp$ for each n. Then ψ_n is in $B_{(Y^\perp)^*}$. Since $Y^\perp$ is finite dimensional, w.l.o.g we assume (ψ_n) converges to ψ in $B_{(Y^\perp)^*}$. Note for $1 \leq k \leq i$,

$$\begin{aligned} f_k(\psi) &= \lim_{n\to\infty} f_k(\psi_n) \\ &= \lim_{n\to\infty} f_k(x_n) \\ &\geq \lim_{n\to\infty} \tilde{N}(f_1, \ldots, f_k) - \frac{1}{n} \\ &= \tilde{N}(f_1, \ldots, f_k) \end{aligned}$$

Again by induction hypothesis $\psi \in J_{(Y^\perp)^*}(f_1, f_2, \ldots, f_{i-1})$ and

$$N(f_1, \ldots, f_i) \geq f_i(\psi) \geq \tilde{N}(f_1, \ldots, f_i).$$

Hence

$$N(f_1, \ldots, f_i) = \tilde{N}(f_1, \ldots, f_i)$$

and this completes the induction and the proof.

The above Proposition, along with theorem A, implies as follows:

COROLLARY 1.2

Let Y be a closed subspace of finite codimension n in a normed linear space X. Then Y is proximinal in X if and only if for every basis $f_1, \ldots, f_n$ *of* $Y^\perp$ *the sets*

$$\bigcap_{i=1}^{j} \{x \in B_X : f_i(x) = \tilde{N}(f_1, \ldots, f_i)\} \neq \emptyset \ for\ 1 \leq j \leq n.$$

Remark 1.3. If $f_1, \ldots, f_n$ is a finite subset of X^* and $f_{n_1}, \ldots, f_{n_k}$ is a maximal linearly independent subset of $f_1, \ldots, f_n$ satisfying $n_1 < n_2 < \ldots < n_k$ then $\bigcap_{i=1}^{j} \{x \in B_X : f_i(x) = \tilde{N}(f_1, \ldots, f_i)\} \neq \emptyset$ for $1 \leq j \leq n$ if and only if $\bigcap_{i=1}^{m} \{x \in B_X : f_{n_i}(x) = \tilde{N}(f_{n_1}, \ldots, f_{n_i})\} \neq \emptyset$ for $1 \leq m \leq k$.

We now recall some known proximinality results that are needed in the sequel. For an normed linear space X, let $NA(X)$ denote the set of norm attaining elements of X^*. Garkav [1] has characterized proximinal subspaces of finite codimension in general normed linea spaces and the following is an easy corollary of his result.

Lemma B [5]. *Let X be a normed linear space and Y be a closed subspace of finit codimension in X. Then Y is proximinal in X if and only if every closed subspace $Z \supseteq$ is proximinal in X.*

Now, if $f \in X^*$ and H is the kernel of f, it is well-known that the hyperplane H i proximinal in X if and only if $f \in NA(X)$. Thus from Lemma B we have the following

Remark 1.4. If Y is a proximinal subspace of finite codimension in X, then $Y^{\perp} \subseteq NA(X$

However $Y^{\perp} \subseteq NA$ is only a necessary but not a sufficient condition for proximinalit of a subspace Y of finite codimension. (See the example of Phelps in [7], p. 309.) B the behaviour of the space $c_0(\Gamma)$ in the above respect is rather special.The following fa is well known, see for instance [6].

Lemma C. Let Y be a closed subspace of finite codimension in $c_0(\Gamma)$ and $f_1, \ldots, f_n$ be basis of $Y^{\perp}$. Then Y is proximinal if and only if $f_i \in NA(c_0(\Gamma))$ for $1 \leq i \leq n$.

Finally we quote a characterization of strongly proximinal subspaces of finite codime sion from [3], which is needed in the proof of our main result.

Theorem B. *Let X be a normed linear space and Y be a proximinal subspace of codime sion n in X. Then Y is strongly proximinal in X if and only if the following hold for eve basis $f_1, \ldots, f_n$ of $Y^{\perp}$.*

Given $\epsilon > 0$ there exists $\delta > 0$ such that for each i, $1 \leq i \leq n$ and for each x $J_X(f_1, \ldots, f_i, \delta)$, we have $d(x, J_X(f_1, \ldots, f_i)) < \epsilon$.

2. Direct sum spaces

We now consider proximinality in c_0-direct sum spaces. Let Λ be an index set and be a Banach space for each $\lambda \in \Lambda$. Let $X = \oplus_{c_0} X_\lambda$. Then $X^* = \oplus_{l_1} X_\lambda^*$. Furt $F = (f_\lambda)_{\lambda \in \Lambda}$ is in $NA(X)$ if and only if $f_\lambda = 0$ but for finite number of indices a $f_\lambda \in NA(X_\lambda)$ whenever $f_\lambda \neq 0$. Also, in this case

$$J_X(F) = \{(x_\lambda) \in X : \|x_\lambda\| = 1 \text{ and } f_\lambda(x_\lambda) = \|f_\lambda\| \, \forall \, \lambda \in \Lambda\} \neq \emptyset.$$

For X defined as above, we have the following Proposition.

PROPOSITION 2.1

Let $F_i \in X^*$ and $F_i = (f_{i\lambda})$ for $1 \leq i \leq n$. Assume further that for each i, $1 \leq i$ n, $f_{i\lambda} = 0$ but for finite indices λ. Then

$$\tilde{N}(F_1, \ldots, F_i) = \sum_{\lambda \in \Lambda} \tilde{N}(f_{1\lambda}, \ldots, f_{i\lambda}) \text{ for } 1 \leq i \leq n.$$

Remark. Observe that the above sum has only finite number of nonzero entries. Also, condition of the above proposition is satisfied if $F_i \in NA(X)$ for $1 \leq i \leq n$.

Proof of the Proposition. Let

$$A = \cup_{i=1}^{n} \{\lambda \in \Lambda : \ f_{i,\lambda} \neq 0\}. \tag{2}$$

Then card $A \ = \ l < \infty$. Set

$$\sigma(\epsilon) \ = \ \max_{\lambda \in \Lambda} \max_{1 \leq i \leq n} [\tilde{N}(f_{1\lambda}, \dots, f_{i\lambda}, \epsilon) - \tilde{N}(f_{1\lambda}, \dots, f_{i\lambda})].$$

For $\epsilon > 0$, let

$$\epsilon_1 \ = \ \epsilon, \ \epsilon_i \ = \ l\sigma(\epsilon_{i-1}) \ + \ \epsilon_{i-1} \text{ for } 2 \leq i \leq n. \tag{3}$$

Then $\epsilon_1 \leq \epsilon_2 \leq \ldots \leq \epsilon_n$. Clearly, $\sigma(\epsilon)$ and therefore ϵ_i, $1 \leq i \leq n$ tend to 0 as $\epsilon \to 0$. Further,

$$\tilde{N}(F_1) \ = \ \|F_1\| \ = \ \sum_{\lambda} \|f_{1\lambda}\| \ = \ \sum_{\lambda} \tilde{N}(f_{1\lambda}).$$

and

$$\begin{aligned} x \in J_X(F_1, \epsilon) \quad &\Leftrightarrow \quad \{x = (x_\lambda) \in B_X : F_1(x) \ > \ \|F_1\| - \epsilon\} && (4) \\ &= \quad \{x = (x_\lambda) \in B_X : \sum_{\lambda} f_{1\lambda}(x_\lambda) \ > \ \Big(\sum_{\lambda} \tilde{N}(f_{1\lambda})\Big) - \epsilon. && (5) \end{aligned}$$

Using (4) and (5) and the fact that $f_{1\lambda}(x_\lambda) \leq \tilde{N}(f_{1\lambda})$ we get

$$\begin{aligned} \textstyle\prod_{\lambda \in \Lambda} J_{X_\lambda}(f_{1\lambda}, \epsilon_1) \quad &= \quad \textstyle\prod_{\lambda \in \Lambda} J_{X_\lambda}(f_{1\lambda}, \epsilon) \\ &\supseteq \quad J_X(F_1, \epsilon) \\ &\supseteq \quad \{x = (x_\lambda) \in B_X : \textstyle\sum_\lambda f_{1\lambda}(x_\lambda) > \\ &\qquad \Big(\textstyle\sum_\lambda \tilde{N}(f_{1\lambda})\Big) - \frac{\epsilon}{l} \ \forall \ \lambda \in \Lambda\} \\ &= \quad \textstyle\prod_{\lambda \in \Lambda} J_{X_\lambda}(f_{1\lambda}, \frac{\epsilon}{l}). \end{aligned}$$

Inductively assume that for some i, $2 \leq i \leq n$ we have

$$\tilde{N}(F_1, \dots, F_k) \ = \ \sum_{\lambda \in \Lambda} \tilde{N}(f_{1\lambda}, \dots, f_{k\lambda}) \text{ for } 1 \leq k \leq i-1$$

and

$$\prod_{\lambda \in \Lambda} J_{X_\lambda}(f_{1\lambda}, \dots, f_{i-1\lambda}, \frac{\epsilon}{l}) \subseteq J_X(F_1, \dots, F_{i-1}, \epsilon) \subseteq \prod_{\lambda \in \Lambda} J_{X_\lambda}(f_{1\lambda}, \dots, f_{i-1\lambda}, \epsilon_{i-1}).$$

Now, observing that the summation below, over λ, involves only finite number of nonzero terms, we have

$$\begin{aligned} \tilde{N}(F_1, \dots, F_i) \quad &= \quad \inf_{\epsilon > 0} \sup\{F_i(x) : x \in J_X(F_1, \dots, F_{i-1}, \epsilon)\} \\ &\leq \quad \inf_{\epsilon > 0} \sup\left\{\textstyle\sum_\lambda f_{i\lambda}(x_\lambda) : J_{X_\lambda}(f_{1\lambda}, \dots, f_{i-1\lambda}, \epsilon_{i-1}) \ \forall \lambda \in \Lambda\right\} \\ &= \quad \inf_{\epsilon > 0} \textstyle\sum_\lambda \sup\{f_{i\lambda}(x_\lambda) : J_{X_\lambda}(f_{1\lambda}, \dots, f_{i-1\lambda}, \epsilon_{i-1}) \ \forall \lambda \in \Lambda\} \\ &= \quad \inf_{\epsilon > 0} \textstyle\sum_\lambda \ \tilde{N}(f_{1\lambda}, \dots, f_{i-1\lambda}, \epsilon_{i-1}) \\ &= \quad \textstyle\sum_\lambda \inf_{\epsilon > 0} \ \tilde{N}(f_{1\lambda}, \dots, f_{i-1\lambda}, \epsilon_{i-1}) \\ &= \quad \textstyle\sum_\lambda \ \tilde{N}(f_{1\lambda}, \dots, f_{i\lambda}) \text{ for } \epsilon_{i-1} \to 0 \text{ as } \epsilon \to 0. \end{aligned}$$

Similarly using the other inclusion we conclude

$$\tilde{N}(F_1, \ldots, F_i) = \sum_{\lambda \in \Lambda} \tilde{N}(f_{1\lambda}, \ldots, f_{i\lambda}).$$

Hence

$$\begin{aligned} D_i &= \{x = (x_\lambda) \in \textstyle\prod_\lambda J_{X_\lambda}(f_{1\lambda}, \ldots, f_{i-1\lambda}, \epsilon_{i-1}) : \\ &\qquad \textstyle\sum_\lambda f_{i\lambda}(x_\lambda) > \sum_{\lambda \in \Lambda} \tilde{N}(f_{1\lambda}, \ldots, f_{i\lambda}) - \epsilon\} \\ &\supseteq \{x = (x_\lambda) \in J_X(F_1, \ldots, F_{i-1}, \epsilon) : \\ &\qquad \textstyle\sum_\lambda f_{i\lambda}(x_\lambda) > \sum_{\lambda \in \Lambda} \tilde{N}(f_{1\lambda}, \ldots, f_{i\lambda}) - \epsilon\} \\ &= \{x = (x_\lambda) \in J_X(F_1, \ldots, F_{i-1}, \epsilon) : F_i(x) > \tilde{N}(F_1, \ldots, F_i) - \epsilon\} \\ &= J_X(F_1, \ldots, F_i, \epsilon). \end{aligned}$$

We have

$$\begin{aligned} x = (x_\lambda) \in D_i \Rightarrow f_{i\lambda}(x_\lambda) &\leq \tilde{N}(f_{1\lambda}, \ldots, f_{i\lambda}, \epsilon_{i-1}) \\ &\leq \tilde{N}(f_{1\lambda}, \ldots, f_{i\lambda}) + \sigma(\epsilon_{i-1}) \ \forall\, \lambda. \end{aligned} \tag{6}$$

Further

$$x = (x_\lambda) \in D_i \Rightarrow \sum_\lambda f_{i\lambda}(x_\lambda) > \textstyle\sum_\lambda \tilde{N}(f_{1\lambda}, \ldots, f_{i\lambda}) - \epsilon. \tag{7}$$

Let $x = (x_\lambda) \in D_i$. If for some λ_0, $f_{i\lambda_0}(x_{\lambda_0}) \leq \tilde{N}(f_{1\lambda_0}, \ldots, f_{i\lambda_0}) - \epsilon_i$ then using (6),

$$\begin{aligned} \textstyle\sum_\lambda f_{i\lambda}(x_\lambda) &\leq \tilde{N}(f_{1\lambda_0}, \ldots, f_{i\lambda_0}) - \epsilon_i \\ &\quad + \sum_{\substack{\lambda \in A \\ \lambda \neq \lambda_0}} \tilde{N}(f_{1\lambda}, \ldots, f_{i\lambda}) + (l-1)\sigma(\epsilon_{i-1}) \\ &\leq \textstyle\sum_\lambda \tilde{N}(f_{1\lambda}, \ldots, f_{i\lambda}) + (l-1)\sigma(\epsilon_{i-1}) \\ &\quad -(l\sigma(\epsilon_{i-1}) + \epsilon_{i-1}) \\ &= \textstyle\sum_\lambda \tilde{N}(f_{1\lambda}, \ldots, f_{i\lambda}) - \sigma(\epsilon_{i-1}) - \epsilon_{i-1} \end{aligned}$$

which contradicts (7) as $\sigma(\epsilon_{i-1}) + \epsilon_{i-1} > \epsilon$. Thus $x = (x_\lambda) \in D_i$ implies

$$f_{i\lambda}(x_\lambda) > \tilde{N}(f_{1\lambda}, \ldots, f_{i\lambda}) - \epsilon_i \ \ \forall \lambda. \tag{8}$$

Now $\epsilon_{i-1} \leq \epsilon_i$ and so

$$J_{X_\lambda}(f_{1\lambda}, \ldots, f_{i-1\lambda}, \epsilon_{i-1}) \subseteq J_{X_\lambda}(f_{1\lambda}, \ldots, f_{i-1\lambda}, \epsilon_i) \ \forall\, \lambda. \tag{9}$$

Since $D_i \subseteq \{x = (x_\lambda) \in \prod_\lambda J_{X_\lambda}(f_{1\lambda}, \ldots, f_{i-1\lambda}, \epsilon_{i-1})$, using (8) and (9) we conclude

$$D_i \subseteq \{x = (x_\lambda) \in \prod_{\lambda \in \Lambda} J_{X_\lambda}(f_{1\lambda}, \ldots, f_{i-1\lambda}, \epsilon_i) \ \ \forall\, \lambda\, \}.$$

But $J_X(F_1, \ldots, F_i, \epsilon) \subseteq D_i$ and therefore

$$J_X(F_1, \ldots, F_i, \epsilon) \subseteq \{x = (x_\lambda) \in \prod_{\lambda \in \Lambda} J_{X_\lambda}(f_{1\lambda}, \ldots, f_{i\lambda}, \epsilon_i)\}. \tag{10}$$

On the other hand

$$\begin{aligned} J_X(F_1, \ldots, F_i, \epsilon) &= \{x = (x_\lambda) \in J_X(F_1, \ldots, F_{i-1}, \epsilon) : \\ &\qquad F_i(x) > \tilde{N}(F_1, \ldots, F_i) - \epsilon\} \\ &= \{x = (x_\lambda) \in J_X(F_1, \ldots, F_{i-1}, \epsilon) : \\ &\qquad \textstyle\sum_\lambda f_{i\lambda}(x_\lambda) > \sum_\lambda \tilde{N}(f_{1\lambda}, \ldots, f_{i\lambda}) - \epsilon\} \\ &\supseteq \{x = (x_\lambda) \in \textstyle\prod_{\lambda \in \Lambda} J_{X_\lambda}(f_{1\lambda}, \ldots, f_{i-1\lambda}, \frac{\epsilon}{l}) : \\ &\qquad \textstyle\sum_\lambda f_{i\lambda}(x_\lambda) > \sum_\lambda \tilde{N}(f_{1\lambda}, \ldots, f_{i\lambda}) - \epsilon\} \\ &\supseteq \{x = (x_\lambda) \in \textstyle\prod_{\lambda \in \Lambda} J_{X_\lambda}(f_{1\lambda}, \ldots, f_{i-1\lambda}, \frac{\epsilon}{l}) : \\ &\qquad f_{i\lambda}(x_\lambda) > \tilde{N}(f_{1\lambda}, \ldots, f_{i\lambda}) - \frac{\epsilon}{l} \ \forall \ \lambda\} \\ &= \textstyle\prod_{\lambda \in \Lambda} J_{X_\lambda}(f_{1\lambda}, \ldots, f_{i\lambda}, \frac{\epsilon}{l}). \end{aligned}$$

This completes the induction and we have

$$J_X(F_1, \ldots, F_i, \epsilon) \supseteq \prod_{\lambda \in \Lambda} J_{X_\lambda}(f_{1\lambda}, \ldots, f_{i\lambda}, \frac{\epsilon}{l})$$

for $1 \leq i \leq n$. This completes the proof of the proposition.We now prove our main result.

Theorem 2.2. *Let X_λ be a normed linear space for each λ in an index set Λ and X be the c_0-direct sum of the spaces X_λ for $\lambda \in \Lambda$. Let Y be a closed subspace of finite codimension n in X. Then Y is (strongly) proximinal in X if and only if the following two conditions hold for every basis $\{F_i : 1 \leq i \leq n\}$ of $Y^\perp$, where $F_i = (f_{i\lambda})_{\lambda \in \Lambda}$, for $1 \leq i \leq n$.*

1. *For each i, $1 \leq i \leq n$, $f_{i\lambda}$ is nonzero only for finite number of indices λ.*
2. *$Y_\lambda = \cap\{\mathrm{Ker}\ f_{i\lambda} : 1 \leq i \leq n\}$ is (strongly) proximinal in X_λ for each $\lambda \in \Lambda$.*

Proof. Necessity. First we observe that by remark 1.4, $F_i \in NA(X)$ for any basis $\{F_i : 1 \leq i \leq n\}$ of $Y^\perp$. This, in particular, implies condition 1 above. Hence Y_λ is a proper subspace of X_λ only for finite number of indices λ.

To prove 2, for each λ such that Y_λ is a proper subspace of X_λ, choose any basis $g_{1\lambda}, \ldots, g_{n\lambda}$ of $(Y_\lambda)^\perp$. If $G_i = (g_{i\lambda})$ for $1 \leq i \leq n$ then $G_1, \ldots, G_n$ is a basis of $Y^\perp$. Since Y is proximinal in X,we can, by Corollary 1, get an element $x = (x_\lambda) \in X$ satisfying $G_i(x) = \tilde{N}(G_1, \ldots, G_i)$ for $1 \leq i \leq n$. In particular,

$$G_1(x) = \sum_\lambda g_{1\lambda}(x_\lambda) = \|G_1\| = \sum_\lambda \|g_{1\lambda}\|.$$

We have $\|x\| = \| \sup_\lambda \|x_\lambda\| \leq 1$ and so the above inequality implies $g_{1\lambda}(x_\lambda) = \|g_{1\lambda}\|$ for all λ. Assume inductively,

$$g_{k\lambda}(x_\lambda) = \tilde{N}(g_{1\lambda}, \ldots, g_{k\lambda}) \text{ for } 1 \leq k \leq i-1 \text{ and } \forall \ \lambda.$$

Now again by Remark 1.4, $G_i \in NA(X)$ for $1 \leq i \leq n$. Hence by Proposition 2.
have,

$$G_i(x) = \sum_{\lambda} g_{i\lambda}(x_\lambda) = \tilde{N}(G_1, \ldots, G_i) = \sum_{\lambda} \tilde{N}(g_{1\lambda}, \ldots, g_{i\lambda}).$$

Also by induction hypothesis, $x_\lambda \in \tilde{N}(g_{1\lambda}, \ldots, g_{i-1\lambda}, \epsilon)$ for every $\epsilon > 0$ and so we

$$g_{i\lambda}(x_\lambda) \;\leq\; \tilde{N}(g_{1\lambda}, \ldots, g_{i\lambda}) \;\forall\, \lambda.$$

This with (11) implies

$$g_{i\lambda}(x_\lambda) \;=\; \tilde{N}(g_{1\lambda}, \ldots, g_{i\lambda}) \;\forall\, \lambda.$$

and completes the process of induction. Hence for all λ, $x_\lambda \in X_\lambda$ satisfies

$$g_{i\lambda}(x_\lambda) \;=\; \tilde{N}(g_{1\lambda}, \ldots, g_{i\lambda}) \,\forall\; 1 \leq i \leq n.$$

By Corollary 1.2, Y_λ is proximinal in X_λ for each λ.

If Y is strongly proximinal in X, then given $\epsilon > 0$ there exists $\delta > 0$ such that f
G in $x = (x_\lambda)$ in $J_X(G_1, \ldots, G_i, \delta)$,

$$d(G, J_X(G_1, \ldots, G_i)) \;<\; \epsilon \;\forall\; 1 \;\leq i \;\leq n.$$

It is easy to verify using (11) and (12) that

$$J_X(G_1, \ldots, G_i) \;=\; \prod_{\lambda \in \Lambda} J_{X_\lambda}(g_{1\lambda}, \ldots, g_{i\lambda})$$

and

$$J_X(G_1, \ldots, G_i, \delta) \;\supseteq\; \prod_{\lambda \in \Lambda} J_{X_\lambda}(g_{1\lambda}, \ldots, g_{i\lambda}, \frac{\delta}{l})$$

for $1 \leq i \leq n$. Now using (13) and (14) we conclude that for any λ and h_i in $J_{X_\lambda}(g_1$
$g_{i\lambda}, \frac{\delta}{l})$ we have

$$d(h_i, J_{X_\lambda}(g_{1\lambda}, \ldots, g_{i\lambda})) \;<\; \epsilon$$

for $1 \leq i \leq n$. Hence Y_λ is strongly proximinal in X_λ for each λ.

Sufficiency. If $G_1, \ldots, G_n$ is any basis of $Y^\perp$ and $G_i = (g_{i\lambda})_{\lambda \in \Lambda}$, then by conditio
each i, $1 \leq i \leq n$, $g_{i\lambda} = 0$ except for finite number of indices λ. So, Proposition
be applied to the basis $\{G_i : 1 \leq i \leq n\}$ of $Y^\perp$.

Since Y_λ is proximinal for each λ, by Remark 1.3 and Corollary 1.2, there
$x_\lambda \in B_{(X_\lambda)}$ satisfying for each λ,

$$g_{i\lambda}(x_\lambda) \;=\; \tilde{N}(g_{1\lambda}, \ldots, g_{i\lambda}) \;\forall\, 1 \leq i \leq n.$$

Now let $x = (x_\lambda)_{\lambda \in \Lambda}$. Clearly $x \in B_X$ and Proposition 2.1 implies

$$G_i(x) = \sum_{\lambda} g_{i\lambda}(x_\lambda) = \sum_{\lambda} \tilde{N}(g_{1\lambda}, \ldots, g_{i\lambda}) = \tilde{N}(G_1, \ldots, G_i) \text{ for } 1 \leq$$

The conclusion now follows from Corollary 1.2.

Assume now Y_λ strongly proximinal in X_λ for each λ. Let $\epsilon > 0$ be given. Since e set A given by (2) is finite we can get $\delta > 0$ such that for each $\lambda \in A$ and h_i in $_\lambda(g_{1\lambda}, \ldots, g_{i\lambda}, \delta))$ we have

$$d(h_i, J_{X_\lambda}(g_{1\lambda}, \ldots, g_{i\lambda})) < \epsilon \text{ for } 1 \leq i \leq n. \tag{15}$$

ow choose $\eta > 0$ small enough so that η_i is given by

$$\eta_1 = \eta, \ \eta_i = l\sigma(\eta_{i-1}) + \eta_{i-1} \text{ for } 2 \leq i \leq n$$

in (3), is less than δ. We have from (10)

$$J_X(G_1, \ldots, G_i, \eta) \subseteq \prod_{\lambda \in \Lambda} J_{X_\lambda}(g_{1\lambda}, \ldots, g_{i\lambda}, \eta_i) \subseteq \prod_{\lambda \in \Lambda} J_{X_\lambda}(g_{1\lambda}, \ldots, g_{i\lambda}, \delta). \tag{16}$$

Clearly (14), (15) and (16) imply that if $x = (x_\lambda) \in J_X(G_1, \ldots, G_i, \eta)$ then

$$d(x, J_X(G_1, \ldots, G_i)) < \epsilon$$

d this completes the proof.

We now give an alternate shorter proof for the proximinality of Y when conditions (1) d (2) of Theorem 2.2 are satisfied. This proof avoids the use of Proposition 2.1 and uses emma B.

Let $\{X_j : 1 \leq j \leq l\} = \{X_\lambda : \lambda \in A\}$, where the set A is given by (2). We set

$$G = X_1 \oplus_\infty X_2 \oplus_\infty \ldots \oplus_\infty X_l$$

d

$$Z_j = \cap_{1 \leq i \leq n} \operatorname{Ker} f_{ij}$$

r $1 \leq j \leq l$. We have Z_j to be proximinal subspace of finite codimension in X_j for $\leq j \leq l$, by (2) of Theorem 2.2. Further if

$$Z = Z_1 \oplus_\infty Z_2 \oplus_\infty \ldots \oplus_\infty Z_l,$$

en Z is a proximinal subspace of finite codimension in G. Set

$$Y_1 = \cap_{i=1}^{n} \{(x_1, \ldots, x_l) : x_j \in X_j \ \forall\, 1 \leq j \leq l \text{ and } \sum_{1 \leq j \leq l} f_{ij}(x_j) = 0\}.$$

en Y_1 is a subspace of G, $Z \subseteq Y_1 \subseteq G$. Now we use Lemma B to conclude Y_1 is oximinal in G. It is easily verified that this, in turn, implies proximinality of Y in X.

emark 2.1. It is easy to see that the above proof goes through when X is taken as a finite -direct sum of normed linear spaces and condition (2) of Theorem 2.2 is satisfied. The ample below shows that this is no longer the case when X is an infinite l_1-direct sum.

emark 2.2. We observe here that the necessity of Theorem 2.2 does not hold even for nite l_1-direct sums. For instance, let X be a non-reflexive Banach space and pick f and g the unit sphere of X^* such that there exists $x \in X$ with $\|x\| = 1 = f(x)$ and g does not tain its norm on X. Now $1 = \max\{\|f\|, \|g\|\} = \|(f, g)\| = 1 = f(x)$. Hence (f, g) tains its norm at $(x, 0)$ and $Z = \{(x, y) : f(x) + g(y) = 0\}$ is a proximinal subspace but er(g) is not proximinal in X.

3. Example

Theorem 2.2 is not true if we replace the c_0-direct sum by, for instance, the l_1 direct sum as the following example shows.

Example. Let $X = \oplus_{l_1} X_n$ where $X_n = c_0$ for $n = 1, 2, \ldots$. Then $X^* = \oplus_{l_\infty} X_n^*$. Select for each positive integer n, $f_{in} \in l_1$ with $\|f_{in}\| \leq 1$ and $f_{in} \in NA(c_0)$. Further set

$$\begin{aligned} f_{12} &= f_{21} = 0, \\ \|f_{11}\| &= \|f_{22}\| = 1, \\ f_{1n} &= f_{2n} \text{ for } n \geq 3, \\ \|f_{in}\| &< 1 \text{ for } n \geq 3, i = 1, 2 \end{aligned}$$

and

$$\lim_{n\to\infty} \|f_{in}\| = 1.$$

Define $F_i \in X^*, i = 1, 2$, as

$$F_i = (f_{i1}, f_{i2} \ldots f_{in} \ldots), \quad i = 1, 2.$$

Let $Y = \cap\{\text{Ker } F_i : i = 1, 2\}$ and $Y_n = \cap\{\text{Ker } f_{in} : i = 1, 2\}$ for $n = 1, 2, \ldots$. Since $f_{in} \in NA(C_0)$ for $i = 1, 2$ and for all n, Y_n is proximinal in $X_n = C_0$ for all n.We will now show that Y is not proximinal in X.

Choose $x_i \in C_0, i = 1, 2$ such that $\|x_i\| = 1$ for $i = 1, 2$ and $f_{11}(x_1) = f_{22}(x_2) = 1$. Consider $x = (x_1, x_2, 0, 0, \ldots)$ in X. Then $\|x\| = 2$. Further $F_i \in NA(X)$ as

$$F_1(x) = f_{11}(x) = 1 = F_2(x) = f_{22}(x_2).$$

So, $d(x, Y) = \|x|Y^\perp\| \geq 1$.We now show that $d(x, Y) = 1$.

To see this, select for $n \geq 3$, $x_n \in C_0$ satisfying $f_{in}(x_n) = -1$ for $i = 1, 2$ and $\lim_{n\to\infty} \|x_n\| = 1$. Define a sequence $(y_k)_{k\geq 3} \in X$ by

$$y_k(n) = \begin{cases} x_n & \text{if } n \in \{1, 2, k\} \\ 0 & \text{otherwise.} \end{cases}$$

Then $F_i(y_k) = f_{i1}(x_1) + f_{ik}(x_k) = 0$ for $i = 1, 2$ and so $y_k \in Y$ for all k. Further

$$\|x - y_k\| = \|x_k\| \to 1 \text{ as } k \to \infty.$$

Hence $d(x, Y) = 1$.

We recall that a nearest element to x from Y exists if and only if there exists y in X satisfying

$$F_i(y) = F_i(x) = 1 \text{ for } i = 1, 2 \ \ \|y\| = d(x, Y) = 1.$$

However $\|y\| = 1 = F_1(y)$ implies $y = (y_1, 0, 0 \ldots)$ where $f_{11}(y_1) = 1$, $\|y_1\| = 1$. But, in this case, $F_2(y) = 0 \neq F_2(x)$ and the above equality can not hold. Therefore, Y is not proximinal in X.

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Common fixed points for weakly compatible maps

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Abstract. The purpose of this paper is to prove a common fixed point theorem, from the class of compatible continuous maps to a larger class of maps having weakly compatible maps without appeal to continuity, which generalizes the results of Jungck [3], Fisher [1], Kang and Kim [8], Jachymski [2], and Rhoades [9].

Keywords. Weakly compatible maps; fixed points.

1. Introduction

In 1976, Jungck [4] proved a common fixed point theorem for commuting maps generalizing the Banach's fixed point theorem, which states that, 'let (X, d) be a complete metric space. If T satisfies $d(Tx, Ty) \leq kd(x, y)$ for each $x, y \in X$ where $0 \leq k < 1$, then T has a unique fixed point in X'. This theorem has many applications, but suffers from one drawback – the definition requires that T be continuous throughout X. There then follows a flood of papers involving contractive definition that do not require the continuity of T. This result was further generalized and extended in various ways by many authors. On the other hand Sessa [11] defined weak commutativity and proved common fixed point theorem for weakly commuting maps. Further Jungck [5] introduced more generalized commutativity, the so-called compatibility, which is more general than that of weak commutativity. Since then various fixed point theorems, for compatible mappings satisfying contractive type conditions and assuming continuity of at least one of the mappings, have been obtained by many authors.

It has been known from the paper of Kannan [7] that there exists maps that have a discontinuity in the domain but which have fixed points, moreover, the maps involved in every case were continuous at the fixed point. In 1998, Jungck and Rhoades [6] introduced the notion of weakly compatible and showed that compatible maps are weakly compatible but converse need not be true. In this paper, we prove a fixed point theorem for weakly compatible maps without appeal to continuity, which generalizes the result of Fisher [1], Jachymski [2], Kang and Kim [8] and Rhoades *et al* [9].

2. Preliminaries

DEFINITION 2.1 [6]

A pair of maps A and S is called weakly compatible pair if they commute at coincidence points.

Example 2.1. Let $X = [0, 3]$ be equipped with the usual metric space $d(x, y) = |x - y|$.

Define $f, g : [0,3] \to [0,3]$ by

$$f(x) = \left| \begin{array}{ll} x & \text{if } x \in [0,1) \\ \\ 3 & \text{if } x \in [1,3] \end{array} \right. \quad \text{and} \quad g(x) = \left| \begin{array}{ll} 3-x & \text{if } x \in [0,1) \\ \\ 3 & \text{if } x \in [1,3] \end{array} \right. .$$

Then for any $x \in [1,3]$, $fgx = gfx$, showing that f, g are weakly compatible maps o $[0,3]$.

Example 2.2. Let $X = R$ and define $f, g : R \to R$ by $fx = x/3$, $x \in R$ and $gx =$ x^2, $x \in R$. Here 0 and 1/3 are two coincidence points for the maps f and g. Note tha f and g commute at 0, i.e. $fg(0) = gf(0) = 0$, but $fg(1/3) = f(1/9) = 1/27$ an $gf(1/3) = g(1/9) = 1/81$ and so f and g are not weakly compatible maps on R.

Remark 2.1. Weakly compatible maps need not be compatible. Let $X = [2, 20]$ and be the usual metric on X. Define mappings $B, T : X \to X$ by $Bx = x$ if $x = 2$ o > 5, $Bx = 6$ if $2 < x \leq 5$, $Tx = x$ if $x = 2$, $Tx = 12$ if $2 < x \leq 5$, $Tx = x -$ if $x > 5$. The mappings B and T are non-compatible since sequence $\{x_n\}$ defined b $x_n = 5 + (1/n)$, $n \geq 1$. Then $Tx_n \to 2$, $Bx_n = 2$, $TBx_n = 2$ and $BTx_n = 6$. But the are weakly compatible since they commute at coincidence point at $x = 2$.

3. Fixed point theorem

Let R^+ denote the set of non-negative real numbers and F a family of all mappings ϕ $(R^+)^5 \to R^+$ such that ϕ is upper semi-continuous, non-decreasing in each coordina variable and, for any $t > 0$,

$$\phi(t, t, 0, \alpha t, 0) \leq \beta t, \phi(t, t, 0, 0, \alpha t) \leq \beta t,$$

where $\beta = 1$ for $\alpha = 2$ and $\beta < 1$ for $\alpha < 2$,

$$\gamma(t) = \phi(t, t, \alpha_1 t, \alpha_2 t, \alpha_3 t) < t,$$

where $\gamma : R^+ \to R^+$ is a mapping and $\alpha_1 + \alpha_2 + \alpha_3 = 4$.

Lemma 3.1 [12]. *For every $t > 0$, $\gamma(t) < t$ if and only if* $\lim_{n\to\infty} \gamma^n(t) = 0$, *where denotes the n times composition of γ.*

Let A, B, S and T be mappings from a metric space (X, d) into itself satisfying t following conditions :

$$A(X) \subset T(X) \text{ and } B(X) \subset S(X), \qquad (3.$$

$$d(Ax, By) \leq \phi(d(Sx, Ty), d(Ax, Sx), d(By, Ty), d(Ax, Ty), d(By, Sx)) \qquad (3.$$

for all $x, y \in X$, where $\phi \in F$. Then for arbitrary point x_0 in X, by (3.1), we choose point x_1 such that $Tx_1 = Ax_0$ and for this point x_1, there exists a point x_2 in X such t $Sx_2 = Bx_1$ and so on. Continuing in this manner, we can define a sequence $\{y_n\}$ in X su that

$$y_{2n} = Ax_{2n} = Tx_{2n+1} \text{ and } y_{2n+1} = Bx_{2n+1} = Sx_{2n+2}, n = 0, 1, 2, 3, \ldots . \qquad (3$$

Lemma 3.2 $\lim_{n\to\infty} d(y_n, y_{n+1}) = 0$, *where $\{y_n\}$ is the sequence in X defined by* (3.3).

Proof. Let $d_n = d(y_n, y_{n+1}), n = 0, 1, 2, \ldots$. Now, we shall prove the sequence $\{d_n\}$ is non-increasing in R^+, that is, $d_n \leq d_{n-1}$ for $n = 1, 2, 3, \ldots$. From (3.2), we have

$$\begin{aligned} d(Ax_{2n}, Bx_{2n+1}) \leq \phi(&d(Sx_{2n}, Tx_{2n+1}), d(Ax_{2n}, Sx_{2n}), d(Bx_{2n+1}, Tx_{2n+1}), \\ &d(Ax_{2n}, Tx_{2n+1}), d(Bx_{2n+1}, Sx_{2n})). \\ d(y_{2n}, y_{2n+1}) &\leq \phi(d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n-1}), d(y_{2n+1}, y_{2n}), d(y_{2n}, y_{2n}), \\ &\quad d(y_{2n+1}, y_{2n-1})) \\ &= \phi(d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n-1}), d(y_{2n+1}, y_{2n}), 0, [d(y_{2n+1}, y_{2n}) \\ &\quad + d(y_{2n}, y_{2n-1})]) \\ &= \phi(d_{2n-1}, d_{2n-1}, d_{2n}, 0, d_{2n} + d_{2n-1}). \end{aligned} \tag{3.4}$$

Suppose that $d_{n-1} < d_n$ for some n. Then, for some $\alpha < 2$, $d_{n-1} + d_n = \alpha d_n$. Since ϕ is non-increasing in each variable and $\beta < 1$ for some $\alpha < 2$. From (3.4), we have

$$d_{2n} \leq \phi(d_{2n}, d_{2n}, d_{2n}, 0, \alpha d_{2n}) \leq \beta d_{2n} < d_{2n}.$$

Similarly, we have $d_{2n+1} < d_{2n+1}$. Hence, for every $n, d_n \leq \beta d_n < d_n$, which is a contradiction. Therefore, $\{d_n\}$ is a non-increasing sequence in R^+. Now, again by (3.2), we have

$$\begin{aligned} d_1 = d(y_1, y_2) &= d(Ax_2, Bx_1) \\ &\leq \phi(d(Sx_2, Tx_1), d(Ax_2, Sx_2), d(Bx_1, Tx_1), \\ &\quad d(Ax_2, Tx_1), d(Bx_1, Sx_2)) \\ &= \phi(d(y_1, y_0), d(y_2, y_1), d(y_1, y_0), d(y_2, y_0), d(y_1, y_1)) \\ &= \phi(d_0, d_1, d_0, d_0 + d_1, 0) \\ &\leq \phi(d_0, d_0, d_0, 2d_0, d_0) \\ &= \gamma(d_0). \end{aligned}$$

In general, we have $d_n \leq \gamma^n(d_0)$, which implies that, if $d_0 > 0$, by Lemma 3.1,

$$\lim_{n\to\infty} d_n \leq \lim_{n\to\infty} \gamma^n(d_0) = 0.$$

Therefore, we have $\lim_{n\to\infty} d_n = 0$. For $d_0 = 0$, since $\{d_n\}$ is non-increasing, we have $\lim_{n\to\infty} d_n = 0$. This completes the proof.

Lemma 3.3. *The sequence* $\{y_n\}$ *defined by* (3.3) *is a Cauchy in* X.

Proof. By virtue of Lemma 3.2, it is a Cauchy sequence in X. Suppose that $\{y_{2n}\}$ is not a Cauchy sequence. Then there is an $\epsilon > 0$ such that for each even integer $2k$, there exist even integers $2m(k)$ and $2n(k)$ with $2m(k) > 2n(k) \geq 2k$ such that

$$d(y_{2m(k)}, y_{2n(k)}) > \epsilon. \tag{3.5}$$

For each even integer $2k$, let $2m(k)$ be the least even integer exceeding $2n(k)$ satisfying (3.5), that is,

$$d(y_{2n(k)}, y_{2m(k)-2}) \leq \epsilon \quad \text{and} \quad d(y_{2n(k)}, y_{2m(k)}) > \epsilon. \tag{3.6}$$

Then for each even integer $2k$, we have

$$\begin{aligned} \epsilon &\leq d(y_{2n(k)}, y_{2m(k)}) \\ &\leq d(y_{2n(k)}, y_{2m(k)-2}) + d(y_{2m(k)-2}, y_{2m(k)-1}) + d(y_{2m(k)-1}, y_{2m(k)}). \end{aligned}$$

By Lemma 3.2 and (3.6), it follows that

$$d(y_{2n(k)}, y_{2m(k)}) \to \epsilon \text{ as } k \to \infty. \tag{3.7}$$

By the triangle inequality, we have

$$|d(y_{2n(k)}, y_{2m(k)-1}) - d(y_{2n(k)}, y_{2m(k)})| \leq d(y_{2m(k)-1}, y_{2m(k)})$$

and

$$\begin{aligned}&|d(y_{2n(k)}, y_{2m(k)-1} - d(y_{2n(k)}, y_{2m(k)})|\\ &\leq d(y_{2m(k)-1}, y_{2m(k)}) + d(y_{2n(k)}, y_{2n(k)+1}).\end{aligned}$$

From Lemma 3.2 and eq. (3.7), as $k \to \infty$,

$$d(y_{2n(k)}, y_{2m(k)-1}) \to \epsilon \text{ and } d(y_{2n(k)+1}, y_{2m(k)-1}) \to \epsilon. \tag{3.8}$$

Therefore, by (3.2) and (3.3), we have

$$\begin{aligned}d(y_{2n(k)}, y_{2m(k)}) &\leq d(y_{2n(k)}, y_{2n(k)+1}) + d(y_{2n(k)+1}, y_{2m(k)})\\ &= d(y_{2n(k)}, y_{2n(k)+1}) + d(Ax_{2m(k)}, Bx_{2n(k)+1})\\ &\leq d(y_{2n(k)}, y_{2n(k)+1}) + \phi(d(Sx_{2m(k)}, Tx_{2n(k)+1}),\\ &\quad d(Ax_{2m(k)}, Sx_{2m(k)}), d(Bx_{2n(k)+1}, Tx_{2n(k)+1}),\\ &\quad d(Ax_{2m(k)}, Tx_{2n(k)+1}), d(Bx_{2n(k)+1}, Sx_{2m(k)}))\\ &= d(y_{2n(k)}, y_{2n(k)+1}) + \phi(d(y_{2m(k)-1}, y_{2n(k)}),\\ &\quad d(y_{2m(k)}, y_{2m(k)-1}), d(y_{2n(k)+1}, y_{2n(k)}), d(y_{2m(k)}, y_{2n(k)}),\\ &\quad d(y_{2n(k)+1}, y_{2m(k)-1})).\end{aligned} \tag{3.9}$$

Since ϕ is upper semi continuous, as $k \to \infty$ as in (3.8), by Lemma 3.2, eqs (3.7), (3.8) and (3.9) we have

$$\epsilon \leq \phi(\epsilon, 0, 0, \epsilon, \epsilon) < \gamma(\epsilon) < \epsilon,$$

which is a contradiction. Therefore, $\{y_{2n}\}$ is a Cauchy sequence in X and so is $\{y_n\}$. This completes the proof.

Theorem 3.1. *Let* (A, S) *and* (B, T) *be weakly compatible pairs of self maps of a complete metric space* (X, d) *satisfying* (3.1) *and* (3.2). *Then A, B, S and T have a unique common fixed point in X.*

Proof. By Lemma 3.3, $\{y_n\}$ is a Cauchy sequence in X. Since X is complete there exists a point z in X such that $\lim_{n\to\infty} y_n = z$. $\lim_{n\to\infty} Ax_{2n} = \lim_{n\to\infty} Tx_{2n+1} = z$ and $\lim_{n\to\infty} Bx_{2n+1} = \lim_{n\to\infty} Sx_{2n+2} = z$ i.e.,

$$\lim_{n\to\infty} Ax_{2n} = \lim_{n\to\infty} Tx_{2n+1} = \lim_{n\to\infty} Bx_{2n+1} = \lim_{n\to\infty} Sx_{2n+2} = z.$$

Since $B(X) \subset S(X)$, there exists a point $u \in X$ such that $z = Su$. Then, using (3.2),

$$\begin{aligned}d(Au, z) &\leq d(Au, Bx_{2n-1}) + d(Bx_{2n-1}, z)\\ &\leq \phi(d(Su, Tx_{2n-1}), d(Au, Su), d(Bx_{2n-1}, Tx_{2n-1}),\\ &\qquad\qquad d(Au, Tx_{2n-1})d(Bx_{2n-1}, Su)).\end{aligned}$$

king the limit as $n \to \infty$ yields

$$\begin{aligned} d(Au, z) &\le \phi(0, d(Au, Su), 0, d(Au, z), d(z, Su)) \\ &= \phi(0, d(Au, z), 0, d(Au, z), 0) \le \beta d(Au, z), \end{aligned}$$

ere $\beta < 1$. Therefore $z = Au = Su$.

Since $A(X) \subset T(X)$, there exists a point $v \in X$ such that $z = Tv$. Then, again using 2),

$$\begin{aligned} z, Bv) = d(Au, Bv) &\le \phi(d(Su, Tv), d(Au, Su), d(Bv, Tv), d(Au, Tv), d(Bv, Su)) \\ &= \phi(0, 0, d(Bv, z), 0, d(Bv, z)) \le \phi(t, t, t, t, t) < t, \end{aligned}$$

ere $t = d(z, Bv)$. Therefore $z = Bv = Tv$. Thus $Au = Su = Bv = Tv = z$. Since ir of maps A and S are weakly compatible, then $ASu = SAu$ i.e, $Az = Sz$. Now we ow that z is a fixed point of A. If $Az \neq z$, then by (3.2),

$$\begin{aligned} d(Az, z) = d(Az, Bv) &\le \phi(d(Sz, Tv), d(Az, Sz), d(Bv, Tv), \\ &\qquad d(Az, Tv), d(Bv, Sz)) \\ &= \phi(d(Az, z), 0, 0, d(Az, z), d(Az, z)) \\ &\le \phi(t, t, t, t, t) < t, \text{ where } t = d(Az, z). \end{aligned}$$

erefore, $Az = z$. Hence $Az = Sz = z$.

Similarly, pair of maps B and T are weakly compatible, we have $Bz = Tz = z$, since

$$\begin{aligned} d(z, Bz) = d(Az, Bz) &\le \phi(d(Sz, Tz), d(Az, Sz), \\ &\quad d(Bz, Tz), d(Az, Tz), d(Bz, Sz)) \\ &= \phi(d(z, Tz), 0, 0, d(z, Tz), d(z, Tz)) \\ &\le \phi(t, t, t, t, t) < t, \text{ where } t = d(z, Tz) = d(z, Bz). \end{aligned}$$

us $z = Az = Bz = Sz = Tz$, and z is a common fixed point of A, B, S and T.

Finally, in order to prove the uniqueness of z, suppose that z and w, $z \neq w$, are common ed points of A, B, S and T. Then by (3.2), we obtain

$$\begin{aligned} (z, w) = d(Az, Bw) &\le \phi(d(Sz, Tw), d(Az, Sz), d(Bw, Tw), d(Az, Tw), d(Bw, Sz)) \\ &= \phi(d(z, w), 0, 0, d(z, w), d(z, w)) \\ &\le \phi(t, t, t, t, t) < t, \text{ where } t = d(z, w). \end{aligned}$$

erefore, $z = w$. The following corollaries follow immediately from Theorem 3.1.

OROLLARY 3.1

et (A, S) *and* (B, T) *be weakly compatible pairs of self maps of a complete metric space* $, d)$ *satisfying* (3.1), (3.3) *and* (3.10)

$$d(Ax, By) \le hM(x, y), 0 \le h < 1, x, y \in X, \textit{ where}$$

$$(x, y) = \max\{d(Sx, Ty), d(Ax, Sx), d(By, Ty), [d(Ax, Ty) + d(By, Sx)]/2\}. \tag{3.10}$$

Then A, B, S and T have a unique common fixed point in X.

Proof. We consider the function $\phi : [0, \infty)^5 \to [0, \infty)$ defined by

$$\phi(x_1, x_2, x_3, x_4, x_5) = h \max \{x_1, x_2, x_3, {}^1\!/_2(x_4 + x_5)\}.$$

Since $\phi \in F$, we can apply Theorem 3.1 and deduce the Corollary.

COROLLARY 3.2

Let (A, S) and (B, T) be weakly compatible pairs of self maps of a complete metric spac (X, d) satisfying (3.1), (3.3) *and* (3.11).

$$d(Ax, By) \le h \max \{d(Ax, Sx), d(By, Ty), {}^1\!/_2 d(Ax, Ty), {}^1\!/_2 d(By, Sx), d(Sx, Ty)\} \textit{ for all } x, y \textit{ in } X, \textit{ where } 0 \le h < 1. \qquad (3.1$$

Then A, B, S and T have a unique common fixed point in X.

Proof. We consider the function $\phi : [0, \infty)^5 \to [0, \infty)$ defined by $\phi(x_1, x_2, x_3, x_4, x_5)$ $h \max \{x_1, x_2, x_3, {}^1\!/_2\, x_4, {}^1\!/_2\, x_5\}$. Since $\phi \in F$, we can apply Theorem 3.1 to obtain t Corollary.

Remark 3.2. Theorem 3.1 generalizes the result of Jungck [3] by using weakly compati maps without continuity at S and T. Theorem 3.1 and Corollary 3.2 also generalize t result of Fisher [1] by employing weakly compatible maps instead of commutativity maps. Further the results of Jachymski [2], Kang and Kim [8], Rhoades *et al* [9] are al generalized by using weakly compatible maps.

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On totally reducible binary forms: I

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Abstract. Let $\nu(n)$ be the number of positive numbers up to a large limit n that are expressible in essentially more than one way by a binary form f that is a product of $\ell > 2$ distinct linear factors with integral coefficients. We prove that

$$\nu(n) = O\left(n^{2/\ell-\eta_\ell+\epsilon}\right),$$

where

$$\eta_\ell = \begin{cases} 1/\ell^2, \text{ if } \ell = 3, \\ (\ell-2)/\ell^2(\ell-1), \text{ if } \ell > 3, \end{cases}$$

thus demonstrating in particular that it is exceptional for a number represented by f to have essentially more than one representation.

Keyword. Binary forms.

1. Introduction

In this publication and its sequel we shall fulfil the undertaking given in our earlier paper [3] to resolve the following problems for binary forms f of degree $\ell > 2$ that are totally reducible as a product of ℓ (disjoint) linear factors with integral coefficients:

(i) to find an asymptotic formula for the number $\Upsilon(n) = \Upsilon_\ell(n)$ of positive integers that are expressible by f and do not exceed n, each such integer being counted just once regardless of multiplicity of representations (no generality is lost by debarring negative numbers because they can be treated by changing the sign of one of the linear factors in f);

(ii) to find an upper bound for the number $\nu(n) = \nu_\ell(n)$ of such integers that are represented in essentially more than one way.

We thus shall extend to a special class of binary forms of arbitrary degree the results obtained for cubics and certain other binary forms in former papers and, in particular, [3], to the last of which the reader is referred for a history of the problem and the relevant citations.

In interpreting the second quest, on which the first will be seen to depend, we must anticipate a later discussion by saying that representations of a number by the form are regarded as being inherently distinct if they be not associated with each other in an obvious way through an automorphic of the form. With this appreciation, we shall shew here that

$$\nu(n) = O\left(n^{2/\ell-\eta_\ell+\epsilon}\right), \tag{1}$$

where

$$\eta_\ell = \begin{cases} 1/\ell^2, \text{ if } \ell = 3, \\ (\ell - 2)/\ell^2(\ell - 1), \text{ if } \ell > 3, \end{cases}$$

from which it will be easily demonstrated that it is extremely rare for a number representable by f to be represented in essentially more than one way.

The derivation from this result of an asymptotic formula for $\Upsilon(n)$ principally depends on the properties of the automorphics of the form. We therefore reserve the treatment of item (i) for a second paper, especially as an exhaustive treatment of the structure of the automorphics occupies some space, is in itself an interesting study, and involves ideas that are somewhat alien to those used in the present work. Suffice it then for the time being to say that we shall ultimately obtain an asymptotic formula of the type

$$\Upsilon(n) \sim A(f)n^{2/\ell}, \quad (A(f) > 0)$$

with a remainder term similar to the right-hand side of (1).

We should mention that the advantage of our present methods – in contrast with those often used in problems of this type – is that they are also applicable to an inhomogeneous situation in which the subject of study is a completely reducible polynomial of degree ℓ consisting of factors of the type $hx+ky+q$. It is hoped to give an account of this extensior to our work in due course.

2. Notation and conventions

As is often the case in the algebra of substitutions as applied to forms or quantics, eacl symbol for a variable therein will denote an indeterminate on some occasions and a specialization of this on other occasions. With this agreement, when not denoting indeterminates r, s, ρ, σ are integers and m, μ with or without distinguishing marks are non-zero integers; p, ϖ are positive prime numbers. The letters $A_1, A_2, \ldots$ denote suitable positiv constants depending at most on the form f under consideration; ϵ is an arbitrarily smal positive number that is not necessarily the same at each occurrence; the constants implie by the O-notation are of type A_i save when they may also depend on ϵ.

Since negative integers may frequently occur, we should mention that they may b moduli in congruences. The terms size, magnitude, modulus are used as synonyms fc absolute value when applied to real numbers. The notation (h, k) indicates the positiv highest common factor (when defined) of integers h, k save when it designates a point wit coordinates h, k; $d(m)$ is the number of positive divisors of m, while $d_r(m)$ is the numb of ways expressing $|m|$ as a product of r positive factors.

3. Prolegomena

Being totally reducible over the rationals with no repeated factors, the binary form f $f(x, y)$ of degree $\ell \geq 3$ under consideration is expressed as

$$\prod_{1 \leq i \leq \ell} (h_i x + k_i y) = \prod_{1 \leq i \leq \ell} L_i(x, y), \text{ say}, \qquad ($$

where the coefficients of the linear forms $L_i(x, y)$ are integers and where, even apart fro order, there is a slight but acceptable ambiguity in their definitions when f is imprimitiv

Of the invariants of the form, the only one that will be needed is the discriminant

$$D = D(f) = \prod_{1 \leq i < j \leq \ell} \left(h_i k_j - h_j k_i\right)^2 > 0, \tag{3}$$

which has the familiar property that, if $f(x, y)$ be transformed into $F(X, Y)$ by a substitution of modulus M, then

$$D(F) = M^{\ell^2 - \ell} D(f). \tag{4}$$

Also, since our investigation only concerns the representations of numbers by f without regard to the size of the variables in f, we may equally well work with any form f' equivalent to f through a rational integral substitution

$$x = \alpha X + \beta Y, \ y = \gamma X + \delta Y \tag{5}$$

with modulus

$$\alpha\delta - \gamma\beta = 1. \tag{6}$$

This means, in particular, that we may certainly assume that

$$h_1, h_2, \ldots, h_\ell \neq 0 \tag{7}$$

because[1], in the opposite instance, having chosen relatively prime numbers α, γ and then β, δ to satisfy $f(\alpha, \gamma) \neq 0$ and (6), we find through (5) a form with non-zero leading coefficient $f(\alpha, \gamma)$ that is equivalent to f.

Closely associated with our study of the representations by $f(x, y)$ of positive numbers up to a large limit n, the curve $C = C(n)$ defined by the equation

$$f(x, y) = n \tag{8}$$

will be encountered together with its asymptotes

$$L_1(x, y) = 0, \ldots, L_\ell(x, y) = 0, \tag{9}$$

which both here and in our second paper will play a not unimportant role in the elucidation of lattice point problems involving regions bounded by $C(n)$. Forming 2ℓ semi-infinite rays emanating from the origin, these asymptotes divide the plane into 2ℓ semi-infinite domains, in each of which $f(x, y)$ has a constant sign opposite to that pertaining to its neighbours. Moreover, from an examination of the configuration formed by (8) and (9), it would be foreseen that the major influence on our situation would be exerted by those x and y having absolute values not substantially larger than $n^{1/\ell}$, which expectation prompts us at once to write

$$N = n^{1/\ell} \tag{10}$$

for notational convenience. Next, elaborating on this line of thought analytically (a geometrical approach is more intuitive but harder to describe), we note by linear relationships that, if

$$\max_{1 \leq i \leq \ell} \left|L_i(x, y)\right| = L_u(x, y) = Q > 0 \tag{11}$$

---

[1] With a little more effort we can shew that we may suppose, in addition, that $k_1, \ldots, k_\ell \neq 0$.

for integer values of x and y, then we always have

$$|x|, |y| < A_1 Q. \tag{12}$$

Also, more significantly, if here

$$Q > A_2 N \tag{13}$$

for a sufficiently large positive constant A_2 and $f(x, y)$ obey the usually assumed inequality $0 < f(x, y) \leq n$, we see first that at least one form $L_\nu(x, y)$ for $\nu \neq u$ has magnitude not less than 1 and not greater than

$$(n/Q)^{1/(\ell-1)} < A_2^{-1/(\ell-1)} N < A_2^{-\ell/(\ell-1)} Q \tag{14}$$

and then deduce from linear relationships and (11) that this form $L_\nu(x, y)$ is unique, all other forms $L_i(x, y)$ having magnitudes greater then $A_3 Q$. Hence $A_3^{\ell-1} Q^{\ell-1} < n$ so that

$$Q < A_4 n^{1/(\ell-1)}, \tag{15}$$

while also the bound (14) is improved to

$$|L_\nu(x, y)| < \frac{n}{A_3^{\ell-1} Q^{\ell-1}} = \frac{A_5 n}{Q^{\ell-1}}. \tag{16}$$

Some amplification of an introductory remark about the automorphics of the form is needed at once even though a full examination of their structure will be delayed until our second paper. Let now

$$x = \alpha X + \beta Y, \ y = \gamma X + \delta Y \tag{17}$$

be a rational automorphic of f, namely, a substitution with rational coefficients $\alpha, \beta, \gamma, \delta$ with the property that $f(x, y) = f(X, Y)$ and, as we confirm from (4), the consequencial property that its modulus $\alpha\delta - \gamma\beta$ is equal to ± 1. Points (x, y), (X, Y) with *integral* coordinates that are connected by means of an automorphic of type (17) will be said to be *associated*, the property of association being denoted by $(x, y) \simeq (X, Y)$. Then, since associated points give rise to linearly connected representations of the same number, we shall agree that representations of a number as $f(x, y)$ and $f(x', y')$ are deemed *essentially different* if $(x, y) \simeq (x', y')$. Thus an unmistakable meaning has been attached to $\nu(n)$, to whose estimation we now attend.

4. The sum *T(n)* and the equation *m F(m, s)* = $\mu G(\mu, \sigma)$

The treatment depends on an analysis of the sum

$$T(n) = \sum_{\substack{0 < f(r,s) = f(\rho,\sigma) \leq n \\ (r,s) \not\simeq (\rho,\sigma)}} 1, \tag{18}$$

through which $\nu(n)$ is bounded by the obvious inequality

$$\nu(n) \leq T(n). \tag{19}$$

First, to dissect the sum into parts that can be appropriately assessed, let $T_1(n)$ be that portion of $T(n)$ that is yielded by values of r, s, ρ, σ in the conditions of summation for which a linear factor of maximal size in the constituents of the equation

$$f(r,s) = \prod_{1 \le i \le \ell} L_i(r,s) = \prod_{1 \le j \le \ell} L_j(\rho,\sigma) = f(\rho,\sigma)$$

occurs on the left. Then, allowing $T_1(n, M)$ to denote the contribution to $T_1(n)$ corresponding to values of r, s, ρ, σ for which the size of this maximal linear factor lies between M inclusive and $2M$ exclusive, we have

$$T(n) \le 2T_1(n)$$

and complete the first phase of the calculations by deducing that

$$T(n) \le 2\sum_i T_1(n, M_i), \tag{20}$$

in which

$$M_i = 2^i \qquad (i \ge 0) \tag{21}$$

is less than

$$A_4 n^{1/(\ell-1)} \tag{22}$$

by (15).

In further preparation for the estimation of $T(n)$ we examine the solutions of the indeterminate equation

$$f(r,s) = f(\rho,\sigma) \tag{23}$$

that are constrained by the conditions

$$L_u(r,s) = m, \qquad L_\nu(\rho,\sigma) = \mu \tag{24}$$

for given subscripts u, ν and non-zero integers m, μ. For this purpose, recalling (7), we employ the substitutions[2]

$$m = h_u r + k_u s, \qquad s = s, \tag{25}$$

$$\mu = h_\nu \rho + k_\nu \sigma, \qquad \sigma = \sigma, \tag{26}$$

to transform (23) into

$$\frac{m}{h_u^{\ell-1}} \prod_{i \ne u} \{h_i m + (h_u k_i - h_i k_u) s\} = \frac{\mu}{h_\nu^{\ell-1}} \prod_{j \ne \nu} \left\{h_j \mu + \left(h_\nu k_j - h_j k_\nu\right) \sigma\right\}, \tag{27}$$

which equation for brevity we express as either

$$mF(m,s) = \mu G(\mu,\sigma) \tag{28}$$

or

$$mF(m,s) - \mu G(\mu,\sigma) = 0, \tag{29}$$

---

[2] Remember the remarks in §2 about entities appearing in linear transformations. We should also comment that it would be pendantic and unhelpful here to introduce symbols s', σ' to substitute for s, σ in (27) and in the left sides of the right hand members of (25) and (26).

where $F(m, s) = F_u(m, s)$ and $G(\mu, \sigma) = G_\nu(\mu, \sigma)(= F_\nu(\mu, \sigma))$ are, respectively, of exact degree $\ell - 1$ in s and σ.

Needing to know when the curve defined by (27) for given non-zero integer values of m and μ is irreducible over $\mathbb{Q}$, we set

$$s = ms', \ \sigma = \mu\sigma', \ \lambda = (m/\mu)^\ell \neq 0$$

to form the equivalent curve

$$\lambda F(1, s') - G(1, \sigma') = 0, \tag{30}$$

which is certainly irreducible (indeed absolutely irreducible) when its projective completion

$$\lambda F(z, s') - G(z, \sigma') = 0$$

is non-singular and hence when the simultaneous equations

$$\frac{\partial F(z, s')}{\partial s'} = 0, \quad \frac{\partial G(z, \sigma')}{\partial \sigma'} = 0, \quad \lambda\frac{\partial F(z, s')}{\partial z} - \frac{\partial G(z, \sigma')}{\partial z} = 0$$

have no non-zero solution. But, if $z = 0$, the first two equations only hold when $s' = \sigma' = 0$, whereas otherwise, since $F(z, s')$ and $G(z, \sigma')$ are each products of real distinct factors, they are only satisfied when $s'/z, \sigma'/z$ each take $\ell - 2$ real values, each combination of which determines λ through the last equation because neither both $\partial F/\partial s'$, $\partial F/\partial z$ nor both $\partial G/\partial \sigma'$, $\partial G/\partial z$ can (non-trivially) simultaneously vanish. Thus reducible curves of type (30) answer to at most $(\ell - 2)^2$ values of λ, and we therefore infer that (29) can only be reducible if

$$Bm = C\mu \tag{31}$$

for one of $O(1)$ sets of relatively prime (bounded) non-zero integers $B = B_{u,\nu}$, $C = C_{u,\nu}$.

Of special importance is the case where the left side of (29) has a rational linear factor in s, σ, in which event for some pair B, C the left side of the corresponding equation (30) with $\lambda = (C/B)^\ell$ contains a linear factor $s' = D\sigma' + E$ with rational coefficients $D \neq 0$ and E that do not depend on m and μ. In this situation, we deduce from (31) that (28) holds identically whenever

$$Bs = Bms' = C\mu s' = CD\mu\sigma' + CE\mu = CD\sigma + CE\mu$$

and hence that the rational substitution

$$Bs = CD\sigma + CE\mu, \ Bm = C\mu$$

transforms $mF(m, s)$ into $\mu G(\mu, \sigma)$. Therefore, compounding this substitution with (26) and the inverse of (25) in the obvious order, we are provided with a rational automorphic of f that takes r, s into ρ, σ, whence any solutions of (29) arising in this way flow from associated points (r, s), (ρ, σ) and are of a type not counted in $T(n)$ and its constituent parts.

In combination with the special features just identified, the main instrument in our treatment of equation (29) is an important theorem due to Bombieri and Pila [1] that we state here as follows.

Lemma 1. *Let* $\Psi(\xi, \eta)$ *be an irreducible polynomial of degree* δ *with rational coefficients. Then the number of solutions of* $\Psi(\xi, \eta) = 0$ *in integers of size not exceeding* z *is*

$$O\left(z^{(1/\delta)+\epsilon}\right) \qquad (z \geq 1),$$

where the constants implied by the O-notation are independent of the coefficients of Ψ.

Proved by Bombieri and Pila when the condition of absolute irreducibility is imposed, the result of the lemma remains true if $\Psi(\xi, \eta)$ be irreducible but not absolutely irreducible because then any integer solution is a zero of an absolutely irreducible factor of $\Psi(\xi, \eta)$ of the form

$$\omega_1 \psi_1(\xi, \eta) + \cdots + \omega_e \psi_e(\xi, \eta) \quad (\psi_i(\xi, \eta) \epsilon \mathbb{Q}[\xi, \eta]),$$

where $\omega_1, \ldots, \omega_e$ is a basis of the field of degree $e > 1$ over which the factor is defined. In fact the zeros of this are the common zeros of the system

$$\psi_1(\xi, \eta), \ldots, \psi_e(\xi, \eta),$$

which belong to a variety of dimension zero and limited degree since clearly $\psi_1, \ldots, \psi_e$ have no common factor. This confirms our extension of the Bombieri–Pila theorem in the context of the present work.

For the case $\ell = 3$ we shall need to augment our armoury with an elementary estimate that is sharper than Lemma 1 when $\Psi(\xi, \eta)$ is a special type of quadratic. This is as follows.

Lemma 2. *Let*

$$\Psi(\xi, \eta) = a_1 \xi^2 + b_1 \xi + a_2 \eta^2 + b_2 \eta + c \quad (a_1, a_2 \neq 0)$$

be an irreducible quadratic polynomial with rational coefficients having bounded denominators and size not exceeding z^{A_7}. *Then the number of zeros of* $\Psi(\xi, \eta)$ *of size not exceeding* z *is* $O(z^\epsilon)$ *for* $z \geq 1$.

Supposing first that $\Psi(\xi, \eta)$ is absolutely irreducible and noting that we may restrict attention to the case where it has integer coefficients, multiply it by $4a_1a_2$ to transform it into

$$a_2\left(2a_1\xi + b_1\right)^2 + a_1\left(2a_2\eta + b_2\right)^2 - \left(a_2b_1^2 + a_1b_2^2 - 4a_1a_2c\right)$$

with the implication that $a_2b_1^2 + a_1b_2^2 - 4a_1a_2c \neq 0$. Hence, since the solutions of $\Psi(\xi, \eta) = 0$ are contained in those of an equation of the type

$$a_3X^2 + a_4Y^2 = a_2b_1^2 + a_1b_2^2 - 4a_1a_2c,$$

we deduce that the solutions to be counted have cardinality

$$O\left\{d\left(a_2b_1^2 + a_1b_2^2 - c\right)\log 2z\right\} = O\left(z^\epsilon\right)$$

by a familiar application of the theory of quadratic forms as used for example in our paper [2].

The case where Ψ is irreducible but not absolutely irreducible is catered for by the argument in the proof of Lemma 1 or, alternatively, is easily handled *a priori* in the present framework by obvious reasoning.

5. Estimation of *T(n)* and the first theorem

In treating the sum $T(n)$ in (17), which we now rejoin, we shall first primarily address the case where $\ell > 3$ and shall delay until later a modified argument for $\ell = 3$ that largely depends on Lemma 2 instead of Lemma 1, although it should be stressed that nothing in the earlier stages of the reasoning is actually invalid for the latter case.

Having indicated the sphere of operation, we first suppose that

$$M \leq A_2 N \tag{32}$$

in the notation of (13) and consider the contribution to $T_1(n, M)$ due to those values of r, s, ρ, σ meeting its conditions of summation for which

$$\max_{1\leq i\leq \ell} \left|L_i(r,s)\right| = \left|L_u(r,s)\right| \text{ and } L_u(r,s) = m \tag{33}$$

for some specified integer m of a size between M inclusive and $2M$ exclusive. In these surroundings the requirement that $0 < f(r,s) = f(\rho, \alpha)$ implies that

$$\left|m\right| = \left(L_u(r,s), \prod_{1\leq j\leq \ell} L_j(\rho,\sigma)\right) \leq \prod_{1\leq j\leq \ell} \left(L_u(r,s), L_j(\rho,\sigma)\right)$$

so that at least one factor $(L_u(r,s), L_\nu(\rho,\sigma))$ on the right above is not less that

$$\left|m\right|^{1/\ell} \geq M^{1/\ell}, \tag{34}$$

the value μ of $L_\nu(\rho,\sigma)$ being governed by the condition $0 < \left|\mu\right| \leq \left|m\right| < 2M$ through the definition of $T_1(n)$. Hence, since $\left|s\right|, \left|\sigma\right| < 2A_1M = A_6M$ by (12), we deduce that

$$T_1(n,M) \leq \sum_{1\leq u,\nu\leq \ell} \sum_{\substack{0<|m|,|\mu|<2M\\ (m,\mu)\geq M^{1/\ell}}} T_{u,\nu}(n,M;m,\mu), \tag{35}$$

where $T_{u,\nu}(n,M;m,\mu$ is the number of solutions of (29) in integers s, σ of size not exceeding A_6M that do not appertain via (25) and (26) to the association $(r,s) \simeq (\rho,\sigma)$.

Let us first depose of the contribution $T_1^*(n,M)$ to the right-hand side of (35) that relates to values of u, ν, m, μ for which the polynomial $mF(m,s) - \mu G(\mu,\sigma)$ is reducible. In this case, by (31), m and μ are connected by an equation $B_{u,\nu}m = C_{u,\nu}\mu$ for one of a finite set of pairs of coprime non-zero integers $B_{u,\nu}, C_{u,\nu}$. Also, the zeros of $mF(m,s) - \mu G(\mu,\sigma)$ are distributed among all its irreducible factors with rational coefficients, each such factor of degree 2 or more having $O\left(M^{\frac{1}{2}+\epsilon}\right)$ zeros in the chosen domain of s and σ by Lemma 1. On the other hand, the zeros of any linear factors are inadmissible because we have shewn earlier that they would not meet the stipulation that $(r,s) \not\simeq (\rho,\sigma)$. Consequently

$$T_1^*(n,M) = O\left(M^{\frac{3}{2}+\epsilon}\right) \tag{36}$$

by (35).

If we write

$$m = dm', \mu = d\mu', \text{ where} (m',\mu') = 1 \text{ and}^3\ d \geq M^{1/\ell} \tag{37}$$

---

[3] When $d > 2M$ all our subsequent calculations are true but trivial, the underlying sums being of course empty.

in the conditions of summation for the remaining portion $T_1^{\dagger}(n, M)$ of the sum in the right of (35), equation (28) takes the form

$$m'F(m, s) = \mu' G(\mu, \sigma), \tag{38}$$

which with (27) and (37) implies both the congruences

$$\Phi(m, s) = \prod_{i \neq u} \left\{h_i m + \left(h_u k_i - h_i k_u\right) s\right\} \equiv 0, \mod \mu'', \tag{39}$$

and

$$\Gamma(\mu, \sigma) = \prod_{j \neq \nu} \left\{h_j \mu + \left(h_\nu k_j - h_j k_\nu\right) \sigma\right\} \equiv 0, \mod m'' \tag{40}$$

for certain coprime moduli

$$m'' = m'/a, \quad \mu'' = \mu'/\alpha$$

derived from the division of m' and μ', respectively, by certain (small) positive divisors a and α of $\left(m', h_u^{\ell-1}\right)$ and $\left(\mu', h_\nu^{\ell-1}\right)$. Even though $\Phi(m, s)$ and $\Gamma(\mu, \sigma)$ are products of rational linear factors, a full discussion of these congruences for general composite moduli having repeated prime factors entails the same sort of difficulties that attend the general theory of polynomial congruencies in one variable as expounded by Nagell ([4], ch. III); these difficulties at the present juncture would in fact involve the prime divisors of the discriminants of $\Phi(m, s)$ and $\Gamma(\mu, \sigma)$ quâ polynomials in s and σ and, therefore, ultimately and especially those of the number d. However, at the expense of a balancing slight lengthening in procedure, we are able here to circumvent these congruential entanglements by reducing our situation to one where the moduli are square-free.

Accordingly, for integers m'' and μ'' in (40) and (39) whose expressions in terms of prime factors are stated as

$$m'' = \pm \prod_p p^b, \quad \mu'' = \pm \prod_{\varpi} \varpi^{\beta}, \text{ say,}$$

we shall first use the *positive* square-free numbers

$$m_3 = \prod_p p, \quad \mu_3 = \prod_{\varpi} \varpi,$$

while later we shall need numbers m_4, μ_4, that similarly originate from m', μ' and bear no relation to a and α; finally, for each given number of type m_4 or μ_4, we let $m_5 = m_5(m_4)$ or $\mu_5 = \mu_5(\mu_4)$ denote positive numbers whose prime factors are divisors of m_4 and μ_4, respectively. Then, the procedure being amply illustrated by reference to the congruence (40), all solutions of this in σ satisfy the corresponding congruence taken to the modulus m_3, the number of incongruent solutions of which we denote by $\kappa(m_3)$. Since $\kappa(m_3)$ is multiplicative, it suffices to consider $\kappa(p)$ when $p \nmid D$ because in the contrary instance we are content with the trivial estimate $\kappa(p) \leq p$. Thus we may assume that the coefficient of σ in each factor of $\Gamma(\mu, \sigma)$ in (40) is indivisible by p and deduce that each such factor is divisible by p when σ belongs to just one residue class, mod p. Consequently, we see that $\kappa(p)$ does not exceed p or $\ell - 1$ according as $p|D$ or $p \nmid D$ and conclude both that

$$\kappa(m_3) = O\left\{(\ell - 1)^{\omega(m_3)}\right\} = O\left(m_3^{\epsilon}\right) \tag{41}$$

and that a similar result holds for the other congruence (39).

In the current circumstances the solutions of (38) in s, σ have been shewn to be distributed into $O(M^\epsilon)$ sets, each of which consists of pairs of numbers of the type

$$s = s_0 + s_1\mu_3, \quad \sigma = \sigma_0 + \sigma_1 m_3 \tag{42}$$

for certain positive numbers s_0, σ_0 not exceeding μ_3, m_3 respectively. The relevant contribution to $T_{u,v}(n, M; m, \mu)$ corresponding to each set is then obtained by substituting (42) in (38) to obtain an irreducible equation is s_1, σ_1 of degree $\ell - 1$, of which, being constrained by the inequalities

$$|s_1| \le \frac{A_6M + \mu_3}{\mu_3} < \frac{2A_6M}{\mu_3}(> 1), \ |\sigma_1| \le \frac{A_6M + m_3}{m_3} < \frac{2A_6M}{m_3}(> 1),$$

the number of qualifying pairs of zeros is

$$O\left(M^{1/(\ell-1)+\epsilon} \max\left(\frac{1}{\mu_3}, \frac{1}{m_3}\right)^{1/\ell-1}\right) = O\left(M^{1/(\ell-1)+\epsilon} \max\left(\frac{1}{\mu_4}, \frac{1}{m_4}\right)^{1/(\ell-1)}\right)$$

by Lemma 1. Therefore, taking stock after this, (41), (35), and (37), we conclude that

$$T_1^\dagger(n, M) = O\left(M^{1/(\ell-1)+\epsilon} \sum_{d \ge M^{1/\ell}} \sum_{0 < m', \mu' \le 2M/d} \max\left(\frac{1}{m_4}, \frac{1}{\mu_4}\right)^{1/(\ell-1)}\right), \tag{43}$$

from which the estimate for $T_1^\dagger(n, M)$ will flow by the way of the simple

Lemma 3. *Let q denote any positive integer composed entirely of prime factors (possibly repeated) that divide a given positive number (possibly* 1) *not exceeding z. Then the number of q not exceeding z is $O(z^\epsilon)$.*

This is a special case of the Lemma 4 in [3]. Evidently the inner sum in (43) does not exceed

$$\begin{aligned} 2 \sum_{0 < m', \mu' \le 2M/d} \frac{1}{m_4^{1/(\ell-1)}} &= O\left(\frac{M}{d} \sum_{0 < m' \le 2M/d} \frac{1}{m_4^{1/(\ell-1)}}\right) \\ &= O\left(\frac{M}{d} \sum_{0 < m_4 \le 2M/d} \frac{1}{m_4^{1/(\ell-1)}} \sum_{m_5 \le 2M/m_4 d} 1\right) \\ &= O\left(\frac{M^{1+\epsilon}}{d} \sum_{0 < m \le 2M/d} \frac{1}{m^{1/(\ell-1)}}\right) \\ &= O\left(\frac{M^{2-1/(\ell-1)+\epsilon}}{d^{2-1/(\ell-1)}}\right) \end{aligned}$$

with the implication that

$$T_1^\dagger(n, M) = O\left(M^{2+\epsilon} \sum_{d \ge M^{1/\ell}} \frac{1}{d^{2-1/(\ell-1)}}\right) = O\left(M^{2-(\ell-2)/\ell(\ell-1)+\epsilon}\right),$$

wherefore on taking this with (36) we have

$$T_1(n, M) = O\left(M^{2-(\ell-2)/\ell(\ell-1)+\epsilon}\right) \tag{44}$$

for $M \leq A_2 N$ as in (32).

Similar principles are successful for the estimation of $T_1(n, M)$ in the complementary range $A_2 N < M < A_4 n^{1/(\ell-1)}$ but are less straightforward to apply. Now, by the definition of $T_1(n, M)$ and (16), we first modify (33) by using the (unique) subscript u for which $L_u(r, s)$ equals a non-zero number m whose size does not exceed

$$M_I = A_5 n / M^{\ell-1} < M, \tag{45}$$

even though the previously used inequalities for s, σ are still valid. Next, following previous thinking, we find there is a subscript ν for which the number $\mu = L_\nu(\rho, \sigma)$ possesses the properties

$$(m, \mu) \geq |m|^{1/\ell} \quad \text{and} \quad |\mu| < 2M. \tag{46}$$

This clears the way for a reconsideration of $T_1^\dagger(n, M)$ because the assessment

$$T_1^*(n, M) = O\left(M^{\frac{1}{2}+\epsilon} M_I\right) \tag{47}$$

is a corollary of (31).

The new surroundings affect the sums bounding $T_1^\dagger(n, M)$ more in regard to the conditions of summation than the summands therein. In the former we still have the first parts of (37) but replace the last part by $d \geq |m|^{1/\ell}$ with the result that

$$0 < m' \leq M_I/d, \quad 0 < m' \leq d^{\ell-1}, \quad 0 < \mu' < 2M/d.$$

Hence, emulating the derivation of (43), we have

$$\begin{aligned} T_1^\dagger(n, M) &= O\left(M^{1/(\ell-1)+\epsilon} \sum_d \sum_{\substack{0<m' \leq M_I/d, d^{\ell-1} \\ 0<\mu' \leq 2M/d}} \max\left(\frac{1}{m_4}, \frac{1}{\mu_4}\right)^{1/(\ell-1)}\right) \\ &= O\left(M^{1/(\ell-1)+\epsilon} \sum_d \sum\nolimits_d\right), \text{ say,} \end{aligned} \tag{48}$$

and then go on to treat $\sum_d$ for the two cases $d > M_I^{1/\ell}$ and $d \leq M_I^{1/\ell}$. In the earlier instance, by Lemma 3 and then (45), we get

$$\begin{aligned} \sum\nolimits_d &\leq \sum_{\substack{0<m' \leq M_I/d \\ 0<\mu' \leq 2M/d}} \frac{1}{m_4^{1/(\ell-1)}} + \sum_{\substack{0<m' \leq M_I/d \\ 0<\mu' \leq 2M/d}} \frac{1}{\mu_4^{1/(\ell-1)}} \\ &\leq \frac{2M}{d} \sum_{m_4 \leq M_I/d} \frac{1}{m_4^{1/(\ell-1)}} \sum_{m_5 \leq M_I/dm_4} 1 \\ &\quad + \frac{2M_I}{d} \sum_{\mu_4 \leq 2M/d} \frac{1}{\mu_4^{1/(\ell-1)}} \sum_{\mu_5 \leq 2M/d\mu_4} 1 \end{aligned}$$

$$= O\left(\frac{M^{1+\epsilon}}{d}\sum_{0<m\leq M_I/d}\frac{1}{m^{1/(\ell-1)}}\right) + O\left(\frac{M_I M^{\epsilon}}{d}\sum_{0<\mu\leq 2M/d}\frac{1}{\mu^{1/(\ell-1)}}\right)$$
$$= O\left(\frac{M^{1+\epsilon}M_I^{1-1/(\ell-1)}}{d^{2-1/(\ell-1)}}\right) + O\left(\frac{M_I M^{1-1/(\ell-1)+\epsilon}}{d^{2-1/(\ell-1)}}\right) = O\left(\frac{M^{1+\epsilon}M_I^{1-1/(\ell-1)}}{d^{2-1/(\ell-1)}}\right)$$

and in the latter instance similarly obtain

$$\sum\nolimits_d = O\left(M^{1+\epsilon}d^{\ell-3}\right)$$

after replacing M_I/d by $d^{\ell-3}$ as a limit for μ' in the summation. Therefore equation (48) can be developed into

$$\begin{aligned} T_1^{\dagger}(n, M) &= O\left(M^{\ell/(\ell-1)+\epsilon}\sum_{d\leq M_I^{1/\ell}} d^{\ell-3}\right) \\ &\quad + O\left(M^{\ell/(\ell-1)+\epsilon}M_I^{(\ell-2)/(\ell-1)}\sum_{d>M_I^{1/\ell}}\frac{1}{d^{2-1/(\ell-1)}}\right) \\ &= O\left(M^{\ell/(\ell-1)+\epsilon}M_I^{(\ell-2)/\ell}\right), \end{aligned} \tag{49}$$

which in combination with (47) furnishes us with the estimate

$$T_1(n, M) = O\left(M^{\ell/(\ell-1)+\epsilon}M_I^{(\ell-2)/\ell}\right) = O\left(n^{(\ell-2)/\ell+\epsilon}M^{1-(\ell-1)(\ell-2)/\ell+1/(\ell-1)}\right) \tag{50}$$

that is the complement of (44) for the range $A_2N < M < A_4n^{1/(\ell-1)}$.

The first part of our initial theorem follows at once because the exponent of M in (50) is negative when $\ell > 3$. Indeed, by embodying (44) and (50) in (20) and then recalling (10), we deduce at once that

$$\begin{aligned} T(n) &= O\left(N^{2-(\ell-2)/\ell(\ell-1)+\epsilon}\right) + O\left(n^{(\ell-2)/\ell+\epsilon}N^{1-(\ell-1)(\ell-2)/\ell+1/(\ell-1)}\right) \\ &= O\left(N^{2-(\ell-2)/\ell(\ell-1)+\epsilon}\right) \end{aligned}$$

or

$$T(n) = O\left(n^{2/\ell-(\ell-2)/\ell^2(\ell-1)+\epsilon}\right) \tag{51}$$

and so estimate $\nu(n)$ because of (19).

When $\ell = 3$ it is only the last part of the analysis leading to (50) that fails to be effective. Yet, if we take the opportunity that arises here to use Lemma 2 instead of Lemma 1, we can not only produce a workable alternative to (50) but find all the relevant revised estimates in the work combine to yield the board

$$T(n) = O\left(N^{2-\frac{1}{3}+\epsilon}\right) = O\left(n^{\frac{2}{3}-\frac{1}{9}+\epsilon}\right) \tag{52}$$

that is better than what would be got by formally putting $\ell = 3$ in (51). Moreover, although the general structure of the previous method is retained, there is the important simplification that all references to the congruences (39) and (40) and to Lemma 3 are avoided.

To indicate briefly what is to be done, we note that revisions are only needed when the polynomial $m'F(m,s) - \mu'G(\mu,\sigma)$ in s, σ is irreducible and hence when it has $O(M^\epsilon)$ zeros of size not exceeding $2A_1M$ by Lemma 2. Hence we can improve (43) to

$$T_1^\dagger(n,M) = O\left(M^\epsilon \sum_{d \geq M^{1/\ell}} \sum_{0<m',\mu' \leq 2M/d} 1\right)$$

and thus (44) to

$$T_1^\dagger(n,M) = O\left(M^{2-1/\ell+\epsilon}\right).$$

Similarly (48) is replaced by

$$T_1^\dagger(n,M) = O\left(M^\epsilon \sum_d \sum_{\substack{0<m' \leq M_1/d, d^{\ell-1} \\ 0<\mu' \leq 2M/d}} 1\right),$$

which leads to the counterpart

$$T_1^\dagger(n,M) = O\left(n^{(\ell-1)/\ell+\epsilon} M^{1-(\ell-1)^2/\ell}\right)$$

of (49). The exponent of M in this being $-\frac{1}{3}$, we then sum over M as before to obtain (52) in place of (51) for $\ell = 3$ and thus complete the proof of

Theorem 1. *Let $f(x,y)$ be a totally reducible binary form of degree ℓ with integral coefficients and non-zero discriminant. Then, if $\nu(n)$ be the number of positive integers up to a large number n that have essentially more than one representation by f, we have*

$$\nu(n) = O\left(n^{2/\ell-\eta_\ell+\epsilon}\right),$$

where

$$\eta_\ell = \begin{cases} 1/\ell^2, & \text{if } \ell = 3, \\ (\ell-2)/\ell^2(\ell-1), & \text{if } \ell > 3. \end{cases}$$

6. The second theorem

To shew it is exceptional for a number to be represented by $f(x,y)$ in essentially more than one way we must foreshadow a simple aspect of our following paper by defining $r(m)$ to be the number of ways of expressing the positive number m by $f(x,y)$, where of course

$$r(m) = O\left\{d_\ell(m)\right\} = O\left(m^\epsilon\right). \tag{53}$$

Let us now take one of the semi-infinite triangular regions described in §3 in which $f(x,y)$ is positive and consider points (x,y) within having integral coordinates for which $|x|, |y| < A_8 n^{1/\ell}$ for a suitably small positive constant A_8. Then $0 < f(x,y) < n$ for all these points, the cardinality of which exceeds $A_9 n^{2/\ell}$ by a standard lattice point argument. Consequently

$$\sum_{0<m\leq n} r(m) > A_8 n^{2/\ell}$$

and thus, by (53),

$$\Upsilon(n) > A(\epsilon)n^{-\epsilon} \sum_{m \le n} r(m) > A(\epsilon)n^{2/\ell-\epsilon},$$

whence, on comparing this with Theorem 1, we gain the following.

Theorem 2. *Almost all the positive numbers represented by the form $f(x, y)$ in Theorem 1 are represented thus in essentially only one way.*

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Stability of Picard bundle over moduli space of stable vector bundles of rank two over a curve

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Abstract. Answering a question of [BV] it is proved that the Picard bundle on the moduli space of stable vector bundles of rank two, on a Riemann surface of genus at least three, with fixed determinant of odd degree is stable.

Keywords. Picard bundle; Hecke lines.

0. Introduction

Let X be a compact connected Riemann surface of genus g, with $g \geq 3$. Let ξ be a holomorphic line bundle over X of odd degree d, with $d \geq 4g-3$. Let M denote the moduli space of stable vector bundles E over X of rank two and $\bigwedge^2 E \cong \xi$. Take a universal vector bundle $\mathcal{E}$ on $X \times M$. Let $p : X \times M \longrightarrow M$ be the projection. The vector bundle $\mathcal{P} := p_*\mathcal{E}$ on M is called the *Picard bundle for M*. In [BV] it was proved that the Picard bundle $\mathcal{P}$ is simple, and a question was asked whether it is stable. In [BHM] a differential geometric criterion for the stability of $\mathcal{P}$ was given. But there is no evidence for this criterion to be valid.

In Theorem 3.1 we prove that the Picard bundle $\mathcal{P}$ over M is stable.

1. Preliminaries

In this section we prove some lemmas that will be needed.

A vector bundle E of rank two and degree d is called *superstable* if for every subline bundle L of E the inequality

$$\deg(L) < \frac{d}{2} - \frac{1}{2}$$

is valid. Clearly, a superstable bundle is stable. The first lemma ensures existence of superstable bundles.

Lemma 1.1. *There is a nonempty open subset U of M corresponding to superstable bundles.*

Proof. Here we need $g \geq 3$. Let T be the subset of M of vector bundles that are not superstable, i.e., $E \in T$ if and only if there exists a subline bundle L such that $\deg(L) \geq$

$(d-1)/2$. Since E is stable, $\deg(L) < d/2$, and since d is odd, $\deg(L) = (d-1)/2$. There is a short exact sequence

$$0 \longrightarrow L \longrightarrow E \longrightarrow \xi \otimes L^{-1} \longrightarrow 0.$$

Note that the quotient is torsion free (hence a line bundle) because E is stable and L has degree $(d-1)/2$.

Therefore, all vector bundles in T can be constructed by choosing a line bundle L of degree $(d-1)/2$ together with an extension class in $\mathrm{Ext}^1(\xi \otimes L^{-1}, L)$. It follows immediately that T is a closed subset of M with dimension

$$\begin{aligned}\dim(T) &\leq g + h^1(\xi^{-1} \otimes L^2) - 1 = g - \chi(\xi^{-1} \otimes L^2) - 1 \\ &= 2g - 1 < 3g - 3 = \dim(M),\end{aligned}$$

and hence the complement $U := M \backslash T$ is open and nonempty. □

Lemma 1.2. *Choose m distinct points $\{x_1, \ldots, x_m\} \subset X$, with $m > d/2$. Let $E \in M$ be a vector bundle and $0 \neq s \in H^0(E)$ a nontrivial section. Then s cannot simultaneously vanish at all the chosen points $\{x_1, \ldots, x_m\}$.*

Proof. If s vanishes at all chosen points $x_1, \ldots, x_m$, then $s : \mathcal{O} \longrightarrow E$ factors as

$$s : \mathcal{O} \longrightarrow E(-D) \hookrightarrow E,$$

where D is the divisor $D = x_1 + \cdots + x_m$. Since $\deg E(-D) = d - 2m < 0$, the stability condition of E forces s to be the zero section. □

2. Hecke lines

Let $U \subset M$ be the open subset of superstable vector bundles (Lemma 1.1). Take a point $x \in X$. Let $E \in U$ and $l \subset E_x$ a line in the fiber of E at x (equivalently, $l \in \mathbb{P}(E_x)$). Define the vector bundle W by

$$0 \longrightarrow W(-x) \longrightarrow E \longrightarrow E_x/l \longrightarrow 0.$$

The vector bundle $W(-x)$ is called the *Hecke transform* of E with respect to x and l. The exact sequence implies $\bigwedge^2 W \cong \xi \otimes \mathcal{O}(x)$. The vector bundle W is stable. Indeed, a line subbundle L of W is realized as a subline bundle of $E(x)$ using the homomorphism $W \longrightarrow E(x)$: Now the superstability condition of E says

$$\deg(L) < \frac{d+1}{2} = \frac{\deg(W)}{2}.$$

In other words, W is stable.

We can reconstruct back E from W by doing another Hecke transform, and E is given as the middle row of the following commutative diagram:

$$\begin{tikzcd}
 & 0 & 0 & & \\
0 \arrow[r] & E_x/l \arrow[r] \arrow[u] & W_x \arrow[r] \arrow[u] & \mathbb{C}_x \arrow[r] & 0 \\
0 \arrow[r] & E \arrow[r] \arrow[u] & W \arrow[r, "f_0"] \arrow[u] & \mathbb{C}_x \arrow[r] \arrow[u, equal] & 0 \\
 & W(-x) \arrow[r, equal] \arrow[u] & W(-x) \arrow[u] & & \\
 & 0 \arrow[u] & 0 \arrow[u] & &
\end{tikzcd} \tag{1}$$

Here $\mathbb{C}_x$ is the skyscraper sheaf at x with stalk $\mathbb{C}$. Instead of $f_0 : W \longrightarrow \mathbb{C}_x$ we may consider an arbitrary nontrivial homomorphism

$$f \in \operatorname{Hom}(W, \mathbb{C}_x) = \operatorname{Hom}(W_x, \mathbb{C}_x) = W_x^\vee$$

and define E_f as the kernel

$$0 \longrightarrow E_f \longrightarrow W \xrightarrow{f} \mathbb{C}_x \longrightarrow 0. \tag{2}$$

This way we obtain a family of vector bundles parametrized by the projective line $\mathbb{P}(W_x^\vee)$, with $E_{f_0} \cong E$. More precisely, there is a short exact sequence on $X \times \mathbb{P}(W_x^\vee)$,

$$0 \longrightarrow \widetilde{E} \longrightarrow \pi_X^* W \xrightarrow{\tilde{f}} \mathcal{O}_{x \times \mathbb{P}(W_x^\vee)}(1) \longrightarrow 0,$$

where $\pi_X : X \times \mathbb{P}(W_x^\vee) \longrightarrow X$ is the projection to X. It has the property that if $f \in W_x^\vee$ and we restrict the exact sequence to the subvariety $X \times [f] \cong X$ of $X \times \mathbb{P}(W_x^\vee)$, then a sequence isomorphic to (2) is obtained.

For every $f \in W_x^\vee$, the vector bundle E_f is stable. Indeed, if L is a subline bundle of E_f, then by composition with the homomorphism $E_f \longrightarrow W$ in (2) it is a subline bundle of W. The stability condition for W says that $\deg(L) < (d+1)/2$. Since d is odd this is equivalent to

$$\deg(L) \leq \frac{d-1}{2} < \frac{d}{2} = \frac{\deg(E_f)}{2}.$$

Note that if E is stable but not superstable, then W is semistable but not necessarily stable. The semistability condition is not enough to ensure the stability of E_f for each f.

The universal property of the moduli space M gives a morphism $\varphi : \mathbb{P}(W_x^\vee) \longrightarrow M$ for the family $\widetilde{E}$.

DEFINITION 2.1

The data consisting of the pair $(\mathbb{P}(W_x^\vee), \varphi)$ is called the Hecke line associated to the triple (E, x, l).

Since φ is determined by W and $\mathbb{P}(W_x^\vee)$, the projective line $\mathbb{P}(W_x^\vee)$ will also be called a Hecke line. The Hecke line $\mathbb{P}(W_x^\vee)$ will also be denoted by $P_{E,x,l}$ or simply by P if the rest of the data is clear from the context. Note that there is a distinguished point $[f_0] \in \mathbb{P}(W_x^\vee)$ that maps to $E \in M$.

For any $f \in \mathbb{P}(W_x^\vee)$, let l_f denote the kernel of the homomorphism $(E_f)_x \longrightarrow W_x$ of fibers in (2). Clearly, the images of the two Hecke lines $P_{E,x,l}$ and P_{E_f,x,l_f} in M coincide.

Therefore, for each $E \in M$, there is a three parameter family of Hecke lines whose image contains E. On the other hand, if we identify two Hecke lines if their images in M coincide, then through each point of M there is a two parameter family of rational curves defined by Hecke lines.

Since the morphism φ is given by the universal property of the moduli space, the pullback of the universal bundle $\mathcal{E}$ on $X \times M$ to $X \times P$ by the map $\mathrm{id}_X \times \varphi$ is isomorphic (up to a twist by a line bundle coming from P) to $\widetilde{E}$. In other words, there is an integer k such that

$$0 \longrightarrow (\mathrm{id}_X \times \varphi)^*\mathcal{E} \longrightarrow W \boxtimes \mathcal{O}_P(k) \longrightarrow \mathcal{O}_{x \times P}(k+1) \longrightarrow 0 \tag{3}$$

is an exact sequence of sheaves on $X \times P$; $\mathcal{O}_P(1)$ is the tautological line bundle on $P = \mathbb{P}(W_x^\vee)$. Applying $(\pi_P)_*$, where π_P is the projection of $X \times P$ to P, the following sequence

$$0 \longrightarrow \varphi^*\mathcal{P} \longrightarrow H^0(W) \otimes \mathcal{O}_P(k) \longrightarrow \mathcal{O}_P(k+1) \longrightarrow 0 \tag{4}$$

on P is obtained, where $\mathcal{P}$ is the Picard bundle. Since $d \geq 4g - 3$, the stability condition ensures that $H^1(X,\ E')$ vanishes for every $E' \in M$.

Let N denote the rank of $\mathcal{P}$. The following proposition describes the pullback $\varphi^*\mathcal{P}$.

PROPOSITION 2.2

The pullback $\varphi^*\mathcal{P}$ of the Picard bundle $\mathcal{P}$ to the $P = P_{E,x,l}$ satisfies

$$\varphi^*\mathcal{P} \cong \mathcal{O}_P(k)^{\oplus N-1} \oplus \mathcal{O}_P(k-1). \tag{5}$$

Hence $\varphi^*\mathcal{P}$ has a canonical subbundle

$$\mathcal{O}_P(k)^{\oplus N-1} \cong \mathcal{V} \hookrightarrow \varphi^*\mathcal{P}.$$

Let $V \subset H^0(X,\ E)$ be the fiber of this subbundle over the distinguished point $[f_0] \in P$. Then $s \in V$ if and only if $s(x) \in l$.

Proof. Grothendieck's theorem [Gr] says that a vector bundle on $\mathbb{P}^1$ is holomorphically isomorphic to a direct sum of line bundles. Hence

$$\varphi^*\mathcal{P} \cong \mathcal{O}_P(a_1) \oplus \cdots \oplus \mathcal{O}_P(a_N).$$

The sequence (4) gives $h^0(W) = 1 + N$, $\sum a_i = Nk - 1$ and $a_i \leq k$ for all i. Combining these, (5) is obtained immediately.

Now we are going to identify the subbundle $\mathcal{V}$. From (3) the following commutative diagram is obtained

$$\begin{tikzcd}
0 \arrow[r] & (\mathrm{id}_X \times \varphi)^*\mathcal{E} \arrow[r] & W \boxtimes \mathcal{O}_P(k) \arrow[r] & \mathcal{O}_{x \times P}(k+1) \arrow[r] & 0 \\
& W(-x) \boxtimes \mathcal{O}_P(k) \arrow[u] \arrow[r, phantom, "="] & W(-x) \boxtimes \mathcal{O}_P(k) \arrow[u] & & \\
& 0 \arrow[u] & 0 \arrow[u] & &
\end{tikzcd}$$

and applying $(\pi_P)_*$ we obtain the following commutative diagram on P:

$$\begin{tikzcd}
0 \arrow[r] & \varphi^*\mathcal{P} \arrow[r] & H^0(W)\otimes\mathcal{O}_P(k) \arrow[r] & \mathcal{O}_P(k+1) \arrow[r] & 0 \\
& H^0(W(-x))\otimes\mathcal{O}_P(k) \arrow[u] \arrow[r, equal] & H^0(W(-x))\otimes\mathcal{O}_P(k) \arrow[u] & & \\
& 0 \arrow[u] & 0 \arrow[u] & &
\end{tikzcd}$$

Since $\varphi^*\mathcal{P} \cong \mathcal{O}_P(k)^{\oplus N-1} \oplus \mathcal{O}_P(k-1)$, we deduce that

$$\mathcal{V} = H^0(W(-x))\otimes\mathcal{O}_P(k) \subset \varphi^*\mathcal{P}.$$

Let V denote the fiber of $\mathcal{V}$ at the point $[f_0] \in P$. So, $V \subset H^0(E)$. Now, $s \in V$ if and only if $s \in H^0(W(-x)) \subset H^0(E)$. Finally, taking global sections for the diagram (2) it is easy to see that this is equivalent to the condition that $s(x) \in l$. This completes the proof of the proposition. □

Proposition 2.2 has the following corollary.

COROLLARY 2.3

The morphism φ is a nonconstant one.

Indeed, if φ were a constant map, then the vector bundle $\varphi^*\mathcal{P}$ would be trivial.

3. Main theorem

In this section we will prove the main theorem of this paper.

Theorem 3.1. *Let $\mathcal{P}$ be the Picard bundle on the moduli space M of stable bundles of rank two and fixed determinant of odd degree d with $d \geq 4g-3$. Then $\mathcal{P}$ is stable.*

Proof. Since $\mathcal{P}$ is a vector bundle, to check stability it is enough to consider reflexive subsheaves of $\mathcal{P}$. Let

$$\mathcal{F} \longrightarrow \mathcal{P}$$

be a reflexive subsheaf of rank $r < N = \text{rank}(\mathcal{P})$. Fix m distinct points $x_1, \ldots, x_m$ in X, with $m > d/2$.

We need the following lemma for the proof of the theorem.

Lemma 3.2. *There is a nonempty open set of M such that if E is a vector bundle corresponding to a point of that open set, then E has the following four properties*:

(i) *E is superstable*;

(ii) *$\mathcal{F}$ is locally free at E*;

(iii) *$\mathcal{F}_E \to \mathcal{P}_E$ is an injection*;

(iv) *Let x_i be one of the fixed points and l any line on E_{x_i}. Let $P = P_{E,x_i,l}$ be the associated Hecke line. Then $\mathcal{F}$ is locally free at all points of the image of $\varphi : P \longrightarrow M$.*

Proof. The subset U of M where property (i) is satisfied is open and nonempty by Lemma 1.1. Let $U' \subset U$ be the subset where also property (ii) is satisfied and $U'' \subset U'$ the subset where furthermore property (iii) is satisfied. Clearly, U'' is a nonempty open subset of M.

Let $S \subset M$ denote the subvariety where $\mathcal{F}$ is not locally free. Since $\mathcal{F}$ is reflexive, $\operatorname{codim}(S) \geq 3$. Let $\widetilde{S}_i$ be the union of the images of all Hecke lines $P_{E,x_i,l}$, when E runs through all points in S and l runs through all lines of E_x. Then

$$\operatorname{codim}\widetilde{S}_i \geq 3 - 1 - 1 = 1 .$$

Finally consider the union

$$\widetilde{S} := \bigcup_{i=1}^{m} \widetilde{S}_i .$$

Since this is a union of a finite number of subvarieties, we still have $\operatorname{codim}\widetilde{S} \geq 1$. Consequently, $U''' := U'' \bigcap (M \backslash \widetilde{S})$ is nonempty and open. By construction, any vector bundle E corresponding to a point in U''' satisfies conditions (i) to (iv). This finishes the proof of the lemma. □

Continuing the proof of Theorem 3.1, fix a vector bundle E satisfying the four properties in the above lemma. Let $v \in \mathcal{F}_E$ be a nonzero vector in the fiber, and let s be its image in the fiber $\mathcal{P}_E \cong H^0(E)$. It is still nonzero because of property (iii).

From the fixed set of chosen points $\{x_1, \ldots, x_m\}$, pick one of them x_i such that the section s does not vanish at x_i. The existence of such a point is ensured by Lemma 1.2. Let $l \subset E_{x_i}$ be a line such that $s(x_i) \notin l$. Consider the Hecke line $P = P_{E,x_i,l}$ defined with this data.

Note that $\varphi^*\mathcal{F}$ is a vector bundle because $\mathcal{F}$ is locally free on all points of the image of P in M (property (iv)), and $\varphi^*\mathcal{F} \longrightarrow \varphi^*\mathcal{P}$ is injective as a sheaf homomorphism because both $\varphi^*\mathcal{F}$ and $\varphi^*\mathcal{P}$ are vector bundles and property (iii).

The Proposition 2.2 says that $\varphi^*\mathcal{P}$ has a canonical subbundle $\mathcal{V}$ with

$$\mathcal{O}_P(k)^{\oplus N-1} \cong \mathcal{V} \subset \varphi^*\mathcal{P} \cong \mathcal{O}_P(k)^{\oplus N-1} \oplus \mathcal{O}_P(k-1). \tag{6}$$

We can think of v and s as vectors in the fibers of $\varphi^*\mathcal{F}$ and $\varphi^*\mathcal{P}$ at $[f_0]$. Since $s(x_i) \notin l$, Proposition 2.2 also gives that $s \notin V = \mathcal{V}_E$. Consequently,

$$\varphi^*\mathcal{F} \not\subseteq \mathcal{V} . \tag{7}$$

By Grothendieck's theorem

$$\varphi^*\mathcal{F} \cong \mathcal{O}_P(b_1) \oplus \cdots \oplus \mathcal{O}_P(b_r) .$$

Since $\varphi^*\mathcal{F} \longrightarrow \varphi^*\mathcal{P}$ is injective, (6) implies that $b_i \leq k$ for all i, and (7) implies that for some i (say $i = 1$), $b_1 \leq k - 1$.

Fix a polarization L on M. Let δ be the degree of φ^*L. The Corollary 2.3 says that $\delta > 0$. Now,

$$\frac{1}{\delta}\frac{\deg(\mathcal{F})}{\operatorname{rank}(\mathcal{F})} = \frac{\deg(\varphi^*\mathcal{F})}{\operatorname{rank}(\varphi^*\mathcal{F})} \leq k - \frac{1}{r} < k - \frac{1}{N} = \frac{\deg(\varphi^*\mathcal{P})}{\operatorname{rank}(\varphi^*\mathcal{P})} = \frac{1}{\delta}\frac{\deg(\mathcal{P})}{\operatorname{rank}(\mathcal{P})}$$

and hence the Picard bundle $\mathcal{P}$ is stable. This completes the proof of the theorem. □

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Principal G-bundles on nodal curves

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Abstract. Let G be a connected semisimple affine algebraic group defined over $\mathbf{C}$. We study the relation between stable, semistable G-bundles on a nodal curve Y and representations of the fundamental group of Y. This study is done by extending the notion of (generalized) parabolic vector bundles to principal G-bundles on the desingularization C of Y and using the correspondence between them and principal G-bundles on Y. We give an isomorphism of the stack of generalized parabolic bundles on C with a quotient stack associated to loop groups. We show that if G is simple and simply connected then the Picard group of the stack of principal G-bundles on Y is isomorphic to $\oplus_m \mathbf{Z}$, m being the number of components of Y.

Keywords. Principal bundles; loop groups; parabolic bundles.

0. Introduction

Let G be a connected semisimple affine algebraic group defined over $\mathbf{C}$. Let Y be a reduced curve with only singularities ordinary nodes y_j, $j = 1, \ldots, J$. Let Y_i, $i = 1, \ldots, I$ be the irreducible components of Y and C_i the desingularization of Y_i. Let C denote the disjoint union of all C_i. We introduce the notions of stability and semistability for principal G-bundles on Y (§2). If Y is reducible these notions depend on parameters $a = (a_1, \ldots, a_I)$. The study of G-bundles on Y is done by extending the notion of (generalized) parabolic vector bundles [U1] to generalized parabolic principal G-bundles (called GPGs in short) on the curve C and using the correspondence between them and principal G-bundles on Y (2.4, 2.11). We study the relation between stable, semistable G-bundles and representations of the fundamental group of Y. Let $\rho : \pi_1(Y) \to G$ be a representation of the fundamental group $\pi_1(Y)$ of Y in G. For $i = 1, \ldots, I$, let $f_i : \pi_1(Y_i) \to \pi_1(Y)$ be the natural maps, $\rho_i = \rho \circ f_i$.

Theorem 1. (I) *If Y is irreducible and $\rho \mid \pi_1(C)$ is unitary* (*resp. irreducible unitary*) *then the principal G-bundle on Y associated to ρ is semistable* (*resp. stable*). *The converse is not true.*
(II) *If Y is reducible then there exist infinitely many I-tuples of positive rational numbers $a_1, \ldots, a_I$ with $\sum a_i = 1$, depending only on the graph of Y and $g(C_i)$ such that for $a = (a_1, \ldots, a_I)$ the following statements are true.*

(1) *If $\rho_{C_i} = \rho_i \mid \pi_1(C_i)$ are unitary representations for all i, then the principal G-bundle $\mathcal{F}$ on Y associated to ρ is a-semistable.*

(2) *If ρ_{C_i} are irreducible unitary representations for all i, then the principal G-bundle $\mathcal{F}$ associated to ρ is a-stable.*

Let Aff/k be the flat affine site over the base field $k = \mathbf{C}$, i.e. the category of k-algebras equipped with $fppf$ topology. Let R denote a k-algebra, $C_{i,R} := C_i \times$ spec R and $C^*_R = C^* \times$ spec R. For each i, fix a point $p_i \in C_i$ such that p_i maps to a smooth point of Y. Let q_i be a local parameter at the point $p_i, i = 1, \ldots, I$. Let $L_{G,i}$ denote the k-group defined by associating to R the group $G(R(q_i))$. Let $L^+_{G,i}$ (resp. $L^{C_i}_G$) be the k-group defined by associating to R the group $G(R[[q_i]])$ (resp. $G(\Gamma(C^*_{i,R}, \mathcal{O}_{C^*_{i,R}}))$). Define $L_G = \prod_i L_{G,i}$, $L^+_G = \prod_i L^+_{G,i}$, $L^C_G = \prod_i L^{C_i}_G$. Let

$$Q_{G,C} = L_G/L^+_G = \prod_i L_{G,i}/L^+_{G,i}, \quad Q^{\mathrm{gpar}}_{G,C} = Q_{G,C} \times \prod_j G(\mathbf{C}).$$

The indgroup L^C_G acts on $Q^{\mathrm{gpar}}_{G,C}$. Let $L^C_G \backslash Q^{\mathrm{gpar}}_{G,C}$ be the quotient stack. Let $\mathrm{Bun}^{\mathrm{gpar}}_{G,C}$ denote the stack of GPGs on C (this is isomorphic to the stack of principal G-bundles on Y.)

Theorem 2. *There exists a canonical isomorphism of stacks*

$$\pi_{\mathrm{par}} : L^C_G \backslash Q^{\mathrm{gpar}}_{G,C} \xrightarrow{\sim} \mathrm{Bun}^{\mathrm{gpar}}_{G,C}.$$

Moreover the projection map $Q^{\mathrm{gpar}}_{G,C} \to \mathrm{Bun}^{\mathrm{gpar}}_{G,C}$ is locally trivial for etale topology.

Theorem 3. *If G is a simple, connected and simply connected affine algebraic group then* (1)

$$\mathrm{Pic}(\mathrm{Bun}^{\mathrm{gpar}}_{G,C}) \approx \oplus_i \mathbf{Z}.$$

(2) *If Y is irreducible and C has genus ≥ 2, then*

$$\mathrm{Pic}(\mathrm{Bun}^{\mathrm{gpar}}_{G,C})^{ss} \approx \mathbf{Z},$$

where ^{ss} denotes semistable points.

The moduli spaces of principal G-bundles on singular curves are not complete. In case $G = GL(n)$ (resp. $G = O(n), Sp(2n)$) the compactifications of these moduli spaces were constructed as moduli spaces of torsionfree sheaves (resp. orthogonal or symplectic sheaves) on Y. For a general reductive group G neither the moduli spaces nor the compactifications have been constructed on Y yet. One way to construct (normal) compactifications of these moduli spaces is to use GPGs on C, for this one needs a good compactification of G. In case G is $GL(n)$, $SL(n)$, $O(n)$ or $Sp(2n)$ we use a compactification F of G obtained by using the natural representation and construct the normal compactifications of moduli spaces ([U1, U2, U4]). In case G is of adjoint type we use the good compactification F of G defined by Deconcini and Procesi. We define 'a compactification' $\overline{\mathrm{Bun}}^{\mathrm{gpar}}_{G,C}$ of $\mathrm{Bun}^{\mathrm{gpar}}_{G,C}$ using F and show that it is isomorphic to the quotient stack $L^C_G \backslash Q_{G,C} \times \prod_j F$. We prove that if further G is simple and simply connected then Pic $\overline{\mathrm{Bun}}^{\mathrm{gpar}}_{G,C} \approx \oplus_i \mathbf{Z} \oplus \oplus_j \mathrm{Pic}\, F$ (Theorem 4).

1. Quasiparabolic bundles

1.1. *Notations.* Let the base field be $\mathbf{C}$ (or an algebraically closed field of characteristic 0). Let I, J be natural numbers. Let Y be a connected reduced (projective) curve with ordinary nodes as singularities. Let Y_i, $i = 1, \dots, I$ be the irreducible components of Y. Let $Y' = Y - \{\text{singular set of } Y\}$, $Y_i' = Y' \cap Y_i$ for all i. Let C be the partial desingularization of Y obtained by blowing up nodes y_j, $j = 1, \dots, J$. Assume that $C = \coprod_1^I C_i$ (a disjoint union). Let $C_i' = C_i - \text{sing}(C_i)$. Fix an orientation of the (dual) graph of Y. In the graph of Y, y_j corresponds to an edge. The initial and terminal points of the edge correspond to curves $Y_{i(j)}$ and $Y_{t(j)}$ respectively, one has $i(j) = t(j)$ if the edge is a loop. Let $x_j \in C_{i(j)}$ and $z_j \in C_{t(j)}$ be the two points of C mapping to $y_j \in Y$ and $D_j = x_j + z_j$, $j = 1, \dots, J$. For each j, D_j is an effective Cartier divisor on C supported outside the singular set of C. We remark that the parabolic structure we shall define in 1.2, 1.4 depends only on these divisors and not on the choice of orientation. Let G denote an affine connected semisimple algebraic group over $\mathbf{C}$ (or an algebraically closed field of characteristic zero). Let $\mathbf{g}$ denote the Lie algebra of G, $n = \dim \mathbf{g}$. A principal G-bundle E on C is an I-tuple (E_i), E_i being a principal G-bundle on C_i.

DEFINITION 1.2

A quasiparabolic structure σ_j on E over the divisor D_j consists of a G-isomorphism $\sigma_j : E_{i(j),x_j} \to E_{t(j),z_j}$ where $E_{i,x}$ denotes the fibre of E_i at x. Let σ be the J-tuple $(\sigma_j)_j$, then (E, σ) is called a quasiparabolic G-bundle, called a QPG in short.

Remark 1.3. A family $(\mathcal{E}, (\sigma_j))$ of QPGs consists of a family of principal G-bundles $\mathcal{E} \to C \times T$ together with an isomorphism of G-bundles $\sigma_j : \mathcal{E} |_{x_j \times T} \to \mathcal{E} |_{z_j \times T}$ for each $j = 1, \dots, J$. Given a family of QPGs $(\mathcal{E}, (\sigma_j)) \to C \times T$ and a representation $\rho : G \to GL(V)$ one can associate to it a family $(\mathcal{E}(V), F_j(V)) \to C \times T$ of generalized parabolic vector bundles [U1] as follows. $\mathcal{E}(V) = \mathcal{E} \times_\rho V$ is a family of vector bundles. For each j, σ_j induces $\sigma_{V,j} : \mathcal{E}(V) \mid x_j \times T \to \mathcal{E}(V) \mid z_j \times T$. Let $F_j(V) =$ graph of $\sigma_{V,j}$ in $\mathcal{E}(V) \mid x_j \times T \oplus \mathcal{E}(V) \mid z_j \times T$. Then $F_j(V)$ and $Q_j(V) = (\mathcal{E}(V) \mid x_j \times T \oplus \mathcal{E}(V) \mid z_j \times T)/F_j(V)$ are vector bundles on T of rank = dim V.

1.4. Let α be a real number, $0 \leq \alpha \leq 1$. Taking ρ the adjoint representation of G we get the associated vector bundle $E(\mathbf{g})$. Then $E(\mathbf{g})$ is the adjoint bundle of E and we often denote it by AdE. The isomorphism σ_j gives an isomorphism $E(\mathbf{g})_{x_j} \to E(\mathbf{g})_{z_j}$ and hence determines an n-dimensional subspace of $E(\mathbf{g})_{x_j} \oplus E(\mathbf{g})_{z_j} = \mathbf{g} \oplus \mathbf{g}$ again denoted by σ_j. Let $\tau_j \in \text{End}_{\mathbf{C}}(\mathbf{g} \oplus \mathbf{g})$ such that τ_j acts on σ_j by $\alpha.Id$ and τ_j restricted to a complement of σ_j in $\mathbf{g} \oplus \mathbf{g}$ is zero. With respect to a suitable basis, $\tau_j = \begin{pmatrix} \alpha I_n & 0 \\ 0 & 0 \end{pmatrix}$, I_n being the unit matrix of rank n. We fix a conjugacy class of τ_j. (This is an analogue of weights in case of (generalized) parabolic vector bundles, the weights in this case being $(0, \alpha)$ for the vector bundle $E(\mathbf{g})$ with induced (generalized) parabolic structure).

We want to define the notions of stability and semistability for QPGs. Since the definitions are rather complicated in the general case, we first define these notions on an irreducible smooth curve C (1.5, 1.6) and later extend these notions to the general case (1.7, 1.8, 1.9).

Assume that C is a nonsingular irreducible curve. Let P be a maximum parabolic subgroup of G and $\mathbf{p}$ its Lie algebra. Let $E/P = E(G/P)$ be the associated fibre bundle with fibres isomorphic to G/P. Let $s : C \to E/P$ be a section i.e. a reduction of the structure group to the maximum parabolic subgroup P. Let Q_j be the stabilizer in

$GL(\mathbf{g} \oplus \mathbf{g})$ of the subspace $E(\mathbf{p})_{x_j} \oplus E(\mathbf{p})_{z_j} = \mathbf{p} \oplus \mathbf{p} \subset \mathbf{g} \oplus \mathbf{g} = E(\mathbf{g})_{x_j} \oplus E(\mathbf{g})_{z_j}$. Let μ_j denote the determinant of the action of Q_j on $\mathbf{g}/\mathbf{p} \oplus \mathbf{g}/\mathbf{p}$. Let $\overline{\mu}_j$ be the form on the Lie algebra $L(Q_j)$ of Q_j corresponding to μ_j. Let $\overline{\tau}_j$ be a conjugate of τ_j in $L(Q_j)$.

DEFINITION 1.5

A QPG $(E, (\sigma_j))$ is α-stable (resp. α-semistable) if for every maximum parabolic P of G and every reduction $s : C \to E/P$, one has

$$\text{degree}\, s^*T(G/P) + \sum_j \overline{\mu}_j(\overline{\tau}_j) > (\text{resp. } \geq)\, \alpha J.\ \text{rank}\, s^*T(G/P). \qquad (*1)$$

Here $T(G/P)$ is the tangent bundle along the fibres of $E/P \to C$.

Lemma 1.6. *With the above notations, the condition* $(*1)$ *is equivalent to the following*

$$\text{par deg}\, E(\mathbf{p}) < (\text{resp. } \leq)\, \alpha J.\ \text{rank}\, E(\mathbf{p}), \qquad (*2)$$

where par deg $E(\mathbf{p})$ *denotes the parabolic degree of the subbundle* $E(\mathbf{p})$ *of the (generalized) parabolic vector bundle* $(E(\mathbf{g}), (\sigma))$ *with weights* $(0, \alpha)$, *each weight being of multiplicity* n.

Proof. One has $s^*T(G/P) = E(\mathbf{g}/\mathbf{p})$, $\sum_j \overline{\mu}_j(\overline{\tau}_j)$ = parabolic weight of the quotient bundle $E(\mathbf{g}/\mathbf{p})$ of $(E(\mathbf{g}), (\sigma_j))$. Thus $(*1)$ can be restated as par deg$E(\mathbf{g}/\mathbf{p}) >$ (resp. $\geq$) αJ rank$E(\mathbf{g}/\mathbf{p})$. Since G is semisimple, deg$E(\mathbf{g}) = 0$ ([R1], Remark 2.2) and hence par deg$E(\mathbf{g}) = \alpha J$ rank $E(\mathbf{g})$. The result now follows from the exact sequence $0 \to E(\mathbf{p}) \to E(\mathbf{g}) \to E(\mathbf{g}/\mathbf{p}) \to 0$ using the additivity of parabolic degrees for exact sequences.

1.7. *Semistable QPGs on reducible curves*. Let the notation be as in 1.1. We consider QPGs $(E, (\sigma_j))$ on C with parabolic structure over $D_j = x_j + z_j$, $j = 1, \ldots, J$. Let $\{\sigma_j\}, \{\tau_j\}, a, \alpha$ be as in 1.4. For $i = 1, \ldots, I$ let P_i denote either a maximum parabolic subgroup of G or the trivial group e or the group G itself. We need to consider the cases $P = \{e\}$ or G also, because a sub-object $N = (N_i)$ of $E = (E_i)$ may have the property that for some i, $N_i = E_i$ or N_i is trivial. For an I-tuple $P = (P_1, \ldots, P_I)$, let $r_i = \dim\ \mathbf{p}_i$, $n_i = \dim\ \mathbf{g}/\mathbf{p}_i$ for all i. For $j = 1, \ldots, J$ denote by Q_j the stabilizer in $GL(\mathbf{g} \oplus \mathbf{g})$ of the subspace $\mathbf{p}_{i(j)} \oplus \mathbf{p}_{t(j)} \subseteq \mathbf{g} \oplus \mathbf{g}$. Let μ_j be the determinant of the action of Q_j on $\mathbf{g}/\mathbf{p}_{i(j)} \oplus \mathbf{g}/\mathbf{p}_{t(j)}$ and $\overline{\mu}_j$ the form on the Lie algebra $L(Q_j)$ of Q_j corresponding to μ_j. Let $\overline{\tau}_j$ be a conjugate of τ_j in $L(Q_j)$. Let $C_i' = C_i$-sing(C_i), $s_i : C_i' \to E(G/P_i) \mid C_i'$ any section, $s = (s_1, \ldots, s_I)$. Let S_{s_i} = the largest subsheaf of Ad $E \mid C_i$ such that $S_{s_i} \mid C_i' = s_i^*(E(\mathbf{p}_i)) \mid C_i'$. Let $S_s = (S_{s_1}, \ldots, S_{s_I})$, $\chi(S_s) = \sum_i \chi(S_{s_i})$, $\chi(\text{Ad } E) = \sum_i \chi(\text{Ad } E \mid C_i)$. Let Q_{s_i} be the (smallest) torsion free quotient sheaf of Ad $E \mid C_i$ with $Q_{s_i} \mid C_i' = s_i^*(E(\mathbf{g}/\mathbf{p}_i)) \mid C_i'$. Let $Q_s = (Q_{s_i})_i$, $\chi(Q_s) = \sum_i \chi(Q_{s_i})$.

DEFINITION 1.8.

A QPG $(E, (\sigma_j))$ is (a, α)-semistable (resp. (a, α)-stable) if for every reduction s of the structure group to P such that $P_i \neq G$ for all i and $P_i \neq \{e\}$ for all i one has

$$\left[\chi(Q_s)+\sum_j \overline{\mu}_j(\overline{\tau}_j)-\alpha\sum_j (n_{i(j)}+n_{t(j)})\right]\Big/\sum_i a_i n_i \geq (>)\chi(\mathrm{Ad}\,E)/n-\alpha J. \quad (*1')$$

Lemma 1.9. (a) *The condition* $(*1')$ *above is equivalent to the following condition*

$$\left[\chi(S_s)-\alpha\sum_j q_j(S_s)\right]\Big/\sum_i a_i r_i \leq (<)\chi(\mathrm{Ad}\;E)/n-\alpha J, \qquad (*2')$$

where $q_j(S_s)=r_{i(j)}+r_{t(j)}-\dim\left(\sigma_j\cap\left((S_s)_{i(j),x_j}\oplus(S_s)_{t(j),z_j}\right)\right)$.
(b) *If* C *is irreducible and smooth, then* $(*2')$ *is same as* $(*2)$.

Proof. (a) The quotient Q_s of $E(\mathbf{g})$ has induced parabolic structure over D_j, $j=1,\ldots,J$ given by $(Q_s)_{x_j}\oplus(Q_s)_{z_j}\supset F_j(Q_s)\supset 0$ with weights $(0,\alpha)$, where $F_j(Q_s)$ is the image of the n-dimensional subspace σ_j of $(E(\mathbf{g})_{x_j}\oplus E(\mathbf{g})_{z_j})$ in $((Q_s)_{x_j}\oplus(Q_s)_{z_j})$. Let $f_j(Q_s)=\dim F_j(Q_s)$. By definition, the parabolic weight of $Q_s=\alpha\sum_j f_j(Q_s)$. Define $q_j(Q_s)=n_{i(j)}+n_{t(j)}-f_j(Q_s)$, it is additive for exact sequences. Then one has

$$\begin{aligned}\text{parabolic weight of } Q_s &= \alpha\sum_j(n_{i(j)}+n_{t(j)}-q_j(Q_s))\\ &= \alpha\sum_j(n_{i(j)}+n_{t(j)}-q_j(\mathrm{Ad}\;E)+q_j(S_s))\\ &= \alpha\sum_j(n_{i(j)}+n_{t(j)}-n+q_j(S_s)).\end{aligned}$$

Note that since $Q_s\approx E(\mathbf{g}/\mathbf{p})$ outside sing(C) and all D_j avoid sing(C), one has parabolic weight of Q_s = the parabolic weight of $E(\mathbf{g}/\mathbf{p})=\sum_j\overline{\mu}_j(\overline{\tau}_j)$. Hence, $\sum_j\overline{\mu}_j(\overline{\tau}_j)-\sum_j\alpha(n_{i(j)}+n_{t(j)})=\alpha\sum_j q_j(S_s)-\alpha Jn$. Using this equality and $\sum a_i r_i=n-\sum a_i n_i$ the first part of the Lemma follows.
(b) If C is a smooth irreducible curve then one has $I=1$, $S_s=E(\mathbf{p})$, $Q_s=E(\mathbf{g}/\mathbf{p})$, $\Sigma a_i r_i=r_1$, $\chi(S_s)=\deg(E(\mathbf{p}))+r_1(1-g)$, $\sum_j\alpha q_j(S_s)=\sum_j\alpha(2r_1-\dim F_j(S_s)=2\alpha Jr_1-$ parabolic weight (S_s). Hence the left hand side of $(*2')$ becomes equal to par deg $E(\mathbf{p})$/rank $E(\mathbf{p})-2\alpha J+(1-g)$. The right hand side of $(*2')=(1-g)-\alpha J$. Hence the result follows.

2. Principal G-bundles on a singular curve Y

2.1. We want to introduce the notions of stability and semistability for principal G-bundles on singular curves. On a smooth curve there are different definitions of stability and semistability of a principal G-bundle, but they all coincide [R1]. The problem is that this is not true on a singular curve. The choice of a representation of G used to define semistability does not matter on a smooth curve essentially because the associated bundles (tensor products etc.) of semistable vector bundles (in characteristic 0) are semistable. This fails if the curve has singularities. For example, if F_1 is the semistable vector bundle of rank 2, degree 0 (on an irreducible nodal curve Y) constructed in Proposition 2.7 of

[U3] then $F_1 \otimes F_1$ and S^2F_1 are not semistable [U5]. This is seen by checking that the corresponding generalized parabolic vector bundles on C are not semistable. Similarly one can show that if F_2 is the stable vector bundle of rank $2m$ constructed in Proposition 2.9, [U3] then $F_1 \otimes F_2$ is not semistable for all $m \geq 2$ [U5].

We give here a notion of semistability for principal G-bundles on singular curves (see Definitions 2.2, 2.3, 2.9, 2.10) which is intrinsic and seems most useful. We first assume that Y is irreducible (the case of a reducible curve will be dealt with later). Let $Y' = Y -$ {singular set of Y}, $i : Y' \to Y$ inclusion map. Let G be a connected reductive algebraic group. Let P be a maximum parabolic subgroup of G and $\mathbf{p}$ the Lie algebra of P. Let $\mathcal{F}$ be a principal G-bundle on Y and $\mathcal{F}/P = \mathcal{F}(G/P)$ the associated fibre bundle with fibres isomorphic to G/P. Let $s' = Y' \to (\mathcal{F}/P) \mid Y'$ be a reduction of the structure group to P (i.e. a section of $\mathcal{F}/P$ restricted to Y'). Let $T(G/P)$ denote the tangent bundle along the fibres of $\mathcal{F}/P \to Y$. Let $Q_{s'}$ be a torsion free quotient of $\mathcal{F}(\mathbf{g})$ such that $Q_{s'} \mid Y' \cong (s')^*(T(G/P)) \mid Y'$ and no further quotient of $Q_{s'}$ has this property. Let $S_{s'}$ be the maximum subsheaf of $\mathcal{F}(\mathbf{g})$ containing $(s')^*\mathcal{F}(\mathbf{p})$.

DEFINITION 2.2

$\mathcal{F}$ is *stable* (resp. *semistable*) if for every reduction s' of the structure group to a maximum parabolic P (over Y'), one has degree $Q_{s'} > 0$ (resp. ≥ 0).

Lemma 2.3. *The above definition is equivalent to the following:* $\mathcal{F}$ *is stable* (*resp. semistable*) *if for every* s' *as above,* degree $S_{s'} < 0$ (*resp.* ≤ 0).

Proof. The exact sequence $0 \to \mathbf{p} \to \mathbf{g} \to \mathbf{g}/\mathbf{p} \to 0$ gives an exact sequence $0 \to s'^*\mathcal{F}(\mathbf{p}) \to \mathrm{Ad}\mathcal{F} \mid Y' \to s'^*T(G/P) \to 0$ and hence $0 \to S_{s'} \to \mathrm{Ad}\mathcal{F} \to Q_{s'} \to 0$. Noting that Ad $\mathcal{F}$ has degree zero, the lemma follows.

We now assume that Y has only ordinary nodes $y_1, \ldots, y_J$ as singularities and $p : C \to Y$ is the normalization map, $D_j = p^{-1}(y_j) = x_j + z_j$, $j = 1, \ldots, J$. Then giving a principal G-bundle $\mathcal{F}$ on Y is equivalent to giving the principal G-bundle $p^*\mathcal{F} = E$ on C together with a G-isomorphism σ_j of the fibres E_{x_j} and E_{z_j} of E for each j. The isomorphisms σ_j induce isomorphisms $E(\mathbf{g})_{x_j} \to E(\mathbf{g})_{z_j}$. We denote the graph of these isomorphisms also by σ_j.

PROPOSITION 2.4

$(E, (\sigma_j))$ *is* 1-*stable* (*resp.* 1-*semistable*) *if and only if the corresponding G-bundle* $\mathcal{F}$ *on* Y *is stable* (*resp. semistable*).

Proof. Suppose that $\mathcal{F}$ is stable (resp. semistable). Let $s : C \to E/P$ be a reduction to a maximum parabolic subgroup P. Since $C - \cup_j D_j \approx Y - \cup_j y_j$, under p and $E \approx p^*\mathcal{F}$, the section s gives a reduction $s' : Y' = Y - \cup_j y_j \to (\mathcal{F}/P)\,|_{Y'}$. One has the exact sequences $0 \to \mathcal{F}(\mathbf{g}) \to p_*s^*E(\mathbf{g}) \to \oplus_j Q_j E(\mathbf{g}) \to 0$, $0 \to S_{s'} \to p_*s^*E(\mathbf{p}) \to \oplus_j Q_j E(\mathbf{p}) \to 0$ where $Q_j(E(\mathbf{g})) = (s^*E(\mathbf{g})_{x_j} \oplus s^*E(\mathbf{g})_{z_j})/\sigma_j$, $Q_j E(\mathbf{p}) = (s^*E(\mathbf{p})_{x_j} \oplus s^*E(\mathbf{p})_{z_j})/(\sigma_j \cap (s^*E(\mathbf{p})_{x_j} \oplus s^*E(\mathbf{p})_{z_j}))$. Note that the quotient of $\mathcal{F}(\mathbf{g})$ by $S_{s'}$ is the torsion free sheaf obtained from $s^*E(\mathbf{g}/\mathbf{p})$ with induced parabolic structure (viz. the image of σ_j in $E(\mathbf{g}/\mathbf{p})_{x_j} \oplus E(\mathbf{g}/\mathbf{p})_{z_j}, \forall j$). The second sequence implies that par deg $s^*E(\mathbf{p}) - J$ rank

$s^*E(\mathbf{p}) = \deg\,(S_{s'})$. Since $\mathcal{F}$ is stable, $\deg(S_{s'}) < 0$. The result follows from Lemma 1.6. The converse follows similarly working backwards in the above argument. One has only to note that if $s' : Y' \to (\mathcal{F}/P)\,|'_Y$ is a reduction to a maximum parabolic P, then s' gives a reduction $s : C \to E/P$ (as G/P is complete). In case of semistability one has to replace strict inequalities in the above proof by inequalities.

2.5 *Bundles associated to representations*

The fundamental group $\pi_1(Y)$ of Y is isomorphic to $H = \pi_1(C) * \mathbf{Z} * \ldots * \mathbf{Z}$, a free product of $\pi_1(C)$ and J copies of $\mathbf{Z}$ (3.5, [U3]). To a representation $\rho : H \to G$ we associate a QPG $(E_\rho, (\sigma_j))$ as follows. E_ρ is the principal G-bundle on C associated to the representation $\rho_C = \rho \mid \pi_1(C)$. If $\widetilde{C}$ is the universal covering of C, then $E_\rho = \widetilde{C} \times_\rho G$. Fixing suitably points x'_j, z'_j of $\widetilde{C}$ lying over x_j, z_j respectively, the fibres $(E_\rho)_{x_j}$ and $(E_\rho)_{z_j}$ can be identified to G. Let $g_j = \rho(1_j)$, 1_j denoting the generator of the jth factor $\mathbf{Z}$ in H. Then g_j gives an isomorphism $h'_j : (E_\rho)_{x_j} \cong (E_\rho)_{z_j}$ and hence $h_j : (E_\rho(\mathbf{g}))_{x_j} \cong (E_\rho(\mathbf{g}))_{z_j}$. Define $\sigma_j =$ graph of h_j. If $\mathcal{F}$ is the principal G-bundle on Y obtained by identifying fibres of E_ρ at x_j and z_j by $g_j \forall_j$, then one has $\mathcal{F} = \mathcal{F}_\rho$, the G-bundle associated to the representation ρ of $\pi_1(Y)$ and $E_\rho = p^*\mathcal{F}_\rho$.

PROPOSITION 2.6

If ρ_C is irreducible unitary (resp. unitary) then $\mathcal{F}_\rho$ is stable (resp. semistable).

Proof. If ρ_C is unitary, so is $\mathrm{Ad}\circ\rho_C$ and hence $\mathcal{F}_{\mathrm{Ad}\circ\rho} = \mathcal{F}_\rho(\mathbf{g})$ is semistable ([U3], Proposition 2.5). Therefore $\mathcal{F}_\rho$ is semistable.

If ρ_C is irreducible unitary, then by Theorem 7.1 of [R1] (in our case $E(\rho, c) = E_\rho$, $c = Id$) E_ρ is a stable G-bundle. We check below that $(E_\rho, (\sigma_j))$ is 1-stable, then $\mathcal{F}_\rho$ is stable by Proposition 2.4. Let s be a reduction of the structure group of E_ρ to a maximum parabolic subgroup P. The stability of E_ρ implies that $\deg\,(s^*E_\rho(\mathbf{p})) < 0$. Note that σ_j maps isomorphically onto $(E_\rho)_{x_j}$, $j = 1, \ldots, J$. Hence $\sigma_j(E_\rho(\mathbf{p})) = \sigma_j \cap (E_\rho(\mathbf{p})_{x_j} \oplus E_\rho(\mathbf{p})z_j)$ maps injectively into $E_\rho(\mathbf{p})_{x_j}$. Therefore $\dim\,\sigma_j(E_\rho(\mathbf{p})) \leq \mathrm{rank}\,(E_\rho(\mathbf{p}))$ for all j. It follows that $\mathrm{par}\deg\,(s^*E_\rho(\mathbf{p})) = \deg\,(s^*E_\rho(\mathbf{p})) + \sum_j \dim\sigma_j(E_\rho(\mathbf{p})) < J\,\mathrm{rank}\,(E_\rho(\mathbf{p}))$. Thus $(E_\rho, (\sigma_j))$ is 1-stable.

Remark 2.7. There may exist stable principal G-bundles on Y which are not associated to any representations of $\pi_1(Y)$. For examples in case $G = GL(n)$ see [U3], similar examples can be constructed in case $G = O(n)$, $Sp(2n)$ also.

Principal G-bundles on a reducible curve Y

Notations 2.8. Let the notation be as in 1.1. Assume further that Y has nodes y_j, $j = 1, \ldots, J$ as only singularities. Let Γ be the graph obtained from the (dual) graph of Y by omitting loops. Let $y_1, \ldots, y_K$ be the nodes of Y such that each y_j lies on two different components of Y. Then $K =$ the number of edges of Γ, $I =$ the number of vertices of Γ.

For $i = 1, \ldots, I$, let P_i denote either a maximum parabolic subgroup of G or the trivial group $\{e\}$ or the group G itself. Let $\mathcal{F}$ denote a principal G-bundle on Y. For each i, let $s'_i : Y'_i \to \mathcal{F}(G/P_i)\,|_{Y'_i}$ be a section. Let $P = (P_i)_i$, $s' = (s'_i)_i$ be I-tuples. We call s' a *reduction of the structure group to* P over Y'. Let $T(G/P_i)$ denote the tangent bundle along the fibres of $\mathcal{F}(G/P_i)\mid_{Y_i}$. If $P_i = \{e\}$ then $s_i^{\prime *}(T(G/P_i)) \approx \operatorname{Ad}\mathcal{F}\mid_{Y_i'}$. If $P_i = G$, then $\mathcal{F}(G/P_i)\mid_{Y_i} \approx Y_i$ and the Euler characteristic $\chi(s_i^{\prime *}(T(G/P_i))) = 0$. Let $Q_{s'}$ be the smallest torsionfree quotient of Ad $\mathcal{F}$ such that $Q_{s'}\mid_{Y_i'} \approx s_i^{\prime *}(T(G/P))\mid_{Y_i'}$ for all i. Let $\mathbf{p}_i$ denote the Lie algebra of P_i and $\mathcal{F}(\mathbf{p}_i), \mathcal{F}(\mathbf{g}), \mathcal{F}(\mathbf{g}/\mathbf{p}_i)$ the fibre bundles (with fibres $\mathbf{p}_i, \mathbf{g}, \mathbf{g}/\mathbf{p}_i$ respectively) associated to the P_i-bundle $\mathcal{F} \to \mathcal{F}(G/P_i)$ via the adjoint representation. Thus $s_i^{\prime *}\mathcal{F}(\mathbf{g}) = \operatorname{Ad}\mathcal{F}\mid_{Y_i'}, s_i^{\prime *}\mathcal{F}(\mathbf{g}/\mathbf{p}_i) = s_i^{\prime *}T(G/P_i)$. Let $S_{s'}$ be the maximum subsheaf of Ad $\mathcal{F}$ such that $S_{s'}\mid_{Y_i'} \approx s'^{*}\mathcal{F}(\mathbf{p})$. Let $a = (a_1, \ldots, a_I)$, where $\{a_i\}$ are positive rational numbers with $\sum a_i = 1$. Recall that for a vector bundle V on Y, a-rank $V = \sum_i a_i \operatorname{rank}(V\mid_{Y_i})$.

DEFINITION 2.9

The principal G-bundle $\mathcal{F}$ on Y is a-semistable (resp. a-stable) if for every reduction s' of the structure group to P with $P_i \neq \{e\}$ for all i and $P_i \neq G$ for all i one has (in the notations of 2.8)

$$\chi(Q_{s'})/a - \operatorname{rank} Q_{s'} \geq (\text{resp.} >) \chi(\operatorname{Ad}\mathcal{F})/a - \operatorname{rank}\operatorname{Ad}\mathcal{F}.$$

Lemma 2.10. *$\mathcal{F}$ is a-semistable (resp. a-stable) if for every reduction s' as above,*

$$\chi(S_{s'})/a - \operatorname{rank} S_{s'} \leq (\text{resp.} <) \chi(\operatorname{Ad}\mathcal{F})/a - \operatorname{rank}\operatorname{Ad}\mathcal{F}.$$

Proof. As in Lemma 2.3, we have the exact sequences $0 \to s_i^{\prime *}\mathcal{F}(\mathbf{p}) \to \operatorname{Ad}\mathcal{F} \mid Y_i' \to s_i^{\prime *}T(G/P) \to 0$ for all i and so $0 \to S_{s'} \to \operatorname{Ad}\mathcal{F} \to Q_{s'} \to 0$. The lemma follows using the fact that both the Euler characteristic and a-rank are additive for an exact sequence.

PROPOSITION 2.11

For $i = 1, \ldots, I$, let C_i be a partial desingularization of Y_i and $C = \coprod C_i$. Suppose that C is obtained by blowing up nodes $y_1, \ldots, y_{J'}$, $J' \leq J$ of Y. Let $(E, (\sigma_j))$ denote a QPG with quasi-parabolic structure σ_j over D_j, $1 \leq j \leq J'$. Then a QPG $(E, (\sigma_j))$ is $(a, 1)$-stable (resp. $(a, 1)$-semistable) if and only if the corresponding principal G-bundle on Y (obtained by identifying fibres of E by σ_j) is a-stable (resp. a-semistable).

Proof. The proof is exactly on same lines as that of Proposition 2.4. Starting with $\mathcal{F}$ a-stable (resp. semistable) and a reduction s' to P, one gets an exact sequence $0 \to S_{s'} \to p_* S_s \to \oplus_j Q_j(S_s) \to 0$, with $q_j(S_s) = \dim Q_j(S_s)$. Then Lemma 1.9 gives $(a, 1)$-stability (resp. semistability) of $(E, (\sigma_j))$. The converse is proved by reversing the argument.

2.12. *G-bundles associated to representations*

Let $\rho : \pi_1(Y) \to G$ be a representation of the fundamental group $\pi_1(Y)$ of Y in G. For $i = 1, \ldots, I$, let $f_i : \pi_1(Y_i) \to \pi_1(Y)$ be the natural maps, $\rho_i = \rho \circ f_i$. Let $\mathcal{F}$ be the G-bundle on Y associated to ρ. Let $p^*\mathcal{F} = E = (E_i)_i$. Then E_i is the G-bundle on Y_i associated to ρ_i. The principal G-bundle $\mathcal{F}$ corresponds to a QPG $(E, (\sigma_j))$ on $\coprod_i Y_i$ where $\{\sigma_j\}$, $j = 1, \ldots, K$ are G-isomorphisms of fibres of E. Finally let C_i denote the desingularization of Y_i, g_i = arithmetic genus of Y_i, $g(C_i)$ = genus of C_i, $g(C_i) \geq 1$. Our aim is to prove the following Theorem.

Theorem 1. *There exist positive rational numbers $a_1, \ldots, a_I$ with $\sum a_i = 1$, depending only on Γ and g_i, such that for $a = (a_1, \ldots, a_I)$ the following statements are true.*

(1) *If $\rho_{C_i} = \rho_i \mid \pi_1(C_i)$ are unitary representations for all i, then the principal G-bundle $\mathcal{F}$ on Y associated to ρ is a-semistable.*

(2) *If ρ_{C_i} are irreducible unitary representations for all i, then the principal G-bundle $\mathcal{F}$ associated to ρ is a-stable.*

For the proof of the theorem, we need the following combinatorial result.

PROPOSITION 2.13

Let Γ be a connected graph without loops. Let I be the number of vertices and K the number of edges of Γ, $K = I + m$, $m \geq -1$. Fix integers $r \geq 1$, $g_i \geq 1$, $i = 1, \ldots, I$ and $g = \sum g_i$. Then there exist positive rational numbers a_i, $i = 1, \ldots, I$ with $\sum a_i = 1$ such that for every I-tuple $\underline{r} = (r_1, \ldots, r_I)$ of integers r_i with $0 \leq r_i \leq r$ and for every K-tuple of integers $\underline{q} = (q_1, \ldots, q_K)$ with $\max(r_{i(j)}, r_{t(j)}) \leq q_j \leq r$, $j = 1, \ldots, K$, *one has*

$$\sum_{i=1}^{I} r_i(g_i - 1) + \sum_{j=1}^{K} q_j \geq (g + m)\left(\sum_{i=1}^{I} a_i r_i\right). \tag{SS}$$

If in addition $r_i = 0$ for some i, $0 \neq \sum_i r_i$, then the inequality (SS) *is a strict inequality.*

Proof. We prove the result by induction on m.

Case $m = -1$: Γ is a tree in this case. Let $a_i = g_i/g$, $i = 1, \ldots, I$. Let $r_{i_0} = \min_i r_i$. Since $K = I - 1$ and $q_j \geq \max(r_{i(j)}, r_{t(j)})$, one has $\sum_j q_j \geq \sum_{i \neq i_0} r_i = \sum_i r_i - r_{i_0}$.

$$\begin{aligned}
\text{L.H.S. of (SS)} &= \textstyle\sum_i r_i g_i - \sum_i r_i + \sum_j q_j \\
&\geq \textstyle\sum_i r_i g_i - r_{i_0} = \sum_i r_i g_i - \sum_i (g_i r_{i_0})/g \\
&\geq \textstyle\sum_i r_i g_i - \sum_i g_i r_i/g \\
&= \textstyle(g-1)\sum r_i g_i/g \\
&= \textstyle(g-1)\sum_i a_i r_i.
\end{aligned}$$

If $0 \neq \sum r_i$ and $r_i = 0$ for some i, then $r_{i_0} = 0$ and

$$\text{L.H.S. of (SS)} \geq \sum r_i g_i = g \sum_i a_i r_i > (g-1) \sum a_i r_i.$$

Case $m \geq 0$: If $m \geq 0$, then Γ contains a cycle. By removing a suitable edge, say e_ℓ, from this cycle in Γ, we get a connected subgraph Γ' of Γ such that $K(\Gamma') - I(\Gamma') = m - 1$. By induction, there exist positive rational numbers a'_i, $i = 1, \ldots, I$ with $\sum a'_i = 1$, such that for all $\underline{r}$ and $\underline{q} = (q_1, \dots, \widehat{q_\ell}, \dots, q_J)$ satisfying the given conditions, one has

$$\begin{aligned} \textstyle\sum_i r_i(g_i - 1) + \sum_{j\neq\ell} q_j &\geq (g+m-1)(\textstyle\sum_i a'_i r_i) \\ &= \textstyle\sum_i b'_i r_i, \quad b'_i = a'_i(g+m-1), \\ \text{L.H.S. of (SS)} &= \textstyle\sum_i r_i(g_i - 1) + \sum_{j=1}^{J} q_j \\ &\geq \textstyle\sum_i b'_i r_i + (r_{i(\ell)} + r_{t(\ell)})/2 \\ &= \textstyle\sum b_i r_i, \end{aligned}$$

where $b_i = b'_i$ if $i \neq i(\ell), t(\ell)$ and $b_i = b'_i + \frac{1}{2}$ if $i = i(\ell)$ or $t(\ell)$. Take $a_i = b_i/(g+m)$ for all i, then (SS) holds. The assertion about strict inequality follows by induction similarly.

Remark 2.14. (1) Note that if both a, a' satisfy (SS) then for $0 \leq t \leq 1$, $a^t = ta + (1-t)a'$ also satisfies (SS). Thus the set of solutions a of (SS) is a convex set.
(2) Given $i_1, i_2,$, $1 \leq i_1, i_2 \leq I$, take $a'_i = (g_i - \frac{1}{2})/(g-1)$ for $i = i_1, i_2$ and $a'_i = g_i/(g-1)$ for $i \neq i_1, i \neq i_2$. Then in case Γ is a tree (i.e. $m = -1$) the inequality (SS) holds (though the strict inequality may not be true eg. for $r_{i_1} = r_{i_2} = 0$). For $K - I \geq 0$, the inductive proof of Proposition 2.13 then gives new $a' = (a'_1, \dots, a'_I)$ satisfying the inequality (SS). It follows that the inequality holds for a^t, $0 \leq t \leq 1$.

PROPOSITION 2.15

Theorem 1 *is true for* $G = GL(r)$.
(1) *If* $\rho_i \mid \pi_1(C_i)$ *are unitary representations for all* i, *then the vector bundle* F *on* Y *associated to* ρ *is* a*-semistable.*
(2) *If* $\rho_i \mid \pi_1(C_i)$ *are irreducible unitary for all* i, *then* F *is* a*-stable.*

Proof.

(1) As in Propositions 3.9 and 3.7(3) of [U2], it can be seen that the vector bundle F on Y corresponds to a QPG $\underline{E} = (E, F_j(E))$ on $\coprod Y_i$ and F is a-semistable (resp. a-stable) if and only if $\underline{E}$ is $(a, 1)$-semistable (resp. $(a, 1)$-stable). Note that $E = \coprod E_i$ is the pull-back of F to $\coprod Y_i$. By Theorem 2, [U3], the vector bundles E_i on Y_i associated to ρ_i are semistable for all i. Hence, for any subsheaf N_i of E_i, one has $\chi(N_i) \leq r_i(1 - g_i)$, $r_i =$ rank N_i (note that degree $(E_i) = 0$). Thus

$$S_N = \left[\sum_i \chi(N_i) - \sum_j q^j(N)\right] \Big/ \sum_i a_i r_i \leq \sum_i r_i(1 - g_i) - \sum_j q^j(N) / \sum_i a_i r_i,$$

where the summation over j is taken for $1 \leq j \leq K$. For the choice of $\{a_i\}$ made in Proposition 2.13, we get

$$S_N \leq I - K - \sum g_i = \left(\sum_i \chi(E_i) - rK\right) \Big/ r.$$

Thus $(E, F_j(E))$ is $(a, 1)$-semistable and hence F is a-semistable.

(2) We need to consider two cases. With the notations in the proof of (1) if $r_i = 0$ for some i then by Proposition 2.13, we have $S_N < I - K - \sum g_i$. If $r_i \neq 0$ for all i, then there exists an i_0 such that $0 \neq r_{i_0} \neq r$. Since E_{i_0} is stable by Theorem 2 [U3], we have $\chi(N_{i_0}) < r_{i_0}(1 - g_{i_0})$. Therefore, $S_N < \sum_i r_i(1 - g_i) - \sum_j q^j(N)/\sum_i a_i r_i \leq I - K - \sum g_i$ (by Proposition 2.13). Thus $(E, F_j(E))$ is $(a, 1)$-stable and so F is a-stable.

Remark 2.16. The proof of Proposition 2.13 shows that there exist curves $Y^m, m = 0, \dots, n+1$ such that (1)$Y^0 = Y$, (2) Y^{n+1} is a curve with ordinary nodes such that the dual graph of Y^{n+1} is a tree after omitting loops (3) Y^{m+1} is obtained from Y^m by blowing up a node which lies on two different components. For $m = 0, \dots, n$, let $\varphi_m = Y^{m+1} \to Y^m$ be the natural surjective maps. Let F denote a unitary (resp. irreducible unitary) vector bundle on Y. The proofs of Propositions 2.13 and 2.15 together show how the 'polarization' $a = (a_1, \dots, a_I)$ for which the vector bundles $\varphi_m^* F$ are a-semistable (resp. a-stable) varies as we go down the tower of curves $\{Y^m\}$.

Proof of Theorem 1.

(1) If ρ_{C_i} is unitary, so is $\mathrm{Ad} \circ \rho_{C_i} = (\mathrm{Ad} \circ \rho)_{C_i}$. Therefore there exist positive rational numbers $a_1, \dots, a_I$ with $\sum a_i = 1$ (depending only on Γ and g_i) such that the vector bundle $\mathcal{F}_{\mathrm{Ad}\circ\rho} \approx \mathrm{Ad}\,\mathcal{F}$ associated to $\mathrm{Ad} \circ \rho$ is a-semistable (Proposition 2.15). Hence $\mathcal{F}$ is a-semistable.

(2) By Proposition 2.6, the principal G-bundles E_i on Y_i associated to ρ_i are stable for all i. We claim that for the choices of $\{a_i\}_i$ as in the proof of (1), the QPG $(E, (\sigma_j))$ corresponding to $\mathcal{F}$ is $(a, 1)$-stable. The result follows from the claim in view of Proposition 2.11. To prove the claim we check that the condition (∗2′) of Lemma 1.9 is satisfied for any reduction s of the structure group to P. Let r_i be the rank of S_{s_i}. Since S_s is a proper subsheaf of $E(\mathbf{g})$, $\sum r_i \neq nI$. Since E_i are stable, by Lemma 2.3, $\chi(S_{s_i}) \leq r_i(1 - g_i)$ and the inequality is strict if $0 < r_i < n$. By Proposition 2.13, for the choices of $\{a_i\}$ as in (1), one has

$$\chi(S_s) - \sum_j q_j(S_s) \Big/ \sum_j a_i r_i < I - K - \sum_i g_i (= \chi(\mathrm{Ad}\ E)/n - K)$$

if $r_{i_0} = 0$ for some i_0 and $0 \neq \sum r_i$. If $r_i \neq 0$ for all i, since $\sum r_i \neq nI$, there exists an i_0 such that $0 < r_{i_0} < n$. Then $\chi(S_s) < \sum_i r_i(1 - g_i)$ and so

$$\begin{aligned}
\chi(S_s) - \sum_j q_j(S_s) &< \sum_i r_i(1 - g_i) - \sum_j q_j(S_s) \\
&\leq (I - K - \textstyle\sum g_i)\left(\displaystyle\sum_i a_i r_i\right), \text{ by Proposition 2.13,} \\
&= (\chi(\mathrm{Ad}\ E)/n - K)\sum_i a_i r_i.
\end{aligned}$$

This proves the claim.

3. The Picard group of the stack of QPGs

In this section Y denotes a reduced connected projective curve with ordinary nodes $\{y_j\}$, $j = 1, \ldots, J$ as only singularities. Let $\{Y_i\}$, $i = 1, \ldots, I$ be the irreducible components of Y and C_i the desingularization of Y_i. Let $C = \coprod_i C_i$ be the desingularization of Y. For convenience of notation, we fix an orientation of the dual graph of Y. For $1 \leq j \leq J$, let $i(j), t(j)$ denote the initial and terminal points of j in the dual graph. They correspond to curves $C_{i(j)}, C_{t(j)}$ intersecting at y_j. Let $x_j \in C_{i(j)}$ and $z_j \in C_{t(j)}$ be the two points of C mapping to $y_j \in Y$ and $D_j = x_j + z_j$, $j = 1, \ldots, J$. Let G denote an affine simply connected simple algebraic group over $\mathbf{C}$ (or an algebraically closed field of characteristic zero). For $i = 1, \ldots, I$, fix points $p_i \in C_i$, p_i not mapping to a singular point of Y. Let $C_i^* = C_i - \{p_i\}$, $C^* = C - \cup_i p_i$.

The results of this section were inspired by [LS]. If G is semisimple, then a principal G-bundle on a smooth curve C is trivial on the complement of a point in C. This no longer holds if C is replaced by a nodal curve Y. The results of [LS] cannot be generalized directly to G-bundles on Y. Hence we work with QPGs on C. Though we closely follow the ideas in [LS], the generalization to QPGs is not straightforward. All the functors involved have to be defined carefully to take care of the additional structure (generalized parabolic structure). Unlike the usual parabolic structure which is supported on isolated points, the generalized parabolic structure is supported on divisors, so one has the action of $G \times G$ rather than G.

3.1 *The stack* $Q^{\mathrm{gpar}}_{G,C}$ *and the stack* $\mathrm{Bun}^{\mathrm{gpar}}_{G,C}$

Let Aff/k be the flat affine site over the base field $k = \mathbf{C}$, i.e. the category of k-algebras equipped with $fppf$ topology. Let R denote a k-algebra, $C_{i,R} := C_i \times$ spec R and $C^*_R = C^* \times$ spec R. Let q_i be a local parameter at the point p_i, $i = 1, \ldots, I$. Let $L_{G,i}$ denote the k-group defined by associating to R the group $G(R(q_i))$. Let $L^+_{G,i}$ (resp. $L^{C_i}_G$) be the k-group defined by associating to R the group $G(R[[q_i]])$ (resp. $G(\Gamma(C^*_{i,R}, \mathcal{O}_{C^*_{i,R}})))$. Define $L_G = \prod_i L_{G,i}$, $L^+_G = \prod_i L^+_{G,i}$, $L^C_G = \prod_i L^{C_i}_G$. Let

$$Q_{G,C} = L_G / L^+_G = \prod_i L_{G,i}/L^+_{G,i}, \quad Q^{\mathrm{gpar}}_{G,C} = Q_{G,C} \times \prod_j G.$$

The indgroup L^C_G acts on $Q_{G,C}$. For each j, the evaluation at x_j and z_j gives an evaluation map $e_j : L^C_G \to G \times G$. $G \times G$ acts on G by $(g_1, g_2)g = g_1^{-1} g g_2$. Thus we have a natural action of L^C_G on $Q^{\mathrm{gpar}}_{G,C}$. Let $L^C_G \backslash Q^{\mathrm{gpar}}_{G,C}$ be the quotient stack.

To an object $R \in Aff/k$, associate the groupoid whose objects are families of QPGs $(E, (\sigma_j))$ on C parametrized by spec R and whose arrows are isomorphisms of the families of QPGs i.e. isomorphisms of E which preserve the parabolic structures (σ_j). For any morphism $R \to R'$ we have a natural functor between the associated groupoids. Thus we get a k-stack of (generalized) quasiparabolic G-bundles on C. We denote this stack by $\mathrm{Bun}^{\mathrm{gpar}}_{G,C}$.

Theorem 2. *There exists a canonical isomorphism of stacks*

$$\tilde{\pi}_{\mathrm{par}} : L^C_G \backslash Q^{\mathrm{gpar}}_{G,C} \xrightarrow{\sim} \mathrm{Bun}^{\mathrm{gpar}}_{G,C}.$$

The projection $\pi_{\mathrm{par}} : Q^{\mathrm{gpar}}_{G,C} \to \mathrm{Bun}^{\mathrm{gpar}}_{G,C}$ *is locally trivial in etale topology.*

Proof. $Q_{G,C}$ represents the functor which associates to every k-algebra R the set of isomorphism classes of pairs (E, ρ) where E is a G-bundle over C_R and ρ is a trivialization of E over C_R^* ([LS], Proposition 3.10). Hence $Q_{G,C}^{\mathrm{gpar}}$ represents the functor P_G which associates to R the isomorphism classes of triples (E, ρ, s) with (E, ρ) as above and $\mathrm{s} \in \prod_j G(R), \mathrm{s} = (s_1, \ldots, s_J), s_j \in G(R) = Mor$ (Spec R, G), G being the jth factor in $\prod_j G$. Such a triple gives a family of QPGs $(E, (\sigma_j))$ parametrized by $S = \mathrm{Spec}\ R$ as follows. Let $\overline{s}_j : S \times x_j \times G \to S \times z_j \times G$ be given by $\overline{s}_j(s, x_j, g) = (s, z_j, gs_j(s))$ for $s \in S, g \in G$. Define $\sigma_j : E_{|S\times x_j} \overset{\approx}{\to} E_{|S\times z_j}$ by $\sigma_j = \rho_{|S\times z_j}^{-1} \circ \overline{s}_j \circ \rho_{|S\times x_j}$. Thus we get a universal QPG over $Q_{G,C}^{\mathrm{gpar}} \times C$, giving a map $\pi_{\mathrm{par}} : Q_{G,C}^{\mathrm{gpar}} \to \mathrm{Bun}_{G,C}^{\mathrm{gpar}}$. Being L_G^C-invariant, this map induces a morphism of stacks $\overline{\pi}_{\mathrm{par}} : L_G^C \backslash Q_{G,C}^{\mathrm{gpar}} \to \mathrm{Bun}_{G,C}^{\mathrm{gpar}}$.

To define a morphism $\mathrm{Bun}_{G,C}^{\mathrm{gpar}} \to L_G^C \backslash Q_{G,C}^{\mathrm{gpar}}$, for each R and $(E, (\sigma_j)) \in \mathrm{Bun}_{G,C}^{\mathrm{gpar}}(R)$ we have to give a L_G^C-bundle $T(R)$ on $\mathrm{Bun}_{G,C}^{\mathrm{par}}(R)$ together with an L_G^C-equivariant map $T(R) \to Q_{G,C}^{\mathrm{gpar}}(R)$. Take $(E, (\sigma_j)) \in \mathrm{Bun}_{G,C}^{\mathrm{gpar}}(R)$. For any R-algebra R', let Spec $R' = S'$ and $T(R') =$ the set of isomorphism classes of pairs $(\rho_{R'}, \sigma')$ where $\rho_{R'}$ is a trivialization of $E_{R'}$ over $C_{R'}^*$ and $\sigma' = (\sigma_j')_j, \sigma_j' : E_{|S'\times x_j} \approx E_{|S'\times z_j}$ is the G-isomorphism which is the pull back of σ_j to R'. This defines an R-space T with the action of the group L_G^C (acting on $\rho_{R'}$). It is an L_G^C-bundle ([DS]; also [LS], Theorem 3.11). As $Q_{G,C}^{\mathrm{gpar}}$ represents the functor P_G, to every element $(\rho_{R'}, \sigma_{R'})$ of $T(R')$ corresponds an element of $Q_{G,C}^{\mathrm{gpar}}(R')$ giving a L_G^C-equivariant map $T \to Q_{G,C}^{\mathrm{par}}$. Hence we get a morphism of stacks $\mathrm{Bun}_{G,C}^{\mathrm{gpar}} \to L_G^C \backslash Q_{G,C}^{\mathrm{gpar}}$ which is clearly the inverse of $\overline{\pi}_{\mathrm{par}}$.

To check the local triviality of π_{par} in etale topology, we have to show that for any morphism f from a scheme S to $\mathrm{Bun}_{G,C}^{\mathrm{gpar}}$ the pull back of the fibration π_{par} to S is etale locally trivial i.e. admits local sections for the etale topology. Such a morphism corresponds to a QPG $(E, (\sigma_j))$ over $S \times C$. For $s \in S$, we can find an etale neighbourhood U of s and a trivialization ρ of $E_{|U\times C^*}$ ([DS]). Using ρ, the G-isomorphism σ_j gives a morphism $s_j : U \to G$. The triple $(E, \rho, (s_j))$ defines a morphism $f' : U \to Q_{G,C}^{\mathrm{gpar}}$ such that $\pi_{\mathrm{par}} \circ f' = f$; i.e. the section over U of the fibration π_{par}. This completes the proof of Theorem 2.

PROPOSITION 3.2

One has

(1) Pic $Q_{G,C} \approx \oplus_i \mathbf{Z}\mathcal{O}_{Q_{G,C_i}}(1)$

(2) Pic $(Q_{G,C}^{\mathrm{gpar}}) \approx \oplus_i \mathbf{Z}\mathcal{O}_{Q_{G,C_i}}(1)$.

Proof. (1) It is known that each Q_{G,C_i} is an ind-scheme which is an inductive limit of reduced projective Schubert varieties $X_{i,w}$, this ind-scheme structure coincides with the one by Kumar and Mathieu ([LS], Proposition 4.7). One has $H^1(X_{i,w}, \mathcal{O}) = 0$ ([KN, M]). It follows that Pic $Q_{G,C} \approx \oplus_i$ Pic Q_{G,C_i}. It is known that Pic $(Q_{G,C_i}) = \mathbf{Z}\mathcal{O}_{Q_{G,C_i}}(1)$ for all i ([LS], 4.10; [M]; [NRS], 2.3) The first assertion follows.
(2) Since $\prod_j G$ is a simply connected affine algebraic group Pic $(\prod_j G)$ is trivial. The ind-scheme Q_{G,C_i} is the inductive limit of integral projective reduced (generalized) Schubert varieties X_{i,w_i} with $H^1(X_{i,w_i}, \mathcal{O}) = 0$. By III, Exer. 12.6 [H] it follows that Pic $(X_{1,w_1} \times \prod_j G) \approx \mathrm{Pic}(X_{1,w_1})$ ([H], III, Exer. 12.6) and therefore by induction on i one sees that

$$\mathrm{Pic}\left(\prod_i X_{i,w_i}\times\prod_j G\right)\approx\oplus_i\mathrm{Pic}(X_{i,w_i})\approx\oplus_i\mathbf{Z}\mathcal{O}_{X_{i,w_i}}(1).$$

Since $(\mathcal{Q}^{\mathrm{gpar}}_{G,C})$ is the inductive limit of $\prod_i X_{i,w_i}\times\prod_j G$ and the restriction $\mathcal{O}_{\mathcal{Q}_{G,C_i}}(1)\mid X_{i,w_i}$ $\approx\mathcal{O}_{X_{i,w_i}}(1)$ it follows that Pic $(\mathcal{Q}^{\mathrm{gpar}}_{G,C})\approx\oplus_i\mathbf{Z}\mathcal{O}_{\mathcal{Q}_{G,C_i}}(1)$.

The following result must be well-known, we are including a proof since we could not fin a reference.

Lemma 3.3. *Let G be a connected semisimple algebraic group. Then any invertible regula function on G is constant.*

Proof. We remark first that the only regular invertible functions on SL_2 and the additi group G_a are constant functions. Let $f: G\to G_m$ be a regular function.

Claim. For any x in a 1-parameter unipotent subgroup U of G, one has $f(gx)=f(g)$ all $g\in G$.

Proof of the claim. Consider the function $U\to G_m$ defined by $x\to f(gx)$. Si $U\approx G_a$, this function is constant i.e. $f(gx)=f(g)$ for all $g\in G$.

Since G is semisimple, G is generated by X_α, α varying over roots of G ([Sp], 9.4.1). Th fore, in view of the claim, one has $f(g)=f(h_{\alpha_1}\dots h_{\alpha_r})$ with $h_{\alpha_i}\in\mathrm{Im}SL_2$. The func $SL_2\to G_m$ defined by $x\to f(h_{\alpha_1}\dots h_{\alpha_{r-1}}x)$ is constant. Hence $f(h_{\alpha_1}\dots h_{\alpha_r}$ $f(h_{\alpha_1}\dots h_{\alpha_{r-1}})$. Repeating this process the result follows.

3.4. For each i, there are morphisms of stacks $\pi_i:\mathcal{Q}_{G,C_i}\to\mathrm{Bun}_{G,C_i}$ inducing iso phisms $\pi_i^*:\mathrm{Pic}(\mathrm{Bun}_{G,C_i})\to\mathrm{Pic}(\mathcal{Q}_{G,C_i})$. If L_i' denotes the generator of Pic $(\mathrm{Bun}_{G,C}$ well as its pull back to $\mathrm{Bun}_{G,C}$, then $\pi_i^*L_i'=\mathcal{O}_{\mathcal{Q}_{G,C_i}}(1)$ ([LS, So, T]). Hence if we de the pull back of $\mathcal{O}_{\mathcal{Q}_{G,C_i}}(1)$ to $\mathcal{Q}_{G,C}$ by $\mathcal{O}_{\mathcal{Q}_{G,C_i}}(1)$ again, we have $\pi^*(L_i')=\mathcal{O}_{\mathcal{Q}_{G,C}}$ Since Pic $\mathcal{Q}_{G,C}=\oplus_i\mathbf{Z}\mathcal{O}_{\mathcal{Q}_{G,C_i}}(1)$ it follows that π^* is surjective. Similar argument Proposition 3.2 shows that π^*_{par} is surjective. One has $\pi^*_{\mathrm{par}}(L_i)=\mathcal{O}_{\mathcal{Q}_{G,C_i}}(1)$, whe denotes the pull back of L_i' under the forgetful morphism $\mathrm{Bun}^{\mathrm{gpar}}_{G,C}\to\mathrm{Bun}_{G,C}$. Not we have a commutative diagram

$$\begin{tikzcd}
\mathrm{Pic}(\mathrm{Bun}_{G,C}) \arrow[r, "\pi^*"] \arrow[d, "\Phi^*"'] & \mathrm{Pic}(\mathcal{Q}_{G,C}) \arrow[d, "\approx"] \\
\mathrm{Pic}(\mathrm{Bun}^{\mathrm{gpar}}_{G,C}) \arrow[r, "\pi^*_{\mathrm{par}}"] & \mathrm{Pic}(\mathcal{Q}^{\mathrm{gpar}}_{G,C}).
\end{tikzcd}$$

We now check that π^*_{par} is injective, the injectivity of π^* follows similarly. De $\mathrm{Pic}^{L^C_G}(\mathcal{Q}^{\mathrm{gpar}}_{G,C})$ the group of L^C_G-linearized line bundles on $\mathcal{Q}^{\mathrm{gpar}}_{G,C}$. Since π_{par} is locall (Theorem 2), for any line bundle L on $\mathrm{Bun}^{\mathrm{gpar}}_{G,C}$, π_{par} induces an isomorphism b the sections of L and L^C_G-invariant sections of $\pi^*_{\mathrm{par}}L$. Therefore we have an i $\mathrm{Pic}(\mathrm{Bun}^{\mathrm{gpar}}_{G,C})\to\mathrm{Pic}^{L^C_G}(\mathcal{Q}^{\mathrm{gpar}}_{G,C})$ induced by π^*_{par}. The kernel of the forgetful m

$\mathrm{Pic}^{L_G^C}(Q_{G,C}^{\mathrm{gpar}}) \to \mathrm{Pic}(Q_{G,C}^{\mathrm{gpar}})$ is the set of L_G^C-linearizations of the trivial line bundle. Any such linearization is given by an invertible (regular) function h on $L_G^C \times Q_{G,C}^{\mathrm{gpar}}$ satisfying a cocycle condition. $Q_{G,C}$ being an inductive limit of integral projective schemes ([LS], 4.6) has no non constant regular functions. Since G is simple, $\prod_j G$ has no invertible nonconstant regular functions (Lemma 3.3). Hence h is the pull back of an invertible function on L_G^C. Since it satisfies a cocycle condition, it is in fact a character on L_G^C. By [LS], Lemma 5.2, h is trivial. Thus the forgetful morphism is injective. Hence the composite $\pi_{\mathrm{gpar}}^* : \mathrm{Pic}(\mathrm{Bun}_{G,C}^{\mathrm{gpar}}) \to \mathrm{Pic}^{L_G^C}(Q_{G,C}^{\mathrm{par}}) \to \mathrm{Pic}(Q_{G,C}^{\mathrm{gpar}})$ is injective. Thus π_{par}^* is an isomorphism. Similarly π^* is an isomorphism and hence Φ^* is also an isomorphism. We have proved the following theorem.

Theorem 3. *Let G be a simple simply connected affine algebraic group over* $\mathbf{C}$. *Then we have the following isomorphisms.*

(1) $\mathrm{Pic}\ (\mathrm{Bun}_{G,C}) \approx \oplus_{i=1}^{l} \mathbf{Z} L_i'$,

(2) $\mathrm{Pic}\ (\mathrm{Bun}_{G,C}^{\mathrm{gpar}}) \approx \oplus_{i=1}^{l} \mathbf{Z} L_i$,

where L_i' and L_i are the pullbacks of the generator of $\mathrm{Pic}\ (\mathrm{Bun}_{G,C_i})$ *to* $\mathrm{Bun}_{G,C}$ *and* $\mathrm{Bun}_{G,C}^{\mathrm{gpar}}$ *respectively.*

Remark 3.5. For $G = GL(n)$, $SL(n)$, $Sp(2n)$, the moduli stack (resp. moduli space) of bundles on Y is isomorphic to the moduli stack (resp. moduli space) of QPGs on C ([U1, U2, U4]). Hence we have

$$\mathrm{Pic}(\mathrm{Bun}_{G,Y}) \approx \oplus_i \mathbf{Z}$$

for $G = GL(n)$, $SL(n)$ or $Sp(2n)$.

PROPOSITION 3.6

Assume that C is irreducible and G as in Theorem 3. *Let* $(\mathrm{Bun}_{G,C}^{\mathrm{gpar}})^{\mathrm{ss}}$ *denote the substack corresponding to α-semistable QPGs. Then*

$$\mathrm{Pic}(\mathrm{Bun}_{G,C}^{\mathrm{gpar}})^{\mathrm{ss}} \approx \mathbf{Z}.$$

Proof. We claim that a QPG (E, σ) is α-semistable (resp. stable) for any α, $0 \le \alpha \le 1$ if the underlying bundle E is semistable (resp. stable). The semistability (resp. stability) of E implies that $\deg E(\mathbf{p}) \le$ (resp. $<$)0. Since σ_j is an isomorphism, the subspace σ_j of $E(\mathbf{g})_{x_j} \oplus E(\mathbf{g})_{z_j}$ maps isomorphically onto $E(\mathbf{g})_{x_j}$ under the projection map. Hence $\sigma_j \cap (E(\mathbf{p})_{x_j} \oplus E(\mathbf{p})_{z_j})$ maps injectively into $E(\mathbf{p})_{x_j}$ and hence has dim.$\le$ rank $E(\mathbf{p})$. It follows that pardeg $E(\mathbf{p}) \le$ (resp. $<$)αJ rank$E(\mathbf{p})$. The claim now follows from Lemma 1.6.

The morphism $\phi : \mathrm{Bun}_{G,C}^{\mathrm{gpar}} \to \mathrm{Bun}_{G,C}$ (forgetting the quasiparabolic structure) is a surjective morphism with isomorphic fibres. It follows from the claim that $\phi^{-1}(\mathrm{Bun}_{G,C} - \mathrm{Bun}_{G,C}^{\mathrm{ss}}) \supseteq \mathrm{Bun}_{G,C}^{\mathrm{gpar}} - (\mathrm{Bun}_{G,C}^{\mathrm{gpar}})^{\mathrm{ss}}$. Hence $\mathrm{codim.}_{\mathrm{Bun}_{G,C}^{\mathrm{gpar}}}(\mathrm{Bun}_{G,C}^{\mathrm{gpar}} - (\mathrm{Bun}_{G,C}^{\mathrm{gpar}})^{\mathrm{ss}}) \ge \mathrm{codim.}_{\mathrm{Bun}_{G,C}}(\mathrm{Bun}_{G,C} - \mathrm{Bun}_{G,C}^{\mathrm{ss}})$. Since the latter is ≥ 2 for $g \ge 2$ ([L-S], 9.3) the same is true for the former. The result now follows from Theorem 3.

3.7. *Results in case $G = GL(n), SL(n)$*

In case of vector bundles we have the following results on Picard groups of moduli spaces ([U5, U6]). Let Y denote an irreducible reduced curve over $\mathbf{C}$ with at most ordinary nodes as singularities. Let $\mathcal{L}$ be a line bundle on Y. Let $U'_Y(n,d)$ (resp. $U'_{\mathcal{L}}(n,d)$) denote the moduli space of semistable vector bundles of rank n and degree d (resp. with fixed determinant $\mathcal{L}$) on Y. Let $U'^s_Y(n,d)$ (resp. $U'^s_{\mathcal{L}}(n,d)$) denote the open subset of $U'_Y(n,d)$ (resp. $U'_{\mathcal{L}}(n,d)$) consisting of stable vector bundles. Let g_C (resp. g_Y) denote the geometric (resp. arithmetic) genus of Y.

I. Assume that $g_C \geq 2$. Then, except possibly for $g_C = 2, n = 2, d$ even, one has

1. Pic $U'^s_{\mathcal{L}}(n,d) \approx$ Pic $U'_{\mathcal{L}}(n,d) \approx \mathbf{Z}$.

2. Pic $U'^s(n,d) \approx$ Pic $U'(n,d) \approx$ Pic $J \oplus \mathbf{Z}$, where J denotes the Jacobian of Y.

II. Assume that $g_Y = 2, n = 2$. Then

$$\text{Pic } U'_{\mathcal{L}}(2,d) \approx \mathbf{Z}.$$

4. Compactifications

In general, the moduli spaces M_G of principal G-bundles on a nodal curve Y are not complete. In case $G = GL(n)$ a compactification of M_G is given by the moduli space of torsionfree sheaves of rank n (and fixed degree) on Y, this compactification is not normal. A normal compactification of M_G is obtained as the moduli space of (generalized) parabolic bundles on the desingularization C of Y ([U1, U2]). This can be done for other classical groups $G = O(n), SO(n), Sp(2n)$ also, we briefly describe the main result (Theorem 5). The details will appear elsewhere [U4]. To construct a normal compactification of M_G, one needs a good compactification of G and hence a good representation of G. In case of classical groups we use their natural representations. For a general group G, a natural choice is the adjoint representation. Unfortunately it gives a compactification of G only if G is of adjoint type ([DP], §6; [S]; [D]). Using this compactification we give a more general definition of QPGs in case G has *trivial centre*. For classical groups and adjoint groups we 'compactify' the stack $\text{Bun}^{\text{gpar}}_{G,C}$ and also compute the Picard group of the compactification. In case of classical groups, the compactifications of moduli spaces obtained are complete normal varieties (see Theorem 5). We do not prove that the 'compactification' is a proper stack in case of adjoint groups. It will be useful to know a natural (canonically defined) compactification of G in the general case.

4.1. Let the notations be as in §3. We further assume that G is a semisimple algebraic group with trivial centre. Let $\mathbf{g}$ denote the Lie algebra of G, $n = \dim \mathbf{g}$. $G \times G$ acts on $\mathbf{g} \oplus \mathbf{g}$ (via adjoint representation) and hence on the Grassmannian $\text{Gr}(n, \mathbf{g} \oplus \mathbf{g})$ of n-dimensional subspaces of $\mathbf{g} \oplus \mathbf{g}$. Let ΔG denote G embedded in $G \times G$ diagonally. Since G has trivial centre the adjoint representation is faithful. Hence $G \approx G \times G/\Delta G$ gets embedded in $\text{Gr}(n, \mathbf{g} \oplus \mathbf{g})$ as $G \times G$-orbit of $\Delta \mathbf{g} \in \text{Gr}(n, \mathbf{g} \oplus \mathbf{g})$. Let F be the closure of the $G \times G$ orbit of $\Delta \mathbf{g}$ in $\text{Gr}(n, \mathbf{g} \oplus \mathbf{g})$.

Given a principal G-bundle E and disjoint divisors $D_j = x_j + z_j$ on C, define

$$E^j = E_{x_j} \times E_{z_j} \cong G \times G, E^j(F) = E^j \times_{(G \times G)} (F), j = 1, \ldots, J.$$

A QPG (quasiparabolic G-bundle) is a pair $(E, (\sigma_j))$ where E is a principal G-bundle and $\sigma_j \in E^j(F)$, $j = 1, \dots, J$.

DEFINITION 4.2

QPGs $(E, (\sigma_j))$ and $(E', (\sigma'_j))$ are isomorphic if there is an isomorphism $f : E \to E'$ of principal G-bundles which maps σ_j to (σ'_j) i.e. for the isomorphism $f_F^j : E^j(F) \to E'^j(F)$ one has $f_F^j(\sigma_j) = \sigma'_j$.

4.3. A family of QPGs $(\mathcal{E}, (\sigma_j)) \to C \times T$ is a family of G-bundles $\mathcal{E} \to C \times T$ together with a section $\sigma_j : T \to \mathcal{E}^j(F)$.

Remark 4.4. (1) The following diagram commutes

$$\begin{tikzcd}
G \times G \arrow[r, "h_1"] \arrow[d, "t_2"'] & F = \overline{G \times G/\Delta G} \arrow[d, "t_1"] \\
GL(\mathbf{g}) \times GL(\mathbf{g}) \arrow[r, hook, "h_2"] & \mathrm{Gr}(n, \mathbf{g} \oplus \mathbf{g}).
\end{tikzcd}$$

Here t_1 is inclusion, t_2 = product of adjoint representations of G in $\mathbf{g}$, $h_2(f_1, f_2)$ = subspace of $\mathbf{g} \oplus \mathbf{g}$ generated by $\{(f_1 v, f_2 v), v \in \mathbf{g}\}$ and $t_1 \circ h_1$ is the map inducing the Demazure embedding of G (by identifying G with $G \times G/\Delta G$).

(2) Recall that a (generalized) quasiparabolic structure (over $D_j = x_j + z_j$, $j = 1, \dots, J$) on a vector bundle N of rank n is given by an n-dimensional subspace of $N_{x_j} \oplus N_{z_j}$, $j \in J$ i.e. by an element of $\Pi_j \mathrm{Gr}(n, N_{x_j} \oplus N_{z_j})$ [U1]. Given a family of QPGs $\mathcal{E} \to C \times T$, let $\mathcal{E}(\mathbf{g})$ be the family of vector bundles of rank n associated to $\mathcal{E}$ via the adjoint representation of G in $\mathbf{g}$. It follows from the above commutative diagram that σ composed with the injection $\Pi_j \mathcal{E}^j(F) \to (\Pi_j \mathcal{E}^j(\mathrm{Gr}(n, \mathbf{g} \oplus \mathbf{g})))$ gives a quasi parabolic structure on $\mathcal{E}(\mathbf{g})$.

4.5 *The stack* $\overline{Q}^{\mathrm{gpar}}_{G,C}$ *and the stack* $\overline{\mathrm{Bun}}^{\mathrm{gpar}}_{G,C}$

Let the notations be as in 3.1. Let $\overline{Q}^{\mathrm{gpar}}_{G,C} = Q_{G,C} \times \prod_j F$. The ind-scheme $Q_{G,C}$ is ind-proper, so is $\overline{Q}^{\mathrm{gpar}}_{G,C}$. The indgroup L_G^C acts on $Q_{G,C}$. For each j, the evaluation at x_j and z_j gives an evaluation map $e_j : L_G^C \to G \times G$. $G \times G$ acts on F naturally. Thus we have a natural action of L_G^C on $\overline{Q}^{\mathrm{gpar}}_{G,C}$. Let $L_G^C \backslash \overline{Q}^{\mathrm{gpar}}_{G,C}$ be the quotient stack.

As in 3.1, we define the k-stack of (generalized) quasiparabolic G-bundles on C (with extended definition of the parabolic structure using F). We denote this stack by $\overline{\mathrm{Bun}}^{\mathrm{gpar}}_{G,C}$. It contains $\mathrm{Bun}^{\mathrm{gpar}}_{G,C}$ as an open substack.

Theorem 4. (1) *There exists a canonical isomorphism of stacks*

$$\overline{\pi}_{\mathrm{par}} : L_G^C \backslash \overline{Q}^{\mathrm{gpar}}_{G,C} \xrightarrow{\sim} \overline{\mathrm{Bun}}^{\mathrm{gpar}}_{G,C}.$$

Moreover the projection map $\overline{Q}^{\mathrm{gpar}}_{G,C} \to \overline{\mathrm{Bun}}^{\mathrm{gpar}}_{G,C}$ *is locally trivial for étale topology.*
(2) *Let G be a simple, simply connected affine algebraic group over* $\mathbf{C}$. *Then there exists an isomorphism*

$$\mathrm{Pic}\,(\overline{\mathrm{Bun}}^{\mathrm{gpar}}_{G,C}) \approx \oplus_i \mathbf{Z}L_i \oplus \oplus_j \mathrm{Pic}F,$$

where L_i *are line bundles coming from* Bun_{G,C_i}.

Proof. The proof is on similar lines as that of Theorem 2 and Theorem 3, we omit some details to avoid repetition.

(1) $\overline{Q}^{\mathrm{gpar}}_{G,C}$ represents the functor $\overline{P}_G$ which associates to every k-algebra R the set of isomorphism classes of triples (E, ρ, s) where E is a principal G-bundle on C_R, ρ is a trivialization of E over C^*_R and $\mathrm{s} \in Mor\ (\mathrm{Spec}\ R, \prod_j F)$. Then $\mathrm{s} = (s_1, \ldots, s_J), s_j \in Mor\ (\mathrm{Spec}\ R, F)$ for all j. We can associate to such a triple a QPG $(E, (\sigma_j))$ on C_R. We only need to define for each j, morphism $\sigma_j : S \to E^j(F), S = \mathrm{Spec}\ R$. The restriction of ρ^{-1} gives isomorphisms $S \times x_j \times G \approx E_{|S\times x_j}, S \times z_j \times G \approx E_{|S\times z_j}$ and hence an isomorphism of $G \times G$-bundles $S \times G \times G = (S \times x_j \times G) \times_S (S \times z_j \times G) \approx E_{|S\times x_j} \times_S E_{|S\times z_j} = E^j$. Therefore we have an isomorphism of associated fibre bundles $\rho_j(F) : S \times F \overset{\approx}{\to} E^j(F)$. Define σ_j by $\sigma_j(s) = \rho_j(F)(s, s_j(s))$. It follows that $\overline{Q}^{\mathrm{gpar}}_{G,C} \times C$ has a universal QPG and we have an L^C_G-equivariant morphism of stacks $\pi_{\mathrm{par}} : \overline{Q}^{\mathrm{gpar}}_{G,C} \to \overline{\mathrm{Bun}}^{\mathrm{gpar}}_{G,C}$. This induces the morphism $\overline{\pi}_{\mathrm{par}}$ on the quotient stack.

To define the inverse of $\overline{\pi}_{\mathrm{par}}$, let $(E, (\sigma_j)) \in \overline{\mathrm{Bun}}^{\mathrm{gpar}}_{G,C}(R)$. Let R' be an R-algebra, $S' = \mathrm{Spec}\ R'$. Let $T(R')$ be the set of pairs $(\rho_{R'}, \sigma')$ where $\rho_{R'}$ is a trivialization of $E_{R'}, \sigma' = (\sigma'_1, \ldots, \sigma'_J)$ where σ'_j is a pull back of $\sigma_j \forall j$. This defines a T-space with an action of L^C_G (via $\rho_{R'}$), it is an L^C_G-bundle [DS]. We now define a L^C_G-equivariant map $T \to \overline{Q}^{\mathrm{gpar}}_{G,C}$. Given $(\rho_{R'}, \sigma') \in T(R')$, we define $s'_j : S' \to F$ by $s'_j = pr_F \circ ((\rho_{R'})_j(F))^{-1} \circ \sigma'_j$. Then $(E_{R'}, \rho_{R'}, (s'_j)) \in \overline{P}_G(R')$. Since $\overline{Q}^{\mathrm{gpar}}_{G,C}$ represents the functor $\overline{P}_G$, this defines a map $\alpha : T \to \overline{Q}^{\mathrm{gpar}}_{G,C}$, it is L^C_G-equivariant. The L^C_G-bundle T together with α give a morphism of stacks from $\overline{\mathrm{Bun}}^{\mathrm{gpar}}_{G,C}$ to the quotient stack $L^C_G \backslash \overline{Q}^{\mathrm{gpar}}_{G,C}$ which is easily seen to be the inverse of $\overline{\pi}_{\mathrm{par}}$.

The assertions about local triviality of $\overline{\pi}_{\mathrm{par}}$ follow similarly as in Theorem 2.

(2) Using the facts that each Q_{G,C_i} is an inductive limit of reduced projective varieties $X_{i,w}$ with $H^1(X_{i,w}, \mathcal{O}) = 0$ and F is a projective variety with $H^1(F, \mathcal{O}) = 0$, it can be proved that $\mathrm{Pic}\ \overline{Q}^{\mathrm{gpar}}_{G,C} \approx \oplus_i \mathbf{Z}\mathcal{O}_{Q_{G,C_i}}(1) \oplus \oplus_j \mathrm{Pic}F$ (similarly as Proposition 3.2). The injectivity of π^*_{par} follows exactly as in Theorem 3. Note that F being a projective variety $\overline{Q}^{\mathrm{gpar}}_{G,C}$ is an inductive limit of integral projective schemes and hence has no nonconstant regular functions.

We now check the surjectivity of π^*_{par}. We have a commutative diagram

$$\begin{tikzcd}
\mathrm{Pic}(\mathrm{Bun}_{G,C}) \arrow[r, "\pi^*"] \arrow[d, "\varphi^*"'] & \mathrm{Pic}(Q_{G,C}) = \oplus \mathbf{Z}\mathcal{O}_{Q_{G,C_i}}(1) \arrow[d] \\
\mathrm{Pic}(\overline{\mathrm{Bun}}^{\mathrm{gpar}}_{G,C}) \arrow[r, "\pi^*_{\mathrm{par}}"] & \oplus_j \mathrm{Pic}F \oplus \mathbf{Z}\mathcal{O}_{Q_{G,C_i}}(1)
\end{tikzcd}$$

with φ the forgetful morphism and the right vertical arrow is the inclusion as direct summand. Hence one has $\pi^*_{\mathrm{par}}(\varphi^* L'_i) = \mathcal{O}_{Q_{G,C_i}}(1)$, L'_i being the pull back of the generator of Pic (Bun_{G,C_i}) to Pic $(\mathrm{Bun}_{G,C})$. Thus for the surjectivity of π^*_{par} it suffices to show that there exist line bundles $\{L'_{i,j}\}$ on $\overline{\mathrm{Bun}}^{\mathrm{gpar}}_{G,C}$ which pullback to the generators of $\oplus_j$ Pic F.

From the construction and results in [S], it follows that Pic F is a lattice of rank r generated by $L_i', i = 1, \dots, r, r =$ rank of G. For each i, there exists a $G \times G$ module W_i and a $G \times G$ equivariant embedding $F \to P(W_i)$ such that $\mathcal{O}_{P(W_i)}(1)$ restricts to L_i' on F. Given a family of QPGs $(E, (\sigma_j))$ on $C \times \operatorname{Spec} R$ one has $E^j(F) \subset E^j(P(W_i))$. Let L'_{ij} denote the line bundle on $E^j(P(W_i))$ (and also its restriction to $E^j(F)$) which restricts to $\mathcal{O}_{P(W_i)}(1)$ on each fibre. The pull-back of L'_{ij} by σ_j: Spec $R \to E^j(F)$ is a line bundle $L'_{ij,R}$ on Spec R. This construction can be done for any R. Hence $\{L'_{ij,R}\}$ define a line bundle L'_{ij} on the stack $\overline{\mathrm{Bun}}^{\mathrm{gpar}}_{G,C}$. By construction, $\overline{\pi}^*_{\mathrm{par}}(L'_{ij})$ is the generator of the jth factor Pic F in Pic $(\overline{Q}^{\mathrm{gpar}}_{G,C})$.

Case of classical groups

For the simple and simply connected classical groups $SL(n)$ and $Sp(2n)$ the compactifications F of G are defined using natural representations (described below). We claim that Theorem 4 holds in these cases also. The existence of the isomorphism $\overline{\pi}_{\mathrm{par}}$ and injectivity of $\overline{\pi}^*_{\mathrm{par}}$ can be seen exactly as in the proof of the Theorem 4. We only need to check the surjectivity of $\overline{\pi}^*_{\mathrm{par}}$, this is done below.

4.6. *Case* $G = SL(n)$. For $G = SL(n)$, the compactification F of G using natural representation of G ([U1, U4]) can be described as follows. $SL(n) \times SL(n)$ is embedded diagonally in $SL(2n) \subset GL(2n)$. Let $G \to GL(V)$ be the natural representation. Let $P \subset SL(2n)$ be the stabilizer of the diagonal in $V \oplus V$; P is a maximum parabolic subgroup. The Grassmannian Gr $= SL(2n)/P$ is embedded in $\mathbf{P}(\overset{n}{\wedge}(V \oplus V))$ by Plücker embedding. Let $\{P_{i_1,\dots,i_n}\}$ denote the Plücker coordinates. Let F be the hyperplane section of Gr defined by $P_{1,\dots,n} = P_{n+1,\dots,2n}$. Then F can be regarded as a compactification of $SL(n)$ with $SL(n)$ identified with the subset of F defined by $P_{1,\dots,n} \neq 0$. The generator of Pic Gr $\approx \mathbf{Z}$ is the line bundle associated to the character w_n on P and its restriction to F is the generator L' of Pic $F \approx \mathbf{Z}$. $F - SL(n)$ is a divisor D' in F to which L' is associated. Given a family of QPGs $(E, (\sigma_j))$ on $C \times$ Spec R, one has $E^j(F) \subset E^j(\mathrm{Gr})$. Let L'_j be the line bundle on $E^j(\mathrm{Gr})$ associated to the P-bundle $E^j(SL(2n)) \to E^j(\mathrm{Gr})$ via the character w_n. The pull back of L'_j by σ_j : Spec $R \to E^j(F) \subset E^j(\mathrm{Gr})$ is a line bundle $L'_{j,R}$ on Spec R, $L'_{j,R}$ define a line bundle L'_j on the stalk $\overline{\mathrm{Bun}}^{\mathrm{gpar}}_{G,C}$. By construction, $\pi^*_{\mathrm{par}}(L'_j)$ is the generator of the jth factor Pic F in Pic $\overline{Q}^{\mathrm{gpar}}_{G,C}$. Hence the morphism in Theorem 4(2) is a surjection and thus an isomorphism for $G = SL(n)$ and F as above.

4.7. *Case* $G = Sp(2n)$. In case $G = Sp(2n)$ also one can use the natural representation of G to define F (§5, [U4]). Let $G \to GL(V)$ be the natural representation. We regard $Sp(2n)$ as the group $Sp(q, V)$ of automorphisms of V preserving a symplectic form (nondegenerate alternating form) q on V. Then F is the variety of maximum isotropic subspaces for $q \oplus (-q)$ on $V \oplus V$. The group $Sp(2n) \times Sp(2n) = Sp(q, V) \times Sp(-q, V)$ is embedded in $Sp(q \oplus (-q), V \oplus V) = \mathrm{Sp}(4n)$ diagonally. Then $F \approx Sp(4n)/P$, P being the maximum parabolic subgroup of $Sp(4n)$ which is the stabilizer of the maximum isotropic subspace Δ_V of $V \oplus V$, Pic $F = \mathbf{Z}L'$, L' being the line bundle associated to the fundamental weight w_{2n}. Given a family of QPGs $(E, (\sigma_j))$ on C parametrized by $S = \operatorname{Spec} R$, $\sigma_j : S \to E^j(F)$, one has $E^j(F) = E^j(Sp(4n)/P)$ and $E^j(Sp(4n)) \to E^j(F)$ is a P-bundle. Let L'_j denote the line bundle on $E^j(F)$ associated to this P-bundle via the character w_{2n}. Let $L'_{j,R}$ denote the line bundle on S which is the pullback of this line bundle by σ_j. This construction being valid for any R, it defines a line bundle L'_j on the stack $\overline{\mathrm{B}}\mathrm{un}^{\mathrm{gpar}}_{G,C}$. Clearly, $\pi^*_{\mathrm{par}}(L'_j$ is the generator of Pic F, the jth factor. It follows that the injection in Theorem 4(2) is an isomorphism for $G = Sp(2n)$ with F defined as above.

The following definitions and results are stated for $O(n)$-bundles, they hold for $Sp(2n)$-bundles also with orthogonal replaced by symplectic and n replaced by $2n$.

DEFINITION 4.8

An orthogonal bundle (E, q) on C is an I-tuple of vector bundles $E = (E_1, \ldots, E_I)$, $E_i =$ a vector bundle on Y_i with a nondegenerate quadratic form q_i and $q = (q_1, \ldots, q_I)$. We assume that rank $E_i = n$ for all i, we call n the rank of E. For a closed point $x \in C$, let q_x denote the induced quadratic form on the fibre E_x.

DEFINITION 4.9

A generalized quasiparabolic orthogonal bundle (*orthogonal QPB* in short) on C is an orthogonal bundle (E, q) of rank n together with n-dimensional vector subspaces $F^j_1(E$ of $E_{x_j} \oplus E_{z_j}$ which are totally isotropic for $q_{x_j} \oplus (-q_{z_j})$.

Theorem 5. *Assume further that Y is irreducible. Then there is a coarse moduli space M for α-semistable orthogonal QPBs of rank n, $\alpha \in (0, 1)$ being rational. M is normal and complete.*

Let $\mathcal{U}$ be the moduli space of orthogonal sheaves of rank n on Y. Assume that $0 < \alpha <$ 1, α is close to 1. Then

(1) there exists a morphism $f : M \to \mathcal{U}$.

(2) Let $\mathcal{U}^s_n$ be the subset of $\mathcal{U}$ corresponding to stable orthogonal bundles. Then the restriction of f to $f^{-1}(\mathcal{U}^s_n)$ is an isomorphism onto $\mathcal{U}^s_n$.

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Uncertainty principles on two step nilpotent Lie groups

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Abstract. We extend an uncertainty principle due to Cowling and Price to two step nilpotent Lie groups, which generalizes a classical theorem of Hardy. We also prove an analogue of Heisenberg inequality on two step nilpotent Lie groups.

Keywords. Uncertainty principles; Hardy's theorem; two step nilpotent Lie groups; Heisenberg's inequality.

1. Introduction

As a meta-theorem in harmonic analysis, the uncertainty principles can be summarized as: *A nonzero function and its Fourier transform cannot both be sharply localized.* When *sharp localization* is interpreted as very rapid decay, this meta-theorem becomes the following theorem due to Hardy ([4]).

Theorem 1.1 (*Hardy*). *Let $f : \mathbb{R} \to \mathbb{C}$ be measurable and for all x, y*
(i) $|f(x)| \leq C\mathrm{e}^{-a\pi x^2}$,
(ii) $|\hat{f}(y)| \leq C\mathrm{e}^{-b\pi y^2}$,
where $C, a, b > 0$. If $ab > 1$ then $f = 0$ almost everywhere. If $ab = 1$ then $f(x) = C\mathrm{e}^{-a\pi x^2}$. If $ab < 1$ then there exist infinitely many linearly independent functions satisfying (i) *and* (ii).

Considerable attention has recently been paid to discover analogues of Hardy's theorem in the context of Lie groups ([28, 27, 5, 1, 16, 25, 24, 12, 8, 22]). Coming back to $\mathbb{R}$, we see that the decay conditions can be stated as $\|e_{a\pi} f\|_{L^\infty(\mathbb{R})} < \infty$ and $\|e_{b\pi}\hat{f}\|_{L^\infty(\mathbb{R})} < \infty$, where $e_k(x) = \mathrm{e}^{kx^2}$. So one reasonable question is to ask: what happens if $\|e_{a\pi} f\|_{L^p(\mathbb{R})} < \infty$ and $\|e_{b\pi}\hat{f}\|_{L^q(\mathbb{R})} < \infty$, where $1 \leq p, q < \infty$? The answer is given by the following theorem due to Cowling and Price ([6]).

Theorem 1.2 (*Cowling and Price*). *Let $f : \mathbb{R} \to \mathbb{C}$ be measurable and*
(i) $\|e_{a\pi} f\|_{L^p(\mathbb{R})} < \infty$,
(ii) $\|e_{b\pi}\hat{f}\|_{L^q(\mathbb{R})} < \infty$,
where $a, b > 0$ and $\min(p, q) < \infty$. If $ab \geq 1$ then $f = 0$ almost everywhere. If $ab < 1$ then there exist infinitely many linearly independent functions satisfying (i) *and* (ii).

The proof of the above theorem uses the following result (see [6]).

Lemma 1.1. *If $g : \mathbb{C} \to \mathbb{C}$ is an entire function and for $1 \leq p < \infty$*

(i) $|g(x+iy)| \leq Ae^{\pi x^2}$,
(ii) $\left(\int_{\mathbb{R}} |g(x)|^p \mathrm{d}x\right)^{1/p} < \infty$,
then $g = 0$.

The importance of Theorem 1.2 is that even if the *pointwise decay* is replaced by *averag decay,* Hardy's uncertainty principle continues to be true. As expected, the case $ab >$ of Hardy's theorem follows trivially from that of Cowling and Price. Actually, if we dro the case $ab = 1$, then, on the real line (more generally on $\mathbb{R}^n$), the above theorems ar equivalent (see [3]).

The following theorem, which follows as a corollary from a deep theorem of Beurlin ([14]), also suggests another generalization of Hardy's theorem.

Theorem 1.3. *Let* $f : \mathbb{R} \to \mathbb{C}$ *be measurable and for all* x, y
(i) $|f(x)| \leq Ce^{-a\pi|x|^p}$,
(ii) $|\hat{f}(y)| \leq Ce^{-b\pi|y|^q}$,
where $C, a, b > 0$, $p^{-1} + q^{-1} = 1$, $1 < p, q < \infty$. *If* $(ap)^{1/p}(bq)^{1/q} > 2$, *then* $f =$ *almost everywhere.*

In this paper our aim is to get analogues of Theorems 1.2, 1.3 on connected, simpl connected, two step nilpotent Lie groups (see [9, 3] for analogues of Theorem 1.2 on othe groups). We also prove an analogue of Heisenberg's inequality on two step nilpotent Li groups which was previously known only in the case of Heisenberg groups (see [29]).

This paper is organized as follows: in §2 we fix notation and describe some backgroun material leading to a proof of the Plancherel theorem via the description of the Hilbert–Schmidt norm of the group Fourier transform, and in §3 we prove the proposed analogu of Theorem 1.2 and indicate a proof of Theorem 1.3. In §4 we prove an analogue o Heisenberg's inequality.

Finally we would like to point out that all the results except Theorem 3.2 are from th author's 1999 Ph.D. thesis of the Indian Statistical Institute.

2. Notation and background material

For a Lie algebra $\mathfrak{g}$ (we will always work with Lie algebras over $\mathbb{R}$), we define $\mathfrak{g}^1 = [\mathfrak{g}, \mathfrak{g}$ and $\mathfrak{g}^n = [\mathfrak{g}, \mathfrak{g}^{n-1}]$, $n \geq 1$.

DEFINITION 2.1

A Lie algebra $\mathfrak{g}$ is called *two step nilpotent* if $\mathfrak{g}^2 = 0$ and $\mathfrak{g}^1 \neq 0$. The connected simply connected Lie group G corresponding to such a $\mathfrak{g}$ is called a *two step nilpotent Lie group.*

We find it more convenient to look at a two step nilpotent Lie algebra in another way. Let $B : \mathbb{R}^{n-m} \times \mathbb{R}^{n-m} \to \mathbb{R}^m$ be a nondegenerate, alternating, bilinear map. Let $\mathfrak{g} =$ $\mathbb{R}^m \oplus \mathbb{R}^{n-m}$, we define

$$[(z, v), (z', v')] = (B(v, v'), 0), \tag{2.1}$$

where $z, z' \in \mathbb{R}^m$ and $v, v' \in \mathbb{R}^{n-m}$. Then $[.,.]$ is a Lie bracket and $\mathfrak{g}$ is a two step nilpotent Lie algebra with $\mathbb{R}^m$ as the center of $\mathfrak{g}$. If on $G = \mathbb{R}^m \oplus \mathbb{R}^{n-m}$ we define the product

$$(z, v).(z', v') = \left(z + z' + \frac{1}{2}B(v, v'), v + v'\right), \tag{2.2}$$

then G is a connected, simply connected, two step nilpotent Lie group with $\mathfrak{g}$ as its Lie algebra and $\exp : \mathfrak{g} \to G$ is the identity diffeomorphism. In this section we will first describe the effective unitary dual of a connected, simply connected, two step nilpotent Lie group $\mathfrak{g}$ following Kirillov theory. Our notations are standard and can be found in [7].

Let $\mathfrak{g}^*$ be the real dual of $\mathfrak{g}$. Then G acts on $\mathfrak{g}^*$ by the coadjoint action, that is $G \times \mathfrak{g}^* \to \mathfrak{g}^*$, $(g, l) \to g.l$ is given by

$$\begin{aligned}(g.l)(X) &= l(\mathrm{Ad}g^{-1}(X)), \qquad g \in G, l \in \mathfrak{g}^*, X \in \mathfrak{g},\\ &= l(\mathrm{Ad}(\exp -Y)(X)), \qquad Y \in \mathfrak{g}, g = \exp Y,\\ &= l(\mathrm{e}^{\mathrm{ad}-y}(X))\\ &= l(X) - l([Y, X]).\end{aligned}$$

Let $l \in \mathfrak{g}^*$. Then we denote O_l = the coadjoint orbit of l. B_l = the skew symmetric matrix corresponding to l, that is, given a basis $\{X_1, \ldots, X_m, X_{m+1}, \ldots, X_n\}$ of $\mathfrak{g}$ through the center (that is, $X_1, \ldots, X_m$ span the centre of $\mathfrak{g}$, we consider the matrix $B_l = (B_l(i, j)) = (l([X_i, X_j])$. r_l = The radical of the bilinear form B_l, that is,

$$r_l = \{X \in \mathfrak{g} : l([X, Y]) = 0 \quad \text{for all } Y \in \mathfrak{g}\}.$$

Clearly r_l is an ideal of $\mathfrak{g}$ and $\mathfrak{z}(= \mathbb{R}^m) \subset r_l$. $\tilde{r}_l = \mathrm{span}_{\mathbb{R}}\{X_{m+1}, \ldots, X_n\} \cap r_l$. $\tilde{B}_l = B_l \mid \mathbb{R}^{n-m} \times \mathbb{R}^{n-m}$ that is restriction of B_l on the complement of the center of $\mathfrak{g}$.

It follows trivially for two step nilpotent Lie groups that all the coadjoint orbits are hyperplanes ([17, 23]). In fact we have from the above, the following.

Theorem 2.1. *Let $l \in \mathfrak{g}^*$. Then $O_l = l + r_l^{\perp}$ where $r_l^{\perp} = \{h \in \mathfrak{g}^* : h \mid r_l = 0\}$.*

In particular, $l' \in O_l$ if and only if $r_l = r_{l'}$ and $l \mid r_l = l' \mid r_{l'}$.

Let $\mathfrak{g}$ be a two step nilpotent Lie algebra such that $\dim \mathfrak{g} = n$ with the basis $\mathcal{B} = \{X_1, \ldots, X_m, X_{m+1}, \ldots, X_n\}$ through the centre. Then B_l is the $n \times n$ matrix whose (i, j)th entry is $l([X_i, X_j])$, $1 \leq i, j \leq n$. Let $\mathcal{B}^* = \{X_1^*, \ldots, X_n^*\}$ be the dual basis of $\mathfrak{g}^*$. This is a Jordan–Hölder basis, that is, $\mathfrak{g}_j^* = \mathrm{span}_{\mathbb{R}}\{X_1^*, \ldots, X_j^*\}$ is $\mathrm{Ad}^*(G)$ stable for $1 \leq j \leq n$.

Let $l \in \mathfrak{g}^*$ and $X_i \in \mathcal{B}$.

DEFINITION 2.2

The term i is called a *jump index* for l if the rank of the $i \times n$ submatrix of B_l, consisting of the first i rows, is strictly greater than the rank of the $(i-1) \times n$ submatrix of B_l, consisting of the first $(i-1)$ rows.

Since an alternating bilinear form has even rank the number of jump indices must be even. The set of jump indices is denoted by $J = \{j_1, \ldots, j_{2k}\}$. Notice that $j_1 \geq m + 1$. The subset of $\mathcal{B}$ corresponding to J is then $\{X_{j_1}, \ldots, X_{j_{2k}}\}$. Notice that if i is a jump index then $\mathrm{rank} B_l^i = \mathrm{rank} B_l^{i-1} + 1$, where B_l^i is the submatrix of B_l consisting of first i rows.

Note 2.1. These jump indices depend on l and on the order of the basis as well. But ultimately we will restrict ourselves to 'generic linear functionals' and they will have the same jump indices.

Now we are going to spell out what we mean by *generic linear functionals*. This is also a basis dependent definition. We work with the basis $\mathcal{B}$ chosen above. Let $R_i(l) = \operatorname{rank} B_l^i$ and $R_i = \max\{R_i(l) : l \in \mathfrak{g}^*\}$.

DEFINITION 2.3

A linear functional $l \in \mathfrak{g}^*$ is called *generic* if $R_i(l) = R_i$ for all i, $1 \leq i \leq n$.

Let $\mathcal{U} = \{l \in \mathfrak{g}^* : l \text{ is generic}\}$. Since for any $l \in \mathfrak{g}^*$, we have $g.l \mid \mathfrak{z} = l \mid \mathfrak{z}$ where $g.l = l \circ \operatorname{Ad} g^{-1}$, we get $R_i(l) = R_i(g.l)$, $1 \leq i \leq n$ and hence,

(i) $\mathcal{U}$ is a G-invariant Zariski open subset of $\mathfrak{g}^*$. So $\mathcal{U}$ is union of orbits.

(ii) If j is a jump index for some $l \in \mathcal{U}$, then j is a jump index for all $l \in \mathcal{U}$.

(iii) Let $l \in \mathcal{U}$, then the number of jump indices for l is the same as the dimension of O_l (as a manifold). For, the rank of the matrix B_l is equal to the number of jump indices ($= 2k$, say) and the dimension of the radical r_l is the nullity of the matrix of B_l, which is $n - 2k$. Since $\mathfrak{g}/r_l$ is diffeomorphic to O_l (see [7]), we have $\dim O_l = 2k$.

(iv) Every orbit in $\mathcal{U}$ is of maximum dimension though not every maximum dimensional orbit may be in $\mathcal{U}$.

Note 2.2. If $l \in \mathfrak{g}^*$ is such that $\tilde{B}_l$ is an invertible matrix, then $r_l = \mathfrak{z}$ and then $m+1, \ldots, n$ are all jump indices and moreover

$$\mathcal{U} = \{l \in \mathfrak{g}^* : \tilde{B}_l \text{ is an invertible matrix}\}.$$

Clearly, if the codimension of $\mathfrak{z}$ in $\mathfrak{g}$ is odd then this cannot happen. Following [18] and [19], we call, the two step nilpotent Lie algebra a *MW algebra*, if there exists $l \in \mathfrak{g}^*$ such that $\tilde{B}_l$ is nondegenerate (or the corresponding matrix is invertible). For example, Heisenberg Lie algebras and $\mathfrak{f}_{2n,2}$, the free nilpotent Lie algebras of step two are MW algebras (see [2]).

Our aim is to parametrize the orbits in $\mathcal{U}$. We will see that they constitute a set of full Plancherel measure. We again describe some notation.

$$N = \{1, \ldots, m, n_1, \ldots, n_r\} \subset \{1, \ldots, n\}$$

is the complement of J in $\{1, \ldots, n\}$, $V_J = \operatorname{span}_{\mathbb{R}}\{X_{j_i} : 1 \leq i \leq 2k, j_i \in J\}$, $V_N = \operatorname{span}_{\mathbb{R}}\{X_1, \ldots, X_m, X_{n_i} : 1 \leq i \leq r, n_i \in N\}$, $V_J^* = \operatorname{span}_{\mathbb{R}}\{X_{j_1}^*, \ldots, X_{j_{2k}}^*\}$, $V_N^* = \operatorname{span}_{\mathbb{R}}\{X_1^*, \ldots, X_m^*, X_{n_i}^* : n_i \in N\}$, $\tilde{V}_N^* = \operatorname{span}_{\mathbb{R}}\{X_{n_i}^* : 1 \leq i \leq r\}$.

The following theorem shows that there exist a vector subspace of $\mathfrak{g}^*$ which intersects *almost every* orbit contained in $\mathcal{U}$ at exactly one point (see [7]). In the two step case one can easily prove it using Theorem 2.1 (see [23]).

Theorem 2.2. (i) *V_N^* intersects every orbit in $\mathcal{U}$ at a unique point.* (ii) *There exist a birational homeomorphism $\Psi : (V_N^* \cap \mathcal{U}) \times V_J^* \to \mathcal{U}$.*

Note 2.3. For each coadjoint orbit in $\mathcal{U}$, we choose their representatives from $V_N^* \cap \mathcal{U}$. Note that $V_N^* \cap \mathcal{U}$ can be identified with the cartesian product of $\tilde{V}_N^*$ and a Zariski open subset $\mathcal{U}'$ of $\mathfrak{z}^*$, where $\mathcal{U}' = \{l \in \mathfrak{z}^* : R_i(l) = R_i, 1 \leq i \leq m\}$.

We begin with a brief discussion of Kirillov theory, for details see [7]. Let G be a connected, simply connected nilpotent Lie group with Lie algebra $\mathfrak{g}$. G acts on $\mathfrak{g}^*$ by the coadjoint action. Given any $l' \in \mathfrak{g}^*$ there exist a subalgebra $\mathfrak{h}_{l'}$ of $\mathfrak{g}$ which is maximal with respect to the property

$$l'([\mathfrak{h}_{l'}, \mathfrak{h}_{l'}]) = 0. \tag{2.3}$$

Thus we have a character $\chi_{l'} : \exp(\mathfrak{h}_{l'}) \to \mathbb{T}$ given by

$$\chi_{l'}(\exp X) = e^{2\pi i l'(X)}, X \in \mathfrak{h}_{l'}.$$

Let $\pi_{l'} = \mathrm{ind}_{\exp(\mathrm{h}_{l'})}^{G} \chi_{l'}$. Then

(1) $\pi_{l'}$ is an irreducible unitary representation of G.

(2) If $\mathfrak{h}'$ is another subalgebra maximal with respect to the property $l'([\mathfrak{h}', \mathfrak{h}']) = 0$, then $\mathrm{ind}_{\exp(\mathrm{h}_l')}^{G} \chi_l \cong \mathrm{ind}_{\exp(\mathrm{h}')}^{G} \chi_{l'}$.

(3) $\pi_{l_1} \cong \pi_{l_2}$ if and only if l_1 and l_2 belong to the same coadjoint orbit.

(4) Any irreducible unitary representation π of G is equivalent to π_l for some $l \in \mathfrak{g}^*$.

So we have a map $\kappa : \mathfrak{g}^*/\mathrm{Ad}^*(G) \to \hat{G}$, which is a bijection. A subalgebra corresponding to $l \in \mathfrak{g}^*$, maximal with respect to (2.3) is called a *polarization*. It is known that the maximality of $\mathfrak{h}$ with respect to (2.3) is equivalent to the following *dimension condition*

$$\dim \mathfrak{h} = \frac{1}{2}(\dim \mathfrak{g} + \dim r_l).$$

Now suppose $\mathfrak{g}$ is a two step nilpotent Lie algebra and $l \in \mathfrak{g}^*$. The following technique for construction of a polarization corresponding to l, seems to be standard: we consider the bilinear form $\tilde{B}_l$ on the complement of the center, we restrict $\tilde{B}_l$ on its nondegenerate subspace, then on that subspace we can choose a basis with respect to which $\tilde{B}_l$ is the canonical symplectic form. With a little modification the basis can be chosen to be orthonormal as well. This is essentially what was done to obtain a canonical polarization in [19, 2, 26, 21]. We will set down the basis change explicitly; our main ingredient for that is the following lemma.

Lemma 2.1. *Let $B : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a nondegenerate, alternating, bilinear form. Then there exists an orthonormal basis $\{X_i, Y_i : 1 \leq i \leq k\}$ of $\mathbb{R}^n$ such that $B(X_i, Y_j) = \delta_{i,j}\lambda_j(B)$, $B(X_i, X_j) = B(Y_i, Y_j) = 0, 1 \leq i, j \leq k, n = 2k$ where $\pm i\lambda_j(B)$ are the eigenvalues of the matrix of B.*

As a consequence we have the following.

COROLLARY 2.2.1

Let $l \in \mathfrak{g}^*$. Then there exist an orthonormal basis

$$\{X_1, \ldots, X_m, Z_1(l), \ldots, Z_r(l), W_1(l), \ldots, W_k(l), Y_1(l), \ldots, Y_k(l)\} \tag{2.4}$$

of $\mathfrak{g}$ such that

(a) $r_l = \mathrm{span}_{\mathbb{R}}\{X_1, \ldots, X_m, Z_1(l), \ldots, Z_r(l)\}$.

(b) $l([W_i(l), Y_j(l)]) = \delta_{ij}\lambda_j(l), 1 \leq i, j \leq k$ *and*

$$l([W_i(l), W_j(l)]) = l([Y_i(l), Y_j(l)]) = 0, 1 \leq i, j \leq k.$$

(c) $\mathrm{span}_{\mathbb{R}}\{X_1, \ldots, X_m, Z_1(l), \ldots, Z_r(l), W_1(l), \ldots, W_k(l)\} = \mathfrak{h}$ *is a polarization for l.*

For a proof see [23]. We call the above basis an *almost symplectic basis*. Given $X \in \mathfrak{g}$ and a basis (2.4) we write

$$X = \sum_{j=1}^{m} x_j X_j(l) + \sum_{j=1}^{r} z_j Z_j(l) + \sum_{j=1}^{k} w_j W_j(l) + \sum_{j=1}^{k} y_j Y_j(l) \equiv (x, z, w, y).$$

Since we are going to use induced representations we need to describe nice sections of G/H and a G-invariant measure on G/H. In our situation H will always be a normal subgroup of G. We identify G and $\mathfrak{g}$ via the exponential map. Let $\mathfrak{h}$ be an ideal of $\mathfrak{g}$ containing $\mathfrak{z}$ and $H = \exp \mathfrak{h}$.

We take $\{X_1, \ldots, X_m, X_{m+1}, \ldots, X_{m+k}, \ldots, X_n\}$ a basis of $\mathfrak{g}$ such that

$$\mathfrak{z} = \operatorname{span}_{\mathbb{R}}\{X_1, \ldots, X_m\}, \qquad \mathfrak{h} = \operatorname{span}_{\mathbb{R}}\{X_1, \ldots, X_m, X_{m+1}, \ldots, X_{m+k}\}.$$

If $L_g(x) = g^{-1}x$ and $R_g(x) = xg$, $x, g \in G$, then it is clear from the group multiplication that the Jacobian matrix for either of the transformations is upper triangular with diagonal entries 1. Thus we have the following lemma whose proof can be found in [7].

Lemma 2.2. *Let* $\mathfrak{g}$, $\mathfrak{h}$, $\{X_1, \ldots, X_m, X_{m+1}, \ldots, X_{m+k}, \ldots, X_n\}$ *be as before. Then*

(i) $dx_1 \ldots dx_n$ *is a left and right invariant measure on* G.

(ii) $\sigma : G/H \to G$ *given by*

$$\sigma\left(\exp\left(\sum_{i=1}^{n} t_i X_i\right) H\right) = \exp\left(\sum_{i=1}^{n-m-k} t_{m+k+i} X_{m+k+i}\right),$$

is a section for G/H.

(iii) $dx_{m+k+1} \ldots dx_n$ *is a* G*-invariant measure on* G/H.

Now we come to the construction of representations corresponding to $l \in V_N^* \cap \mathcal{U}$. Let $\dim r_l = m+r$ and $\dim O_l = 2k$ so $m+r+2k = n$. We choose an almost symplectic basis (2.4) of $\mathfrak{g}$ corresponding to l and get hold of $\mathfrak{h}_l$ as in Corollary 2.2.1, c). On $H_l = \exp(\mathfrak{h}_l)$ we have the character $\chi_l : H_l \to \mathbb{T}$. Let $\pi_l = \operatorname{ind}_{H_l}^{G} \chi_l$. We do not use the standard model for the induced representation as given in chapter 2 of [7], rather using the continuous section σ given in Lemma 2.2.2 and computing the unique splitting of a typical group element

$$(x, z, w, y) = (0, 0, 0, y)\left(x - \frac{1}{2}[(0, 0, 0, y), (0, z, w, 0)], z, w, 0\right),$$

corresponding to σ, the representation π_l is realized on $L^2(\mathbb{R}^k)$ and is given by

$$\begin{aligned}&(\pi_l(x, z, w, y) f)(\bar{y}) \qquad f \in L^2(\mathbb{R}^k),\\ &= \mathrm{e}^{2\pi i (l(x)+l(z)+l(w)-\frac{1}{2}\Sigma_{j=1}^{k} y_j w_j \lambda_j(\ell)+\Sigma_{j=1}^{k} \bar{y}_j w_j \lambda_j(\ell))} f(\bar{y} - y),\end{aligned} \tag{2.5}$$

for $\bar{y} \in \mathbb{R}^k$.

DEFINITION 2.4

For $l \in \mathfrak{g}^*$ we define

$$Pf(l) = \sqrt{\det((B_l')_{js})}$$

called the *Pfaffian* of l, where $(B_l')_{is} = l([X_{j_i}, X_{j_s}]), X_{j_i}, X_{j_s} \in V_J$.

Note 2.4. If J is the set of jump indices for l, then B_l' is nondegenerate on V_J and then $Pf(l)$ is the Pfaffian of B_l' (see [15]). It is easy to show that

(a) $\det((B_l')_{is})$ is always a square of a polynomial and hence $Pf(l)$ is a homogeneous polynomial in $l \mid \mathfrak{z}$.

(b) $Pf(l) \neq 0$ if $l \in \mathcal{U}$ and is Ad^*G invariant.

We restrict our attention to the representations π_l for $l \in V_N^* \cap \mathcal{U}$ and, motivated by the example of the Heisenberg groups ask the following question: suppose $f \in L^1(G) \cap L^2(G)$. What is the relation between $\hat{f}(\pi_l)$ and $\mathcal{F}_1 f(l \mid \mathfrak{z}, v)$? Here $\hat{f}$ is the operator valued group Fourier transform, (z, v) are elements of the group with $z \in \mathfrak{z}$ and $v \in \mathrm{Span}_{\mathbb{R}}\{X_{m+1}, \ldots, X_n\}$ and $\mathcal{F}_1 f(l \mid \mathfrak{z}, v)$ means the partial (Euclidean) Fourier transform of f in the central variables at the point $l \mid \mathfrak{z}$.

In the case of the Heisenberg groups H_n, with Lie algebra

$$\mathfrak{h}_n = \mathrm{Span}_{\mathbb{R}}\{Z, W_1, \ldots, W_n, Y_1, \ldots, Y_n\}$$

with the only nontrivial Lie brackets $[W_i, Y_i] = Z$, $1 \leq i \leq n$, we have $V_N = \mathrm{Span}_{\mathbb{R}}\{Z\}$, $V_J = \mathrm{Span}_{\mathbb{R}}\{W_1, \ldots, Y_n\}$ and $V_N^* \cap \mathcal{U} = \{l \in \mathfrak{h}_n^* : l(Z) = \lambda \neq 0\}$. Then it can be proved easily (see [10]), that for $f \in L^1(H_n) \cap L^2(H_n)$ and $l \in V_N^* \cap \mathcal{U}$,

$$\|\hat{f}(\pi_l)\|_{HS}^2 = |\lambda|^{-n} \int_{\mathbb{R}^{2n}} |\mathcal{F}_1 f(\lambda, w, y)|^2 \mathrm{d}w\mathrm{d}y. \tag{2.6}$$

To find an analogue of (2.6), the most important thing is to find the Jacobian of a transformation which we are now going to describe.

Let $l \in \tilde{V}_N^* = \mathrm{Span}_{\mathbb{R}}\{X_{n_1}^*, \ldots, X_{n_r}^*\}$. Notice that for H_n and $F_{n,2}$, where n is even, $\tilde{V}_N^* = \{0\}$, so the transformation we are going to describe, appears only for those two step nilpotent Lie groups whose Lie algebras are not MW. Suppose $l_{n_i} = l(X_{n_i})$, $1 \leq i \leq r$; we also have $l(X_{j_i}) = 0$, $1 \leq i \leq 2k$. From $\tilde{B}_l$ we have constructed an orthonormal basis $\{Z_1(l), \ldots, Z_r(l), W_1(l), \ldots, W_k(l), Y_1(l), \ldots, Y_k(l)\}$ with respect to which the matrix of $\tilde{B}_l$ is of the following form

$$\begin{pmatrix} 0 & 0 \\ 0 & S \end{pmatrix}, \tag{2.7}$$

where the $2k \times 2k$ matrix S is given by

$$\begin{pmatrix} 0 & \begin{matrix} \lambda_1(l) & & \\ & \ddots & \\ & & \lambda_k(l) \end{matrix} \\ \begin{matrix} -\lambda_1(l) & & \\ & \ddots & \\ & & -\lambda_k(l) \end{matrix} & 0 \end{pmatrix}, \tag{2.8}$$

where $\lambda_i(l) > 0$, $1 \leq i \leq k'$. Let $l(Z_i(l)) = \bar{l}_i$, $1 \leq i \leq r$. We consider the map

$$\begin{aligned} &\phi : \tilde{V}_N^*(\cong \mathbb{R}^r) \to \mathbb{R}^r \\ &\phi(l_{n_1}, \ldots, l_{n_r}) = (\bar{l}_1, \ldots, \bar{l}_r). \end{aligned} \tag{2.9}$$

Lemma 2.3. *The modulus of the Jacobian determinant of* ϕ *is given by*

$$|\det J_\phi| = \frac{|Pf(l)|}{\lambda_1(l)\lambda_2(l)\ldots\lambda_k(l)},$$

where J_ϕ *is the Jacobian matrix of* ϕ.

Proof. First we systematically describe the transformations which gave the almost symplectic basis. We restrict ourselves only to the complement of the center, because it is there that the change of basis takes place.

$$\begin{aligned} A_1 &: \{X_{m+1}, X_{m+2}, \ldots, X_n\} &\to\ & \{X_{n_1}, \ldots, X_{n_r}, X_{j_1}, \ldots, X_{j_{2k}}\} \\ A_2 &: \{X_{n_1}, \ldots, X_{n_r}, X_{j_1}, \ldots, X_{2k}\} &\to\ & \{\tilde{X}_{n_1}, \ldots, \tilde{X}_{n_r}, X_{j_1}, \ldots, X_{j_{2k}}\} \\ A_3 &: \{\tilde{X}_{n_1}, \ldots, \tilde{X}_{n_r}, X_{j_k}, \ldots, X_{j_{2k}}\} &\to\ & \{Z_1(l), \ldots, Z_r(l), W_1(l), \ldots, \\ &&& W_k(l), Y_1(l), \ldots, Y_k(l)\}, \end{aligned}$$

where $\tilde{X}_{n_i} = X_{n_i} - \sum_{s=1}^{2k} c_s^i(l) X_{j_s}$, $1 \leq i \leq r$, so that each $\tilde{X}_{n_i} \in r_l$. A_1 is just a rearrangement of basis and hence is given by an orthogonal matrix. A_2 is clearly given by a lower triangular matrix with diagonal entries equal to one. The matrix of A_3 looks like

$$\begin{pmatrix} A' & C' \\ 0 & D' \end{pmatrix},$$

where A' is a $r \times r$ matrix, C' is a $r \times 2k$ matrix and D' is a $2k \times 2k$ matrix, because A_3 is obtained from the following operations:

(i) Gram–Schmidt orthogonalization of $\{\tilde{X}_{n_i} : 1 \leq i \leq r\}$.
(ii) Finding the orthogonal complement of the span of $\{\tilde{X}_{n_i} : 1 \leq i \leq r\}$.
(iii) Choosing an almost symplectic basis on the nondegenerate subspace of $\tilde{B}_l$.

Notice that for $l \in \tilde{V}_N$, $l(X_{j_i}) = 0$, $1 \leq i \leq 2k$; thus $l(\tilde{X}_{n_i}) = l(X_{n_i})$, $1 \leq i \leq r$. Hence

$$|\det J_\phi| = |\det A'|.$$

Since $|\det A_1 . \det A_2 . \det A_3| = 1$, we have $|\det A_3| = 1$. But

$$|\det A_3| = |\det A'| |\det D'|.$$

So $|\det J_\phi| = |\det D'|^{-1}$. If we write $\tilde{B}_l$ in terms of the basis $\{\tilde{X}_{n_1}, \ldots, \tilde{X}_{n_r}, X_{j_1}, \ldots, X_{j_{2k}}\}$, then the matrix of $\tilde{B}_l$ looks like

$$\begin{pmatrix} 0 & 0 \\ 0 & B_l' \end{pmatrix},$$

where $(B_l')_{is} = l([X_{j_i}, X_{j_s}])$. Thus clearly $|\det B_l'| = |Pf(l)|^2$. Because of A_3 the above matrix changes to

$$\begin{pmatrix} 0 & 0 \\ 0 & D' B_l' (D')^t \end{pmatrix}$$

which is nothing but the matrix in (2.7). So

$$|\det D'|^2 = \frac{|\lambda_1(l)\ldots\lambda_k(l)|^2}{|Pf(l)|^2} \quad \Longrightarrow \quad |\det D'| = \frac{|\lambda_1(l)\ldots\lambda_k(l)|}{|Pf(l)|}.$$

Thus $|\det J_\phi| = \dfrac{|Pf(l)|}{|\lambda_1(l)\ldots\lambda_k(l)|}$ as claimed.

Now we come to the analogue of (2.6). Given $f \in L^1(G) \cap L^2(G)$ and $\pi \in \hat{G}$ the so called group Fourier transform at π is the bounded linear transformation (realized on the Hilbert space $\mathcal{H}_\pi$) given by

$$\langle \hat{f}(\pi)\xi, \eta\rangle = \int_G f(g)\langle \pi(g^{-1})\xi, \eta\rangle \mathrm{d}\mu(g), \qquad \xi, \eta \in \mathcal{H}_\pi.$$

We recall, for $l \in V_N^* \cap \mathcal{U}$, the almost symplectic basis (2.4) and because of the orthonormal basis change, $\mathrm{d}x\mathrm{d}z\mathrm{d}w\mathrm{d}y$ is the normalized Haar measure on G we started with, where

$$(x, z, w, y) = \sum_{i=1}^{m} x_i X_i + \sum_{i=1}^{r} z_i Z_i(l) + \sum_{i=1}^{k} w_i W_i(l) + \sum_{i=1}^{k} y_i Y_i(l).$$

The representation π_l corresponding to l is now given by (2.5). Let $\mathrm{d}l_{n_1}\ldots\mathrm{d}l_{n_r}$ denote the usual Lebesgue measure on $\tilde{V}_N^*$ (after we identify $\tilde{V}_N^*$ with $\mathbb{R}^r$ through the basis $\{X_{n_1}^*, \ldots, X_{n_r}^*\}$).

Theorem 2.3. *Let* $f \in L^1(G) \cap L^2(G)$. *Then*

$$\begin{aligned} &|Pf(l)| \int_{\tilde{V}_N^*} \|\hat{f}(\pi_l)\|_{HS}^2 \mathrm{d}l_{n_1}\ldots\mathrm{d}l_{n_r} \\ &= \int_{\mathrm{R}^{r+2k}} |\mathcal{F}_1 f(l_1, \ldots, l_m, x_{n_1}, \ldots, x_{n_r}, u, v)|^2 \mathrm{d}x_{n_1}\ldots\mathrm{d}v \end{aligned} \tag{2.10}$$

for almost every $l \in V_N^* \cap \mathcal{U}$, *where*

$$\begin{aligned} &\mathcal{F}_1 f(l_1, \ldots, l_m, x_{n_1}, \ldots, x_{n_r}, u, v) \\ &= \int_{\mathrm{R}^m} f(x_1, \ldots, x_m, x_{n_1}, \ldots, x_{n_r}, u, v) \mathrm{e}^{-2\pi i \Sigma_{j=1}^m l_j x_j} \mathrm{d}x_1\ldots\mathrm{d}x_m \end{aligned}$$

and $l(X_j) = l_j$, $1 \le j \le m$.

Proof. Let $\phi \in L^2(\mathbb{R}^k)$. Then from (2.5),

$$\begin{aligned} &(\hat{f}(\pi_l)\phi)(\bar{y}) \\ &= \int_{\mathrm{R}^{m+r+2k}} f(x, z, w, y)(\pi_l(-x, -z, -w, -y)\phi)(\bar{y})\mathrm{d}x\mathrm{d}z\mathrm{d}w\mathrm{d}y \\ &= \int_{\mathrm{R}^{m+r+2k}} f(x, z, w, y)\mathrm{e}^{2\pi i[-l(x)-l(z)-\Sigma_{j=1}^k w_j \bar{y}_j \lambda_j(l)-(1/2)\Sigma_{j=1}^k w_j y_j \lambda_j(l)]} \\ &\quad \times\phi(\bar{y}+y)\mathrm{d}x\mathrm{d}z\mathrm{d}w\mathrm{d}y \\ &= \int_{\mathrm{R}^{r+m+2k}} f(x, z, w, y-\bar{y})\mathrm{e}^{2\pi i[-l(x)-l(z)-\Sigma_{j=1}^k w_j \bar{y}_j \lambda_j(l)-(1/2)\Sigma_{j=1}^k w_j (y_j-\bar{y}_j)\lambda_j(l)]} \\ &\quad \times\phi(y)\mathrm{d}x\mathrm{d}z\mathrm{d}w\mathrm{d}y \qquad \text{(by the change of variable } y' = y + \bar{y}) \end{aligned}$$

$$
\begin{aligned}
&= \int_{\mathsf{R}^{r+m+2k}} f(x,z,w,y-\bar{y})\mathrm{e}^{2\pi i[-l(x)-l(z)-(1/2)\Sigma_{j=1}^{k} w_j y_j \lambda_j(l)-(1/2)\Sigma_{j=1}^{k} w_j \bar{y}_j \lambda_j(l)]} \\
&\quad \times \phi(y)\mathrm{d}x\mathrm{d}z\mathrm{d}w\mathrm{d}y \\
&= \int_{\mathsf{R}^{r+m+2k}} f(x,z,w,y-\bar{y})\mathrm{e}^{-2\pi i l(x)}\mathrm{e}^{-2\pi i l(z)}\mathrm{e}^{-\pi i \Sigma_{j=1}^{k}(y_j+\bar{y}_j)w_j\lambda_j(l)} \\
&\quad \times \phi(y)\mathrm{d}x\mathrm{d}z\mathrm{d}w\mathrm{d}y.
\end{aligned}
$$

Let

$$K_l^f(y,\bar{y}) = \int_{\mathsf{R}^{m+r+k}} f(x,z,w,y-\bar{y})\mathrm{e}^{-2\pi i l(x)}\mathrm{e}^{-2\pi i l(z)}\mathrm{e}^{-\pi i \Sigma_{j=1}^{k}(y+\bar{y})\lambda_j(l)w_j}\mathrm{d}x\mathrm{d}z\mathrm{d}w.$$

Since $f \in L^1(G) \cap L^2(G)$, it follows that $K_l^f \in L^2(\mathbb{R}^k \times \mathbb{R}^k)$ for almost every $l \in \tilde{V}_N^* \cap \mathcal{U}$. Let $l \mid \mathfrak{z} = (l_1,\ldots,l_m)$ and $l \mid \operatorname{span}_{\mathsf{R}}\{Z_1(l),\ldots,Z_r(l)\} = (\bar{l}_1,\ldots,\bar{l}_r)$. Then

$$K_l^f(y,\bar{y}) = \mathcal{F}_{123}f\left(l_1,\ldots,l_m,\bar{l}_1,\ldots,\bar{l}_r,\frac{y_1+\bar{y}_1}{2}\lambda_1(l),\ldots,\frac{y_k+\bar{y}_k}{2}\lambda_k(l),y-\bar{y}\right),$$

where $\mathcal{F}_{123}$ stands for the partial Fourier (Euclidean) transform in the variables x, z, w. Thus $\hat{f}(\pi_l)$ is a Hilbert–Schmidt operator on $L^2(\mathbb{R}^k)$ with the kernel K_l^f. Hence

$$
\begin{aligned}
\|\hat{f}(\pi_l)\|_{HS}^2 &= \int_{\mathsf{R}^{2k}} |k_l^f(y,\bar{y})|^2\mathrm{d}y\mathrm{d}\bar{y} \\
&= \int_{\mathsf{R}^{2k}} \Bigg|\mathcal{F}_{123}f\Bigg(l_1,\ldots,\bar{l}_r,\frac{y_1+\bar{y}_1}{2}\lambda_1(l),\ldots,\frac{y_k+\bar{y}_k}{2} \\
&\qquad \times \lambda_k(l),y-\bar{y}\Bigg)\Bigg|^2\mathrm{d}y\mathrm{d}\bar{y}.
\end{aligned}
$$

If we do the change of variables

$$
\begin{aligned}
u_j &= \frac{y_j+\bar{y}_j}{2}\lambda_j(l), \quad 1 \le j \le k, \\
v_j &= y_j-\bar{y}_j, \quad 1 \le j \le k,
\end{aligned}
$$

then the modulus of the Jacobian determinant is $|\lambda_1(l)\ldots\lambda_k(l)|$ and the above integral reduces to

$$|\lambda_1(l)\ldots\lambda_k(l)|^{-1}\left(\int_{\mathsf{R}^{2k}}|\mathcal{F}_{123}f(l_1,\ldots,l_m,\bar{l}_1,\ldots\bar{l}_r,u,v)|^2\mathrm{d}u\mathrm{d}v\right),$$

where $u = (u_1,\ldots,u_k)$ and $v = (v_1,\ldots,v_k)$. By applying the Euclidean Plancherel theorem in the variable u we get

$$\|\hat{f}(\pi_l)\|^2 = |\lambda_1(l)\ldots\lambda_k(l)|^{-1}\int_{\mathsf{R}^{2k}}|\mathcal{F}_{12}f(l_1,\ldots,l_m,\bar{l}_1,\ldots,\bar{l}_r,u,v)|^2\mathrm{d}u\mathrm{d}v.$$

If we integrate both sides of the above equation on $\tilde{V}_N^*$ with respect to the usual Lebesgue measure and use change of variables by the map ϕ defined in (2.9), we get

$$
\begin{aligned}
&\int_{\tilde{V}_N^*}\|\hat{f}(\pi_l)\|_{HS}^2\mathrm{d}l_{n_1}\ldots\mathrm{d}l_{n_r} \\
&= |\lambda_1(l)\ldots\lambda_k(l)|^{-1}\frac{|\lambda_1(l)\ldots\lambda_k(l)}{|Pf(l)|} \\
&\quad \times \int_{\mathsf{R}^{r+2k}}|\mathcal{F}_{12}f(l_1,\ldots,l_m,l_{n_1},\ldots,l_{n_r},u,v)|^2\mathrm{d}l_{n_1}\ldots\mathrm{d}l_{n_r}\mathrm{d}u\mathrm{d}v.
\end{aligned}
$$

Then by applying the Euclidean Plancherel theorem on the variables $(l_{n_1}, \ldots, l_{n_r}) \in \mathbb{R}^r$ we get

$$\begin{aligned}
&|Pf(l)| \int_{\tilde{V}_N^*} \|\hat{f}(\pi_l)\|_{HS}^2 \mathrm{d}l_{n_1} \ldots \mathrm{d}l_{n_r} \\
&= \int_{\mathbb{R}^{r+2k}} |\mathcal{F}_1 f(l_1, \ldots, l_m, x_{n_1}, \ldots, x_{n_r}, u, v)|^2 \mathrm{d}x_{n_1} \ldots \mathrm{d}x_{n_r} \mathrm{d}u \mathrm{d}v.
\end{aligned}$$

This completes the proof.

Theorem 2.4 (*Plancherel theorem*). *For $f \in L^1(G) \cap L^2(G)$*

$$\int_{V_N^* \cap \mathcal{U}} \|\hat{f}(\pi_l)\|_{HS}^2 |Pf(l)| \mathrm{d}l = \|f\|_{L^2(G)}^2,$$

where $\mathrm{d}l$ *is the standard Lebesgue measure on $V_N^* (\cong \mathbb{R}^{m+r})$ with respect to the basis $\{X_1^*, \ldots, X_m^*, X_{n_1}^*, \ldots, X_{n_r}^*\}$.*

Proof. Regarding $V_N^* \cap \mathcal{U}$ as the Cartesian product of $\mathcal{U}'$ and $\mathbb{R}^r$ as in Note 2.3, we integrate both sides of (2.10) with respect to the standard Lebesgue measure on $\mathfrak{z}^*$ (upon identification with $\mathbb{R}^m$ via the basis $\{X_1^*, \ldots, X_m^*\}$) to get

$$\begin{aligned}
&\int_{V_N^* \cap \mathcal{U}} \|\hat{f}(\pi_l)\|_{HS}^2 |Pf(l)| \mathrm{d}l \\
&= \int_{\mathcal{U}'} \left(|Pf(l)| \int_{\tilde{V}_N^*} \|\hat{f}(\pi_l)\|_{HS}^2 \mathrm{d}l_{n_1} \ldots \mathrm{d}l_{n_r} \right) \mathrm{d}l_1 \ldots \mathrm{d}l_m \\
&= \int_{\mathcal{U}'} \left(\int_{\mathbb{R}^{r+2k}} |\mathcal{F}_c f(l_1, \ldots, l_m, x_{n_1}, \ldots, x_{n_r}, u, v)|^2 \mathrm{d}x_{n_1} \ldots \mathrm{d}v \right) \mathrm{d}l_1 \ldots \mathrm{d}l_m \\
&\qquad \text{(by (2.10))} \\
&= \int_{\mathbb{R}^m} \int_{\mathbb{R}^{r+2k}} |f(x_1, \ldots, x_m, x_{n_1}, \ldots, x_{n_r}, u, v)|^2 \mathrm{d}x_1 \ldots \mathrm{d}x_m \mathrm{d}x_{n_1} \ldots \mathrm{d}x_{n_r} \mathrm{d}u \mathrm{d}v,
\end{aligned}$$

by using the Euclidean Plancherel theorem in the outer integral ($\mathcal{U}'$ is a set of full Lebesgue measure in $\mathfrak{z}^*$). The last integral is, of course, $\|f\|_{L^2(G)}^2$ and the proof is complete.

Note 2.5. The situation is simpler if we consider the case of MW groups. In this case $V_N^* \cap \mathcal{U} \subseteq \mathfrak{z}^*$ is Zariski open and for $l \in \mathcal{U} \subseteq \mathfrak{z}^*$, the representation π_l is given by

$$(\pi_l(x, z, y) f)(\bar{y}) = \mathrm{e}^{2\pi i [l(x) + \sum_{j=1}^k \bar{y}_j \bar{w}_j \lambda_j(l) - (1/2) \sum_{j=1}^k y_j w_j \lambda_j(l)]} f(\bar{y} - y),$$

where $\bar{y} \in \mathbb{R}^k$, $f \in L^2(\mathbb{R}^k)$ and $\dim \mathfrak{g}/\mathfrak{z} = 2k$. Then it follows from the calculations done in theorem (2.3) that

$$\|\hat{f}(\pi_l)\|_{HS}^2 = \frac{1}{|\lambda_1(l) \ldots \lambda_k(l)|} \int_{\mathbb{R}^{2k}} |\mathcal{F}_1 f(l_1, \ldots, l_m, u, v)|^2 \mathrm{d}u \mathrm{d}v.$$

Clearly $|\lambda_1(l) \ldots \lambda_k(l)| = |Pf(l)|$, since $\tilde{B}_l$ is nondegenerate. The Plancherel theorem again follows from integrating both sides on $U \subseteq \mathfrak{z}^*$. *So the change of variables through the map ϕ is not needed for MW groups.*

Let $\mathfrak{g}$ be a two step nilpotent Lie algebra with a basis $\mathcal{B}$ as before. Now we consider elements of $\mathfrak{g}$ as left invariant differential operators acting on $C^\infty(G)$ where the action is given by

$$X(f)(g) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} f(g \exp tX).$$

We define

$$\mathcal{L} = -\sum_{i=1}^{n-m} X_{m+i}^2 \tag{2.11}$$

and as on the Heisenberg groups, call it the sub-Laplacian of G.

Given an irreducible, unitary representation π of G, we look at the matrix functions of π given by

$$\begin{aligned} &\phi_{u,v}^\pi : G \to \mathbb{C}, \qquad u, v \in \mathcal{H}_\pi \\ &\phi_{u,v}^\pi(g) = \langle \pi(g)u, v\rangle. \end{aligned} \tag{2.12}$$

Our aim is to find: which matrix functions of representations are joint eigenfunctions of $\mathcal{L}$ and $\{X_i : 1 \le i \le m\}$?

Given $\pi \in \hat{G}$ and $X \in \mathfrak{g}$, we have

$$\mathrm{d}\pi(X)(u) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \pi(\exp tX)u, \tag{2.13}$$

where u, v are C^∞ vectors for π. If $A \in \mathcal{U}(\mathfrak{g})$ then it follows that

$$A\langle \pi(g)u, v\rangle = \langle \pi(g)\mathrm{d}\pi(A)u, v\rangle$$

(see [7]). Thus if u is an eigenvector for $\mathrm{d}\pi(A)$ then $\phi_{u,v}^\pi$ is an eigenfunction for A. Since for $1 \le i \le m$, $X_i \in \mathcal{Z}(\mathcal{U}(\mathfrak{g}))$, the center of the universal enveloping algebra, then $\mathrm{d}\pi(X_i)$ acts as a scalar (see [7]) and hence $\phi_{u,v}^\pi$ is an eigenfunction for X_i for any u, v. Thus our job reduces to finding the eigenfunctions of $\mathrm{d}\pi(\mathcal{L})$ which are also matrix functions of π. Looking at the case of the Heisenberg groups and the group $F_{2n,2}$ (see [26]) it is reasonable to expect that $\mathrm{d}\pi(\mathcal{L})$ is closely related to the Hermite operator and, indeed, that is the case.

We use on G the exponential coordinates given by the above chosen basis. Given $x = \sum_{i=1}^n x_i X_i$ and $x' = \sum_{i=1}^n x_i' X_i$ denote

$$[x, x']_p = \langle [x, x'], X_p\rangle, \quad 1 \le p \le m,$$

where $\langle .,. \rangle$ is the Euclidean inner product on $\mathfrak{g}$ for which $\{X_i : 1 \le i \le n\}$ is an orthonormal basis. Then it follows that, for $1 \le i \le m$,

$$(X_i f)(x) = \frac{\partial f}{\partial x_i}(x), \tag{2.14}$$

and for $m + 1 \le i \le n$,

$$(X_i f)(x) = \left(\frac{\partial}{\partial x_i} + \frac{1}{2}\sum_{j=1}^m [x, X_i]_j \frac{\partial}{\partial x_j}\right) f(x). \tag{2.15}$$

Now we start with a representation $\pi_l \in \hat{G}$ such that $l \mid \mathfrak{z} \neq 0$. We get hold of an almost symplectic basis (2.4) with $\dim r_l = m + r$ and $\dim O_l = 2k$, so $n = 2k + m + r$. The representations π_l are realized on $L^2(\mathbb{R}^k)$ and are given by (2.5). Using the explicit description (2.5), it is easy to see that $C^\infty(\pi_l) = \mathcal{S}(\mathbb{R}^k)$, the Schwartz class functions on $\mathbb{R}^k$. By direct calculation we find the effect of applying $\mathrm{d}\pi_l$ on the elements of the almost symplectic basis, which in turn describes $\mathrm{d}\pi_l(\mathcal{L})$.

Lemma 2.4. *For* $\phi \in \mathcal{S}(\mathbb{R}^k)$ *and* $\xi \in \mathbb{R}^k$

(i) $\mathrm{d}\pi_l(Z_j(l))\phi(\xi) = 2\pi i \bar{l}_j \phi(\xi), \qquad 1 \leq j \leq r.$

(ii) $\mathrm{d}\pi_l(W_j(l))\phi(\xi) = 2\pi i \xi_j \lambda_j(l)\phi(\xi), \qquad 1 \leq j \leq k.$

(iii) $\mathrm{d}\pi_l(Y_j(l))\phi(\xi) = -\frac{\partial \phi}{\partial \xi_j}(\xi), \qquad 1 \leq j \leq k.$

(iv) $\mathrm{d}\pi_l(\mathcal{L})\phi(\xi) = \{4\pi^2 \sum_{j=1}^r \bar{l}_j^2 + L_l\}\phi(\xi)$, *where*

$$L_l = \sum_{j=1}^{k} \left(-\frac{\partial^2}{\partial \xi_j^2} + 4\pi^2 \lambda_j(l)^2 \xi_j^2 \right).$$

Because of (iv) now it is easy to describe the eigenfunctions of $\mathrm{d}\pi_l(\mathcal{L})$. Let $\mu(l) = 4\pi^2 \sum_{j=1}^r \bar{l}_j^2$. Then $\mathrm{d}\pi_l(\mathcal{L}) = \mu(l) + L_l$, and $\mu(l) \geq 0$. If ϕ is an eigenfunction of L_l with eigenvalue $c(l)$, then ϕ is an eigenfunction of $\mathrm{d}\pi_l(\mathcal{L})$ with eigenvalue $c(l) + \mu(l)$. Again, if ϕ_j^l is an eigenfunction of $-\frac{\partial^2}{\partial x^2} + 4\pi^2 \lambda_j(l)^2 x^2$ on $\mathbb{R}$, then clearly

$$\phi^l(\xi_1, \ldots, \xi_k) = \phi_i^l(\xi_1) \ldots \phi_k^l(\xi_k)$$

is an eigenfunction of L_l. Since for $s \in \mathbb{N}$, the sth normalized Hermite function h_s is an eigenfunction of $-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + x^2$ with eigenvalue $2s + 1$, it is clear that

$$h_s^l(x) = (2\pi \lambda_j(l))^{\frac{1}{4}} h_s(\sqrt{2\pi}\lambda_j(l)^{\frac{1}{2}} x)$$

is an eigenfunction of $-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + 4\pi^2 \lambda_j(l)^2 x^2$ with eigenvalue $2\pi\lambda_j(l)(2s + 1)$ and also $\|h_s^l\|_2 = 1$. So for $(\alpha_1, \ldots, \alpha_k) \in \mathbb{N}^k$ we define

$$h_\alpha^l(\xi_1, \ldots, \xi_k) = \Pi_{j=1}^k h_{\alpha_j}^l(\xi_j), \tag{2.16}$$

where

$$h_{\alpha_j}^l(\xi_j) = (2\pi\lambda_j(l))^{\frac{1}{4}} h_{\alpha_j}(\sqrt{2\pi}\lambda_j(l)^{\frac{1}{2}}\xi_j).$$

Then

$$L_l(h_\alpha^l) = \left(\sum_{j=1}^{k} 2\pi\lambda_j(l)(2\alpha_j + 1) \right) h_\alpha^l. \tag{2.17}$$

Thus

$$\mathrm{d}\pi_l(\mathcal{L})(h_\alpha^l) = \left(\mu(l) + \sum_{j=1}^{k} 2\pi\lambda_j(l)(2\alpha_j + 1) \right) h_\alpha^l. \tag{2.18}$$

Now we state a mild generalization of Theorem 1.2, which follows from Lemma 2.3 of [20].

Theorem 2.5. *Suppose* $f : \mathbb{R}^n \to \mathbb{C}$ *be a measurable function such that*
(i) $\int_{\mathbb{R}^n} e^{pa\pi\|x\|^2}|f(x)|^p dx < \infty$,
(ii) $\int_{\mathbb{R}^n} e^{qb\pi\|y\|^2}|\hat{f}(y)|^q|Q(y)|^r dy < \infty$,
where $a, b > 0$, Q *is a polynomial and* $r > 0$ *is any real number. If* $ab \geq 1$ *then* $f = 0$.

3. Extensions of Hardy's theorem

The principal result in this section is the analogue of the Theorem 1.2 for two step nilpotent Lie groups. Along the way we also talk about the analogue of Theorem 1.3. Hardy's theorem for Heisenberg groups was proved in [28] and its L^p-analogue (Theorem 1.2) and the analogue of Theorem 1.3 was proved in [3]. An analogue of Hardy's theorem on two step nilpotent Lie groups was proved in [1].

Remark 3.1. Our treatment in this section tacitly assumes that G is not MW. For the case of MW groups the treatment needs only obvious modifications using the description of $\|\hat{f}(\pi_l)\|_{HS}$ given in Note 2.5.

In the case of Heisenberg groups, Hardy's theorem and Cowling–Price theorem actually reduce to the corresponding problems on the center of the group by an application of (2.6). Two step nilpotent Lie groups having reasonable analogue of (2.6) in Theorem 2.3, it is expected that the same technique may work here also; and it does, as we shall show presently. Since we are going to talk about exponential decay of the group Fourier transform, we need a growth parameter on the dual, where usual exponential makes sense, but that has been addressed in §1. In our parametrization the dual is essentially a vector subspace (actually a Zariski open subset of that subspace) of $\mathfrak{g}^*$, which is good enough for us.

Let $\mathfrak{g}$ be a two step nilpotent Lie algebra with basis $\mathcal{B}$ as before. G is the corresponding connected, simply connected, Lie group. We write elements of $\mathfrak{g}$ (as well as G) by $(x, v) \equiv \sum_{i=1}^{m} x_i X_i + \sum_{i=1}^{n-m} v_i X_{m+i}$. The set $V_N^* \cap \mathcal{U}$ serves as the effective dual (that is, it is a set of full Plancherel measure in $\hat{G}$) of G and we put Euclidean norm there such that $\{X_1^*, \ldots, X_m^*, X_{n_i}^* : 1 \leq i \leq r\}$ is an orthonormal basis. We write elements of V_N^* as

$$(\lambda, \gamma) \equiv \sum_{i=1}^{m} \lambda_i X_i^* + \sum_{i=1}^{r} \gamma_i X_{n_i}^*.$$

To prove an analogue of Theorem 1.2, we need the following trivial lemma.

Lemma 3.1. *Let* G *be a two step nilpotent Lie group. Then there exists a constant* C *such that*

$$\|(x, v).(x_1, v_1)^{-1}\| \geq \|(x, v)\| - \|(x_1, v_1)\| - C\|(x, v)\|\,\|(x_1, v_1)\|, \tag{3.1}$$

for all $(x, v), (x_1, v_1) \in G$.

Now we come to the proposed analogue of Theorem 1.2.

Theorem 3.1. *Let* $f \in L^1(G) \cap L^2(G)$ *satisfy*
(i) $\int_G e^{pa\pi\|(x,v)\|^2}|f(x, v)|^p dx dv < \infty$,
(ii) $\int_{V_N^*} e^{qb\pi\|(\lambda,\gamma)\|^2}\|\hat{f}(\pi_{\lambda,\gamma})\|_{HS}^q|Pf(\lambda)|d\lambda d\gamma < \infty$,
where $1 \leq p \leq \infty$ *and* $2 \leq q < \infty$. *If* $ab > 1$, *then* $f = 0$ *almost everywhere.*

roof. We first prove the case $p = \infty$ and later, use this result for the case $1 \leq p < \infty$.

ase 1. $p = \infty$. In this case we interpret (i) as

$$|f(x, v)| \leq A\mathrm{e}^{-a\pi \|(x,v)\|^2}. \tag{3.2}$$

e define

$$\tilde{f}(x, v) = \overline{f(-x, v)},$$

d

$$h(x) = \int_{\mathbb{R}^{n-m}} (f_v * \tilde{f}_v)(x)\mathrm{d}v, \tag{3.3}$$

here $f_v(x) = f(x, v)$ and $*$ is the convolution on $\mathbb{R}^m$. Since $f \in L^1(G)$, $h \in L^1(\mathbb{R}^m)$ d the Euclidean Fourier transform of h is given by

$$\begin{aligned} \hat{h}(\lambda) &= \int_{\mathbb{R}^m} h(x)\mathrm{e}^{-2\pi i \langle \lambda, x\rangle}\mathrm{d}x \\ &= \int_{\mathbb{R}^{n-m}} |\mathcal{F}_1 f(\lambda, v)|^2 \mathrm{d}v \\ &= |Pf(\lambda)| \int_{\tilde{V}_N^*} \|\hat{f}(\pi_{\lambda,\gamma})\|_{HS}^2 \mathrm{d}\gamma \qquad \text{(by (2.10))}. \end{aligned} \tag{3.4}$$

ow writing $\mathrm{e}^x = \exp x$,

$$\begin{aligned} h(x)| &\leq \int_{\mathbb{R}^{n-m}} |(f_v * \tilde{f}_v)(x)|\mathrm{d}v \\ &\leq A^2 \int_{\mathbb{R}^n} \exp(-a\pi[2\|v\|^2 + \|x - y\|^2 + \|y\|^2])\mathrm{d}y\mathrm{d}v \\ &\leq A^2 \exp\left(-a\pi \frac{\|x\|^2}{2}\right) \int_{\mathbb{R}^n} \exp\left(-a\pi \left[2\|v\|^2 + 2\left(\|y\| - \frac{\|x\|}{2}\right)^2\right]\right) \mathrm{d}y\mathrm{d}v \\ &\leq A_1 \mathrm{e}^{-(a'\pi/2)\|x\|^2}, \end{aligned} \tag{3.5}$$

here $a' < a$ with $a'b > 1$ (the integral in the last line but one being a polynomial in $\|x\|$). oosing $b' < b$ such that $a'b' > 1$ we have, on the other hand,

$$\begin{aligned} &\int_{\mathbb{R}^m} \exp\left(\frac{q}{2}\pi 2b'\|\lambda\|^2\right) |\hat{h}(\lambda)|^{q/2}\mathrm{d}\lambda \\ &\quad = \int_{\mathbb{R}^m} \left(\int_{\tilde{V}_N^*} \exp(2b'\pi\|\gamma\|^2)\|\hat{f}(\pi_{(\lambda,\gamma)})\|_{HS}^2 \exp(-2b'\pi\|\gamma\|^2)\mathrm{d}\gamma\right)^{q/2} \\ &\qquad\qquad \times \exp(q\pi b'\|\lambda\|^2)|Pf(\lambda)|^{q/2}\mathrm{d}\lambda \\ &\quad \leq \int_{\mathbb{R}^m} \exp(qb'\pi\|\lambda\|^2) \left\{\left(\int_{\tilde{V}_N^*} \exp\left(\frac{q}{2}2b'\pi\|\gamma\|^2\right) \|\hat{f}(\pi_{\lambda,\gamma})\|_{HS}^q \mathrm{d}\gamma\right)^{2/q}\right. \\ &\qquad \left. \times \left(\int_{\tilde{V}_N^*} \exp(-2b'\pi\alpha\|\gamma\|^2)\mathrm{d}\gamma\right)^{1/\alpha}\right\}^{q/2} |Pf(\lambda)|^{q/2}\mathrm{d}\lambda \\ &\qquad\qquad \text{(by Hölder's inequality, where } 2/q + 1/\alpha = 1) \end{aligned}$$

$$= B \int_{V_N^*} \exp(qb\pi \|(\lambda, \gamma)\|^2) \|\hat{f}(\pi_{(\lambda,\gamma)})\|_{HS}^q$$
$$\times \{\exp((b' - b)\pi \|(\lambda, \gamma)\|^2) |Pf(\lambda)|^{q/2}\} \mathrm{d}\lambda \mathrm{d}\gamma < \infty \quad \text{(by (ii))}. \tag{3.6}$$

Since $(a'/2)2b' = a'b' > 1$, by 3.5, 3.6 and Theorem 1.2 for the case $p = \infty$ and $q/2$ (which is ≥ 1 as $q \geq 2$) we get that $h = 0$ almost everywhere. Thus $\|\hat{f}(\lambda, \gamma)\|_{HS} = 0$ for almost every (λ, γ) and hence $f = 0$ almost everywhere by the Plancherel theorem.

Case 2. $p < \infty$. Let $e_k(x, v) = \mathrm{e}^{k\|(x,v)\|^2}$ for $k \in \mathbb{R}^+$. Suppose $g \in C_c(G)$ is such that $\operatorname{supp} g \subset \{(x_1, v_1) : \|(x_1, v_1)\| \leq \frac{1}{m}\}$, where $m \in \mathbb{N}$. We choose $(x, v) \in G$ with $\|(x, v)\| > 1$. Thus, if $(x_1, v_1) \in \operatorname{supp} g$ we have $\|(x_1, v_1)\| \leq \|(x, v)\|/m$ and hence by Lemma 3.1,

$$\|(x, v)(x_1, v_1)^{-1}\| \geq \|(x, v)\| \left(1 - \frac{d}{m}\right), \tag{3.7}$$

where $d = 1 + C$. Thus for $(x, v) \in G$ with $\|(x, v)\| > 1$ we have

$$\begin{aligned}
&(e_{a\pi}|f| * |g|)(x, v) \\
&= \int_{\operatorname{supp} g} \mathrm{e}^{a\pi\|(x,v)(x_1,v_1)^{-1}\|} |f((x, v)(x_1, v_1)^{-1})||g(x_1, v_1)| \mathrm{d}x_1 \mathrm{d}v_1 \\
&\geq \mathrm{e}^{a\pi(1-(d/m))^2\|(x,v)\|^2} (|f| * |g|)(x, v) \quad \text{(by (3.7))}.
\end{aligned} \tag{3.8}$$

By (i) we have that $e_{a\pi}|f|$ is a L^p function ($p < \infty$) on G and $g \in C_c(G)$, thus $e_{a\pi}|f| * |g|$ is a continuous function vanishing at infinity. Thus from (3.8) we have that

$$|(f * g)(x, v)| \leq \beta \mathrm{e}^{-a\pi(1-(d/m))^2\|(x,v)\|^2}$$

for all $(x, v) \in G$ with Euclidean norm greater than 1. By continuity of $f * g$ we have

$$|(f * g)(x, v)| \leq \beta \mathrm{e}^{-a\pi(1-(d/m))^2\|(x,v)\|^2}, \tag{3.9}$$

for all $(x, v) \in G$ (possibly with a different constant). Since

$$\begin{aligned}
\|(\widehat{f * g})(\pi_{(\lambda,\gamma)})\|_{HS} &\leq \|\hat{g}(\pi_{(\lambda,\gamma)})\|_{Op} \|\hat{f}(\pi_{(\lambda,\gamma)})\|_{HS} \\
&\leq \|g\|_{L^1(G)} \|\hat{f}(\pi_{(\lambda,\gamma)})\|_{HS},
\end{aligned}$$

from (ii) we get that

$$\int_{V_N^*} \mathrm{e}^{qb\pi\|(\lambda,\gamma)\|^2} \|(\widehat{f * g})(\pi_{(\lambda,\gamma)})\|_{HS}^q |Pf(\lambda)| \mathrm{d}\lambda \mathrm{d}\gamma < \infty. \tag{3.10}$$

We choose m so large that $ab(1 - (d/m))^2 > 1$. Then by (3.9) and (3.10) we are reduced to case 1. Hence $f * g = 0$ almost everywhere. Now by choosing g from an approximate identity we get $f = 0$ almost everywhere. This completes the proof.

Note 3.1. For general two step nilpotent Lie groups we are unable to answer the case $q < 2$. But if G is a MW group then we have a complete answer, as is shown in the following theorem.

Theorem 3.2. *Let G be a connected, simply connected, two step nilpotent Lie group which is MW. Let $f \in L^1(G) \cap L^2(G)$. Suppose that for $a, b > 0$ and* $\min(p, q) < \infty$

(i) $\int_G \mathrm{e}^{pa\pi\|(z,v)\|^2}|f(z,v)|^p \mathrm{d}z\mathrm{d}v < \infty$,
(ii) $\int_{V_N^*} \mathrm{e}^{qb\pi l^2}\|\hat{f}(\pi_l)\|_{HS}^q|Pf(l)|\mathrm{d}l < \infty$.

Then
(a) *If* $q \geq 2$, *then* $f = 0$ *for* $ab > 1$.
(b) *If* $1 \leq q < 2$ *then* $f = 0$ *if* $ab \geq 1$.

Proof. Part (a) is essentially in Theorem 3.1. So we prove (b). In this case, for $f \in L^1(G) \cap L^2(G)$ we have

$$\|\hat{f}(\pi_l)\|_{HS} = |Pf(l)|^{-1}\int_{\mathbb{R}^{2n}} |\mathcal{F}_1 f(l,z)|^2 \mathrm{d}z,$$

(see Note 2.5). Starting from (ii) we have

$$\begin{aligned}
&\int_{V_N^*} \mathrm{e}^{qb\pi\|l\|^2}\|\hat{f}(\pi_l)\|_{HS}^q|Pf(l)|\mathrm{d}l \\
&= \int_{V_N^*} \mathrm{e}^{qb\pi\|l\|^2}\left(|Pf(l)|^{-1}\int_{\mathbb{R}^{2n}} |\mathcal{F}_1 f(l,v)|^2\mathrm{d}v\right)^{\frac{q}{2}} |Pf(l)|\mathrm{d}l \\
&= \int_{V_N^*}\left(\int_{\mathbb{R}^{2n}} g(l,v)^{\frac{2}{q}}\mathrm{d}v\right)^{\frac{q}{2}} \mathrm{d}\mu(l)
\end{aligned}$$

(where $g(l,v) = |\mathcal{F}_1 f(l,v)|^q$ and $\mathrm{d}\mu(l) = \mathrm{e}^{qb\pi\|l\|^2}|Pf(l)|^{(1-\frac{q}{2})}\mathrm{d}l$)

$$\geq \left(\int_{\mathbb{R}^{2n}}\left(\int_{V_N^*} g(l,v)\mathrm{d}\mu(l)\right)^{\frac{2}{q}} \mathrm{d}v\right)^{\frac{q}{2}} \qquad \text{(by Minkowski's inequality).}$$

Thus for almost every v, $\int_{V_N^*} g(l,v)\mathrm{d}\mu(l) < \infty$, that is

$$\int_{V_N^*} \mathrm{e}^{qb\pi\|l\|^2}|\mathcal{F}_1 f(l,v)|^q|Pf(l)|^{(1-\frac{q}{2})}\mathrm{d}l < \infty. \tag{3.11}$$

But from (i) it follows that for almost every v,

$$\int_Z \mathrm{e}^{pa\pi\|z\|^2}|f(z,v)|^p\mathrm{d}z < \infty. \tag{3.12}$$

Thus for almost every v, the function $f(.,v)$ satisfies the condition of Theorem 2.5 and hence for $ab \geq 1$, $f = 0$ after all.

Going back to connected, simply connected, two step nilpotent Lie groups G, we observe that the same technique using the functions $\tilde{f}$ and h, as in Theorem 3.1, yields the following theorem.

Theorem 3.3. *Let* $f : G \to \mathbb{C}$ *be a measurable function. Suppose*
(i) $|f(x,v)| \leq Cg(v)\mathrm{e}^{-a\pi\|x\|^p}$,
(ii) $\|\hat{f}(\pi_{\lambda,\gamma})\|_{HS} \leq Ch(\gamma)\mathrm{e}^{-b\pi\|\lambda\|^q}$,
where $C > 0$, $p \geq 2$, $1/p + 1/q = 1$ *and* g, h *are nonnegative functions with* $g \in L^1(\mathbb{R}^{n-m}) \cap L^2(\mathbb{R}^{n-m})$ *and* $h \in L^1(\mathbb{R}^r) \cap L^2(\mathbb{R}^r)$. *If* $(ap)^{1/p}(bq)^{1/q} > 2$, *then* $f = 0$ *almost everywhere.*

4. Heisenberg's inequality

The classical inequality of Heisenberg for L^2 functions on $\mathbb{R}$ says that

$$\left(\int_{\mathbb{R}} |x|^2|f(x)|^2 \mathrm{d}x\right)^{1/2} \left(\int_{\mathbb{R}} |y|^2|\hat{f}(y)|^2 \mathrm{d}y\right)^{1/2} \geq C\|f\|_2^2, \tag{4.1}$$

where $\hat{f}$ is defined by

$$\hat{f}(y) = \int_{\mathbb{R}} f(x)\mathrm{e}^{-2\pi i x}\mathrm{d}x$$

and C is a constant independent of f.

In this section our aim is to extend the version of Heisenberg's inequality proved in [29] for the Heisenberg groups to all connected, simply connected, step two nilpotent Lie groups. Two other variants of Heisenberg's inequality on Heisenberg groups are available in [13] and [28], but since these results use the existence of rotations on Heisenberg groups, it is not clear how, without the notion of rotation, one should proceed to extend them to a general two step nilpotent Lie group (see [2]).

We state (4.1) in a slightly different way. Let $\Delta = -\sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ be the Laplacian on $\mathbb{R}^n$. Then $(\widehat{\Delta f})(y) = 4\pi^2\|y\|^2\hat{f}(y)$ for any Schwartz class function on $\mathbb{R}^n$. We may relate Δ to the character $\gamma_y(x) = \mathrm{e}^{2\pi i y.x}$ of $\mathbb{R}^n$ by $\mathrm{d}\gamma_y\left(\frac{\partial}{\partial x_j}\right) = 2\pi i y_j$, and hence $\mathrm{d}\gamma_y(\Delta) = 4\pi^2\|y\|^2$. Thus we have

$$(\widehat{\Delta f})(y) = \mathrm{d}\gamma_y(\Delta)\hat{f}(y).$$

Since $\mathrm{d}\gamma_y(\Delta)$ is a positive, self adjoint operator, it has a (visible) square root, which is *multiplication by* $2\pi\|y\|$. Thus we define

$$(\widehat{\Delta^{\frac{1}{2}} f})(y) = 2\pi\|y\|\hat{f}(y) = (\mathrm{d}\gamma_y(\Delta))^{\frac{1}{2}}\hat{f}(y),$$

for all Schwartz class functions on $\mathbb{R}^n$. Since the Fourier transform is an isomorphism on Schwartz class functions, the operator $(\Delta)^{\frac{1}{2}}$ is defined completely. Then we can restate (4.1) as

$$\left(\int_{\mathbb{R}^n} \|x\|^2|f(x)|^2 \mathrm{d}x\right)^{1/2} \left(|(\widehat{\Delta^{\frac{1}{2}}(f)})(y)|^2 \mathrm{d}y\right)^{1/2} \geq C\|f\|_2^2, \tag{4.2}$$

for all f of Schwartz class on $\mathbb{R}^n$, where C is a constant independent of f. It is (4.2), whose analogue on connected, simply connected, two step nilpotent Lie groups we are looking for. As in the case of Heisenberg groups, here also the proof, in principle, is close to the proof on $\mathbb{R}^n$ (see [11]) having the same basic ingredients, namely, integration by parts, Cauchy–Schwartz inequality and the Plancherel theorem.

We call *a function f on G a Schwartz class function if $f \circ \exp$ is a Schwartz class function on* $\mathfrak{g}$. We denote the Schwartz class functions by $\mathcal{S}(G)$.

Replacing $\Delta^{\frac{1}{2}}$ by $\mathcal{L}^{\frac{1}{2}}$, the main result of this section is as follows.

Theorem 4.1. *Let G be a connected, simply connected, step two nilpotent Lie group and $f \in \mathcal{S}(G)$. Then*

$$\left(\int_G \|v\|^2|f(x,v)|^2\mathrm{d}x\mathrm{d}v\right)^{1/2}\left(\int_{V_N^*\cap\mathcal{U}} \|\widehat{(\mathcal{L}^{\frac{1}{2}}f)}(\pi_l)\|_{HS}^2|Pf(l)|\mathrm{d}l\right)^{1/2}$$
$$\geq C\|f\|^2_{L^2(G)}, \tag{4.3}$$

where C is a constant independent of f and $\mathcal{L} = -\Sigma_{i=1}^{n-m}X_{m+i}^2$ is the sub-Laplacian.

Let us explain the meaning of $\widehat{(\mathcal{L}^{\frac{1}{2}}f)}(\pi_l)$. We view $X \in \mathfrak{g}$ as a left invariant differential operator on $C^\infty(G)$. Then in view of our definition of the group Fourier transform, we have for $f \in \mathcal{S}(G)$

$$\widehat{(Xf)}(\pi_l) = \mathrm{d}\pi_l(X) \circ \hat{f}(\pi_l), \tag{4.4}$$

where $\mathrm{d}\pi_l(X)$ is given by (2.13). We view the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ as the algebra of all left invariant differential operators on $C^\infty(G)$. Since $\mathrm{d}\pi_l$ is a representation of $\mathfrak{g}$, it extends to a representation of $\mathcal{U}(\mathfrak{g})$ realized on $C^\infty(\pi_l)$. By (4.4) we have

$$\widehat{(\mathcal{L}f)}(\pi_l) = \mathrm{d}\pi_l(\mathcal{L}) \circ \hat{f}(\pi_l),$$

as $\mathcal{L} \in \mathcal{U}(\mathfrak{g})$. In §2 we have seen that the eigenfunctions of $\mathrm{d}\pi_l(\mathcal{L})$ are parametrized by $\mathbb{N}^k$ and are given by (2.16). Let $\{t_i(l) > 0 : i = 0, \ldots\}$ be an enumeration of those real numbers such that there exist $\alpha \in \mathbb{N}^k$ with

$$t_i(l) = \mu(l) + \sum_{j=1}^{k} 2\pi\lambda_j(l)(2\alpha_j + 1), \tag{4.5}$$

as α varies over $\mathbb{N}^k$. Let $E_i(l) = \mathrm{span}_{\mathbb{C}}\{h_\alpha^l : \mathrm{d}\pi_l(\mathcal{L})(h_\alpha^l) = t_i(l)h_\alpha^l\}$, that is, $E_i(l)$ is the eigenspace corresponding to the eigenvalue $t_i(l)$, which is clearly finite dimensional. If $P_i(l) : L^2(\mathbb{R}^k) \to E_i(l)$ is the projection, we have

$$\mathrm{d}\pi_l(\mathcal{L}) = \sum_{j=0}^{\infty} t_j(l)P_j(l). \tag{4.6}$$

Thus we define

$$\mathrm{d}\pi_l(\mathcal{L})^{\frac{1}{2}} = \sum_{j=0}^{\infty} t_j(l)^{\frac{1}{2}}P_j(l) \tag{4.7}$$

and

$$\mathrm{d}\pi_l(\mathcal{L})^{-\frac{1}{2}} = \sum_{j=0}^{\infty} t_j(l)^{-\frac{1}{2}}P_j(l). \tag{4.8}$$

Analogous to the Euclidean spaces, we define

$$\widehat{(\mathcal{L}^{\frac{1}{2}}f)}(\pi_l) = \mathrm{d}\pi_l(\mathcal{L})^{\frac{1}{2}} \circ \hat{f}(\pi_l), \tag{4.9}$$

for all $f \in \mathcal{S}(G)$ and $l \in V_N^* \cap \mathcal{U}$. Thus the statement in theorem 4.1 makes sense.

It follows from (4.5) that the eigenvalues of $\mathrm{d}\pi_l(\mathcal{L})^{-\frac{1}{2}}$ are bounded by $\lambda_0(l)^{-\frac{1}{2}}$ where $\lambda_0(l) = \min\{\lambda_j(l) : 1 \leq j \leq k\}$. As a consequence we get the following Lemma.

Lemma 4.1. *The operator* $\mathrm{d}\pi_l(\mathcal{L})^{-\frac{1}{2}}$ *is bounded on* $L^2(\mathbb{R}^k)$.

Let us consider the following elements of $\mathfrak{g}_{\mathbb{C}}$, the complexification of $\mathfrak{g}$,

$$D_j(l) = Y_j(l) - iW_j(l), \qquad 1 \leq j \leq k, \tag{4.10}$$

$$\bar{D}_j(l) = Y_j(l) + iW_j(l), \qquad 1 \leq j \leq k. \tag{4.11}$$

Because of Lemma 2.4 we have

$$\mathrm{d}\pi_l(D_j(l))\phi(\xi) = \left(-\frac{\partial}{\partial \xi_j} + 2\pi\lambda_j(l)\xi_j\right)\phi(\xi), \tag{4.12}$$

$$\mathrm{d}\pi_l(\bar{D}_j(l))\phi(\xi) = \left(-\frac{\partial}{\partial \xi_j} - 2\pi\lambda_j(l)\xi_j\right)\phi(\xi). \tag{4.13}$$

For h_s the sth normalized hermite function on $\mathbb{R}$, we define $h_s^c(x) = c^{1/4}h_s(c^{1/2}x)$, then

$$\left(-\frac{\mathrm{d}}{\mathrm{d}x} + cx\right)h_s^c = c^{1/2}(2s+2)^{1/2}h_{s+1}^c,$$

$$\left(\frac{\mathrm{d}}{\mathrm{d}x} + cx\right)h_s^c = c^{1/2}(2s)^{1/2}h_{s-1}^c.$$

Using this with (4.12) and (4.13) we get for $\alpha \in \mathbb{N}^k$,

$$\mathrm{d}\pi_l(D_j(l))(h_\alpha^l) = (2\pi\lambda_j(l))^{1/2}(2\alpha_j+2)^{1/2}h_{\alpha+e_j}, \tag{4.14}$$

$$\mathrm{d}\pi_l(\bar{D}_j(l))(h_\alpha^l) = -(2\pi\lambda_j(l))^{1/2}(2\alpha_j)^{1/2}h_{\alpha-e_j}^l, \tag{4.15}$$

where

$$\alpha + e_j = (\alpha_1, \ldots, \alpha_{j-1}, \alpha_j + 1, \alpha_{j+1}, \ldots, \alpha_k) \in \mathbb{N}^k,$$

$$\alpha - e_j = (\alpha_1, \ldots, \alpha_{j-1}, \alpha_j - 1, \alpha_{j+1}, \ldots, \alpha_k) \in \mathbb{N}^k.$$

Lemma 4.2. *The operators* $\mathrm{d}\pi_l(D_j(l)) \circ \mathrm{d}\pi_l(\mathcal{L})^{-\frac{1}{2}}$ *and* $\mathrm{d}\pi_l(\bar{D}_j(l)) \circ \mathrm{d}\pi_l(\mathcal{L})^{-\frac{1}{2}}$ *are bounded operators on* $L^2(\mathbb{R}^k)$, $1 \leq j \leq k$.

Proof. We consider the orthonormal basis $\{h_\alpha^l : \alpha \in \mathbb{N}^k\}$ of $L^2(\mathbb{R}^k)$. By (4.8), (4.14) and (4.15) we have

$$\mathrm{d}\pi_l(D_j(l)) \circ \mathrm{d}\pi_l(\mathcal{L})^{-\frac{1}{2}}(h_\alpha^l) = \left(\frac{2\pi\lambda_j(l)(2\alpha_j+2)}{\mu(l) + \sum_{p=1}^k 2\pi\lambda_p(l)(2\alpha_p+1)}\right)^{\frac{1}{2}} h_{\alpha+e_j}^l,$$

and

$$\mathrm{d}\pi_l(\bar{D}_j(l)) \circ \mathrm{d}\pi_l(\mathcal{L})^{-\frac{1}{2}}(h_\alpha^l) = -\left(\frac{2\pi\lambda_j(l)2\alpha_j}{\mu(l) + \sum_{p=1}^k 2\pi\lambda_p(l)(2\alpha_p+1)}\right)^{\frac{1}{2}} h_{\alpha-e_j}^l.$$

Since

$$\left(\frac{2\pi\lambda_j(l)(2\alpha_j+2)}{\mu(l) + \sum_{p=1}^k 2\pi\lambda_p(l)(2\alpha_p+1)}\right)^{\frac{1}{2}} \leq \sqrt{2}$$

and

$$\left(\frac{2\pi\lambda_j(l)2\alpha_j}{\mu(l)+\sum_{p=1}^{k}2\pi\lambda_p(l)(2\alpha_p+1)}\right)^{\frac{1}{2}}\leq 1,$$

the operators $\mathrm{d}\pi_l(D_j(l))\circ\mathrm{d}\pi_l(\mathcal{L})^{-\frac{1}{2}}$ and $\mathrm{d}\pi_l(\bar{D}_j(l))\circ\mathrm{d}\pi_l(\mathcal{L})^{-\frac{1}{2}}$ are bounded operators on $L^2(\mathbb{R}^k)$. This completes the proof.

Suppose $f\in\mathcal{S}(G)$ and let $l\in V_N^*\cap\mathcal{U}$ be arbitrary but fixed. So we have an almost symplectic basis (2.4) of $\mathfrak{g}$. Let $l\mid\mathfrak{z}=\lambda$. We define

$$\mathcal{F}_c f(\lambda,v)=\int_{\mathrm{R}^m}f(x,v)\mathrm{e}^{-2\pi i\lambda(x)}\mathrm{d}x, \tag{4.16}$$

that is, the partial Fourier transform in the central component. So $v\to\mathcal{F}_c f(\lambda,v)$ is a Schwartz class function on $\mathbb{R}^{n-m}$. On Euclidean spaces, differentiation and multiplication are intertwined by the Fourier transform. On two step groups, as analogues of differentiation we consider the operators $D_l(l)$ and $\bar{D}_j(l)$ and as analogue of Fourier transform we consider the partial Fourier transform defined in (4.16). We want to find what plays the role of multiplication?

Let $f\in\mathcal{S}(G)$ and $X_j\in\mathcal{B}\subset\mathfrak{g}, m+1\leq j\leq n$. By (2.15) it is clear that $X_jf\in\mathcal{S}(G)$, and an easy calculation shows that

$$\mathcal{F}_c(X_jf)(\lambda,v)=\left(\frac{\partial}{\partial x_j}+\pi iB_\lambda(v,X_j)\right)(\mathcal{F}_cf)(\lambda,v).$$

Thus using the basis in (2.4) we have

$$\begin{aligned}&\mathcal{F}_c(W_j(l)f)(\lambda,z,w,y)\\ &=\left(\frac{\partial}{\partial w_j}-\pi i\lambda_j(l)y_j\right)(\mathcal{F}_cf)(\lambda,z,w,y), &(4.17)\\ &\mathcal{F}_c(Y_j(l)f)(\lambda,z,w,y)\\ &=\left(\frac{\partial}{\partial y_j}+\pi i\lambda_j(l)w_j\right)(\mathcal{F}_cf)(\lambda,z,w,y) &(4.18)\end{aligned}$$

for $1\leq j\leq k$. Thus writing

$$V_j(l)=\left(\frac{\partial}{\partial y_j}-i\frac{\partial}{\partial w_j}\right)-\pi\lambda_j(l)(y_j-iw_j), \tag{4.19}$$

$$\bar{V}_j(l)=\left(\frac{\partial}{\partial y_j}+i\frac{\partial}{\partial w_j}\right)+\pi\lambda_j(l)(y_j+iw_j), \tag{4.20}$$

we have from (4.17) and (4.18)

$$\mathcal{F}_c(D_j(l)f)(\lambda,z,w,y)=V_j(l)(\mathcal{F}_cf)(\lambda,z,w,y), \tag{4.21}$$

$$\mathcal{F}_c(\bar{D}_j(l)f)(\lambda,z,w,y)=\bar{V}_j(l)(\mathcal{F}_cf)(\lambda,z,w,y). \tag{4.22}$$

Thus $V_j(l)$ and $\bar{V}_j(l)$ play the role of multiplication.

Now we come to the proof of Theorem 4.1.

Proof of theorem 4.1. Let $f \in \mathcal{S}(G)$ and $l \mid \mathfrak{z} = \lambda$. Now

$$\begin{aligned}
&\int_{\mathbb{R}^{n-m}} |\mathcal{F}_c f(\lambda, z, w, y)|^2 \mathrm{d}z\mathrm{d}w\mathrm{d}y \\
&= \int_{\mathbb{R}^{n-m}} \mathcal{F}_c f(\lambda, z, w, y)\overline{\mathcal{F}_c f(\lambda, z, w, y)}\mathrm{d}z\mathrm{d}w\mathrm{d}y.
\end{aligned}$$

Since

$$\begin{aligned}
&\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right)(x_j + iy_j)g(x, y) \\
&= 2g(x, y) + (x_j + iy_j)\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right)g(x, y),
\end{aligned}$$

we have from the above equality

$$\begin{aligned}
&\int_{\mathbb{R}^{n-m}} |\mathcal{F}_c f(\lambda, z, w, y)|^2 \mathrm{d}z\mathrm{d}w\mathrm{d}y \\
&= \int_{\mathbb{R}^{n-m}} \frac{1}{2}\left\{\left(\frac{\partial}{\partial y_j} - i\frac{\partial}{\partial w_j}\right)(y_j + iw_j)\mathcal{F}_c f(\lambda, z, w, y)\right. \\
&\qquad \left. -(y_j + iw_j)\left(\frac{\partial}{\partial y_j} - i\frac{\partial}{\partial w_j}\right)\mathcal{F}_c f(\lambda, z, w, y)\right\} \\
&\qquad \times\overline{\mathcal{F}_c f(\lambda, z, w, y)}\mathrm{d}z\mathrm{d}w\mathrm{d}y \\
&= -\frac{1}{2}\int_{\mathbb{R}^{n-m}} (y_j + iw_j)\mathcal{F}_c f(\lambda, z, w, y) \\
&\qquad \times\overline{\left(\frac{\partial}{\partial y_j} + i\frac{\partial}{\partial w_j}\right)\mathcal{F}_c f(\lambda, z, w, y)}\mathrm{d}z\mathrm{d}w\mathrm{d}y \\
&\qquad -\frac{1}{2}\int_{\mathbb{R}^{n-m}} (y_j + iw_j)\left(\frac{\partial}{\partial y_j} - i\frac{\partial}{\partial w_j}\right)\mathcal{F}_c f(\lambda, z, w, y) \\
&\qquad \times\overline{\mathcal{F}_c f(\lambda, z, w, y)}\mathrm{d}z\mathrm{d}w\mathrm{d}y \qquad \text{(by integration by parts)} \\
&= -\frac{1}{2}\int_{\mathbb{R}^{n-m}} (y_j + iw_j)\mathcal{F}_c f(\lambda, z, w, y) \\
&\qquad \times\overline{(\bar{V}_j(l) - \pi\lambda_j(l)(y_j + iw_j))\mathcal{F}_c f(\lambda, z, w, y)}\mathrm{d}z\mathrm{d}w\mathrm{d}y \\
&\qquad -\frac{1}{2}\int_{\mathbb{R}^{n-m}} (y_j + iw_j)(V_j(l) + \pi\lambda_j(l)(y_j - iw_j))\mathcal{F}_c f(\lambda, z, w, y) \\
&\qquad \times\overline{\mathcal{F}_c f(\lambda, z, w, y)}\mathrm{d}z\mathrm{d}w\mathrm{d}y \qquad \text{(by (4.19) and (4.20))} \\
&= -\frac{1}{2}\int_{\mathbb{R}^{n-m}} (y_j + iw_j)\mathcal{F}_c f(\lambda, z, w, y)\overline{\mathcal{F}_c(\bar{D}_j(l)f)(\lambda, z, w, y)}\mathrm{d}z\mathrm{d}w\mathrm{d}y \\
&\qquad -\frac{1}{2}\int_{\mathbb{R}^{n-m}} (y_j + iw_j)\mathcal{F}_c(D_j(l)f)(\lambda, z, w, y)\overline{\mathcal{F}_c f(\lambda, z, w, y)}\mathrm{d}z\mathrm{d}w\mathrm{d}y \\
&\qquad \text{(by (4.21) and (4.22)).}
\end{aligned} \tag{4.23}$$

Let us recall, if l varies over $V_N^* \cap \mathcal{U}$ then $l \mid \mathfrak{z} = \lambda$ varies over the Zariski open subset $\mathcal{U}'$ of $\mathfrak{z}^*$ (see Note 2.3). Hence

$$\begin{aligned}
\int_{\mathsf{z}}\int_{\mathrm{R}^{n-m}} |f(x,v)|^2 \mathrm{d}x\mathrm{d}v &= \int_{\mathcal{U}'}\int_{\mathrm{R}^{n-m}} |\mathcal{F}_c f(\lambda,v)|^2 \mathrm{d}\lambda\mathrm{d}v \\
&= \int_{\mathcal{U}'}\left(\int_{\mathrm{R}^{n-m}} |\mathcal{F}_c f(\lambda,z,w,y)|^2 \mathrm{d}z\mathrm{d}w\mathrm{d}y\right)\mathrm{d}\lambda
\end{aligned}$$

(by Fubini's theorem and the orthogonal basis change on $\mathbb{R}^{n-m}$ by $T_l : \mathrm{span}_{\mathrm{R}}\{X_{m+1}, \ldots, X_n\} \to \mathrm{span}_{\mathrm{R}}\{Z_1(l), \ldots, Y_k(l)\}$)

$$\begin{aligned}
&= \int_{\mathcal{U}'}\Bigg(-\frac{1}{2}\int_{\mathrm{R}^{n-m}} (y_j + iw_j)\mathcal{F}_c f(\lambda,z,w,y)\overline{\mathcal{F}_c(\bar{D}_j(l)f)(\lambda,z,w,y)}\mathrm{d}z\mathrm{d}w\mathrm{d}y \\
&\quad -\frac{1}{2}\int_{\mathrm{R}^{n-m}} (y_j + iw_j)\mathcal{F}_c(D_j(l)f)(\lambda,z,w,y)\overline{\mathcal{F}_c f(\lambda,z,w,y)}\mathrm{d}z\mathrm{d}w\mathrm{d}y\Bigg)\, d\lambda \\
&\qquad \Big(\text{by } (4.23)\Big) \\
&= \int_{\mathcal{U}'}\Bigg(-\frac{1}{2}\int_{\mathrm{R}^{n-m}} T_l^{-1}(y_j + iw_j)\mathcal{F}_c f(\lambda,v)\overline{\mathcal{F}_c(\bar{D}_j(l)f)(\lambda,v)}\mathrm{d}v \\
&\quad -\frac{1}{2}\int_{\mathrm{R}^{n-m}} T_l^{-1}(y_j + iw_j)\mathcal{F}_c(D_j(l)f)(\lambda,v)\overline{\mathcal{F}_c f(\lambda,v)}\mathrm{d}v\Bigg)\,\mathrm{d}\lambda \\
&\qquad \Big(\text{by change of variables}\Big) \\
&\le \frac{1}{2}\left(\int_{\mathcal{U}'}\int_{\mathrm{R}^{n-m}} |T_l^{-1}(y_j + iw_j)|^2|\mathcal{F}_c f(\lambda,v)|^2 \mathrm{d}v\mathrm{d}\lambda\right)^{\frac{1}{2}} \\
&\quad \times\left\{\left(\int_{\mathcal{U}'}\int_{\mathrm{R}^{n-m}} |\mathcal{F}_c(\bar{D}_j(l)f)(\lambda,v)|^2 \mathrm{d}v\mathrm{d}\lambda\right)^{\frac{1}{2}}\right. \\
&\quad \left.+\left(\int_{\mathcal{U}'}\int_{\mathrm{R}^{n-m}} |\mathcal{F}_c(D_j(l)f)(\lambda,v)|^2 \mathrm{d}v\mathrm{d}\lambda\right)^{\frac{1}{2}}\right\} \\
&\qquad \Big(\text{by Cauchy–Schwartz inequality and nonnegativity} \\
&\qquad \text{of the integral}\Big) \\
&\le \frac{1}{2}\left(\int_{\mathcal{U}'}\int_{\mathrm{R}^{n-m}} \|v\|^2|\mathcal{F}_c f(\lambda,v)|^2 \mathrm{d}v\mathrm{d}\lambda\right)^{\frac{1}{2}} \\
&\quad \times\left\{\left(\int_{\mathcal{U}'}\int_{\mathrm{R}^{n-m}} |\mathcal{F}_c(\bar{D}_j(l)f)(\lambda,v)|^2 \mathrm{d}v\mathrm{d}\lambda\right)^{\frac{1}{2}}\right. \\
&\quad \left.+\left(\int_{\mathcal{U}'}\int_{\mathrm{R}^{n-m}} |\mathcal{F}_c(D_j(l)f)(\lambda,v)|^2 \mathrm{d}v\mathrm{d}\lambda\right)^{\frac{1}{2}}\right\} \\
&= \frac{1}{2}\left(\int_{\mathsf{z}}\int_{\mathrm{R}^{n-m}} \|v\|^2|f(x,v)|^2 \mathrm{d}x\mathrm{d}v\right)^{\frac{1}{2}} \\
&\quad \times\left\{\left(\int_{\mathcal{U}'}\int_{\mathrm{R}^{n-m}} |\mathcal{F}_c(\bar{D}_j(l)f)(\lambda,v)|^2 \mathrm{d}v\mathrm{d}\lambda\right)^{\frac{1}{2}}\right. \\
&\quad \left.+\left(\int_{\mathcal{U}'}\int_{\mathrm{R}^{n-m}} |\mathcal{F}_c(D_j(l)f)(\lambda,v)|^2 \mathrm{d}v\mathrm{d}\lambda\right)^{\frac{1}{2}}\right\}, \qquad (4.24)
\end{aligned}$$

by the Euclidean Plancherel theorem on $\mathfrak{z}$.

By Theorem (2.3) we have

$$\int_{\tilde{V}_N^*} \|\hat{f}(\pi_{(\lambda,\gamma)}\|_{HS}^2 \mathrm{d}l_{n_1}\ldots\mathrm{d}l_{n_r} = |Pf(l)|^{-1}\int_{\mathsf{R}^{n-m}} |\mathcal{F}_c f(\lambda, v)|^2 \mathrm{d}v,$$

where $l \mid \mathfrak{z} = \lambda$ and $l \mid \tilde{V}_N = \gamma = (l_{n_1}, \ldots, l_{n_r})$. Thus

$$\begin{aligned}
&\left(\int_{\mathcal{U}'}\int_{\mathsf{R}^{n-m}} |\mathcal{F}_c(\bar{D}(l)f)(\lambda, v)|^2 \mathrm{d}v\mathrm{d}\lambda\right)^{\frac{1}{2}}\\
&= \left(\int_{\mathcal{U}'}\int_{\tilde{V}_N^*} \|(\widehat{\bar{D}_j(l)}f)(\pi_l)\|_{HS}^2 |Pf(l)| \mathrm{d}l_{n_1}\ldots\mathrm{d}l_{n_r}\mathrm{d}\lambda\right)^{\frac{1}{2}}\\
&= \left(\int_{\mathcal{U}'}\int_{\tilde{V}_N^*} \|\mathrm{d}\pi_l(\bar{D}_j(l)) \circ \hat{f}(\pi_l)\|_{HS}^2 |Pf(l)| \mathrm{d}l_{n_1}\ldots\mathrm{d}l_{n_r}\mathrm{d}\lambda\right)^{\frac{1}{2}}.\\
&= \left(\int_{\mathcal{U}'}\int_{\tilde{V}_N^*} \|\mathrm{d}\pi_l(\bar{D}_j(l)) \circ \mathrm{d}\pi_l(\mathcal{L})^{-\frac{1}{2}} \circ \mathrm{d}\pi_l(\mathcal{L})^{\frac{1}{2}} \circ \hat{f}(\pi_l)\|_{HS}^2\right.\\
&\qquad \left.\times |Pf(l)| \mathrm{d}l_{n_1}\ldots\mathrm{d}l_{n_r}\mathrm{d}\lambda\right)^{\frac{1}{2}}\\
&\le \left(\int_{\mathcal{U}'}\int_{\tilde{V}_N^*} \|\mathrm{d}\pi_l(\bar{D}_j(l) \circ \mathrm{d}\pi_l(\mathcal{L})^{-\frac{1}{2}}\|_{Op}^2 \|(\widehat{\mathcal{L}^{\frac{1}{2}}f})(\pi_l)\|_{HS}^2\right.\\
&\qquad \left.\times |Pf(l)| \mathrm{d}l_{n_1}\ldots\mathrm{d}l_{n_r}\mathrm{d}\lambda\right)^{\frac{1}{2}}\\
&\le 2\left(\int_{\mathcal{U}'}\int_{\tilde{V}_N^*} \|(\widehat{\mathcal{L}^{\frac{1}{2}}f})(\pi_l)\|_{HS}^2 |Pf(l)| \mathrm{d}l_{n_1}\ldots\mathrm{d}l_{n_r}\mathrm{d}l_1\ldots\mathrm{d}l_m\right)^{\frac{1}{2}}\\
&\qquad\qquad \big(\text{by Lemma 4.2}\big).
\end{aligned}$$

Similarly as above we can show that

$$\begin{aligned}
&\left(\int_{\mathcal{U}'}\int_{\mathsf{R}^{n-m}} |\mathcal{F}_c(D_j(l)f)(\lambda, v)|^2 \mathrm{d}v\mathrm{d}\lambda\right)^{\frac{1}{2}}\\
&\le \left(\int_{\mathcal{U}'}\int_{\tilde{V}_N^*} \|(\widehat{\mathcal{L}^{\frac{1}{2}}f})(\pi_l)\|_{HS}^2 |Pf(l)| \mathrm{d}l_{n_1}\ldots\mathrm{d}l_{n_r}\mathrm{d}l_1\ldots\mathrm{d}l_m\right)^{\frac{1}{2}}\\
&= \left(\int_{V_N^*\cap\mathcal{U}} \|(\widehat{\mathcal{L}^{\frac{1}{2}}f})(\pi_l)\|_{HS}^2 |Pf(l)| \mathrm{d}l\right)^{\frac{1}{2}}.
\end{aligned}$$

Thus from (4.24) we have

$$\begin{aligned}
&\int_G |f(x, v)|^2 \mathrm{d}x\mathrm{d}v\\
&\le C\left(\int_G \|v\|^2 |f(x, v)|^2 \mathrm{d}x\mathrm{d}v\right)^{\frac{1}{2}} \left(\int_{V_N^*\cap\mathcal{U}} \|(\widehat{\mathcal{L}^{\frac{1}{2}}f})(\pi_l)\|_{HS}^2 |Pf(l)| \mathrm{d}l\right)^{\frac{1}{2}},
\end{aligned}$$

where C is a constant independent of f. This completes the proof.

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On property (β) in Banach lattices, Calderón–Lozanowskiĭ and Orlicz–Lorentz spaces

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Abstract. The geometry of Calderón–Lozanowskiĭ spaces, which are strongly connected with the interpolation theory, was essentially developing during the last few years (see [4, 9, 10, 12, 13, 17]). On the other hand many authors investigated property (β) in Banach spaces (see [7, 19, 20, 21, 25, 26]). The first aim of this paper is to study property (β) in Banach function lattices. Namely a criterion for property (β) in Banach function lattice is presented. In particular we get that in Banach function lattice property (β) implies uniform monotonicity. Moreover, property (β) in generalized Calderón–Lozanowskiĭ function spaces is studied. Finally, it is shown that in Orlicz–Lorentz function spaces property (β) and uniform convexity coincide.

Keywords. Banach lattice; Calderón–Lozanowskiĭ space; Orlicz–Lorentz space; property (β).

1. Introduction

Let $(X, \|\cdot\|)$ be a real Banach space, and let $B(X)$, $S(X)$ be the closed unit ball, unit sphere of X, respectively. For any subset A of X, we denote by $\mathrm{conv}(A)$ the convex hull of A.

We denote by $\varepsilon_0(X)$ the *characteristic of convexity* and by $\delta_X(\varepsilon)$ the *modulus of convexity* of the space X, i.e.

$$\begin{aligned}\delta_X(\varepsilon) &= \inf\left\{1 - \tfrac{1}{2}\|x+y\|_X : x, y \in S(X),\ \|x-y\|_X \geq \varepsilon\right\},\\ \varepsilon_0(X) &= \inf\{\varepsilon > 0 : \delta_X(\varepsilon) > 0\}.\end{aligned}$$

We say that X is *uniformly convex* ($X \in$ (UC) for short) if $\varepsilon_0(X) = 0$ (see [22]).

Define for any $x \notin B(X)$ the *drop* $D(x, B(X))$ determined by x by $D(x, B(X)) = \mathrm{conv}(\{x\} \cup B(X))$.

Recall that for any subset C of X, the *Kuratowski measure of non-compactness* of C is the infimum $\alpha(C)$ of those $\varepsilon > 0$ for which there is a covering of C by a finite number of sets of diameter less than ε.

Rolewicz in [25] has proved that $X \in$ (UC) iff for any $\varepsilon > 0$ there exists $\delta > 0$ such that $1 < \|x\| < 1 + \delta$ implies $\mathrm{diam}(D(x, B(X)) \setminus B(X)) < \varepsilon$. In connection with this he has introduced in [26] the following property.

A Banach space X has the *property* (β) ($X \in (\beta)$ for short) if for any $\varepsilon > 0$ there exists $\delta > 0$ such that $\alpha(D(x, B(X)) \setminus B(X)) < \varepsilon$ whenever $1 < \|x\| < 1 + \delta$.

We say that a sequence $\{x_n\} \subset X$ is *ε-separated for some $\varepsilon > 0$* if $\mathrm{sep}\{x_n\} = \inf\{\|x_n - x_m\| : n \neq m\} > \varepsilon$.

The following characterization of the property (β) is very useful (see [20]).

A Banach space X has property (β) if and only if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for each element $x \in B(X)$ and each sequence (x_n) in $B(X)$ with $\operatorname{sep}\{x_n\} \geq \varepsilon$ there is an index k for which

$$\left\| \frac{x + x_k}{2} \right\| \leq 1 - \delta.$$

A Banach space is *nearly uniformly convex* ($X \in$ (NUC)) if for every $\varepsilon > 0$ there exists $\delta \in (0, 1)$ such that for every sequence $\{x_n\} \subseteq B(X)$ with $\operatorname{sep}\{x_n\} > \varepsilon$, we have $\operatorname{conv}(\{x_n\}) \cap (1 - \delta)B(X) \neq \phi$. Rolewicz proved the following implications (UC) $\Rightarrow$ (β) $\Rightarrow$ (NUC) (see [26]). Moreover, the class of Banach spaces with an equivalent norm with property (β) coincides neither with that of super-reflexive spaces ([21]) nor with the class of nearly uniformly convexifiable spaces ([19]).

A Banach space X is said to have the *Kadec–Klee property* ($X \in$ (H) for short) if every weakly convergent sequence on the unit sphere is convergent in norm. The Banach space X is called to have *uniformly Kadec–Klee property* ($X \in$ (UKK) for short) if for every $\varepsilon > 0$ there exists $\delta \in (0, 1)$ such that $\|x\|_X < 1 - \delta$ whenever $(x_n) \subset B(X)$, $x_n \overset{w}{\to} x$ and $\operatorname{sep}\{x_n\}_X \geq \varepsilon$. For any Banach space we have (NUC) $\Rightarrow$(UKK) $\Rightarrow$(KK). Moreover $X \in$ (NUC) iff $X \in$ (UKK) and X is reflexive ([15]).

Denote by $\mathcal{N}$, $\mathcal{R}$ and $\mathcal{R}_+$ the sets of natural, real and non-negative real numbers, respectively. Let (T, Σ, μ) be a measure space with a σ-finite, complete, non-atomic measure μ. By $L^0 = L^0(T)$ we denote the set of all μ-equivalence classes of real valued measurable functions defined on T.

Let $E = (E, \leq, \|\cdot\|_E)$ be a function Banach lattice over the measure space (T, Σ, μ), where $\leq$ is the usual semi-order relation in the space L^0 and $(E, \|\cdot\|_E)$ is a Banach function space (i.e. E is a linear subspace of L^0 and norm $\|\cdot\|_E$ is complete in E). Let E satisfy two conditions:

(i) if $x \in E$, $y \in L^0$, $|y| \leq |x|$ μ-a.e., then $y \in E$ and $\|y\|_E \leq \|x\|_E$,

(ii) there exists function x in E that is positive on whole T (see [18] and [22]).

Denote by E_+, L^0_+ the positive cone of E, L^0 respectively, i.e. $L^0_+ = \{x \in L^0 : x \geq 0\}$.

Recall that E satisfies the *Fatou property* ($E \in$ (FP)) if $x \in L^0$ and $(x_m) \in E$ are such that $0 \leq x_m \nearrow x$ and $\sup_m \|x_m\|_E < \infty$, then $x \in E$ and $\|x\|_E = \lim_{m\to\infty} \|x_m\|_E$ (see [18] and [22]).

We say that Banach lattice E is *uniformly monotone* ($E \in$ (UM)) if for every $q \in (0, 1)$ there exists $p \in (0, 1)$ such that for all $0 \leq y \leq x$ satisfying $\|x\|_E \leq 1$ and $\|y\|_E \geq q$ we have $\|x - y\|_E \leq 1 - p$. Then the modulus $p(\cdot)$ of the uniform monotonicity of E is defined as follows:

$$p(q) = \inf\left\{1 - \|x - y\|_E : \|x\|_E \leq 1, \|y\|_E \geq q, 0 \leq y \leq x\right\}.$$

A Banach lattice E is called *order continuous* ($E \in$ (OC)) if for every $x \in E$ and every sequence $(x_m) \in E$ such that $0 \leftarrow x_m \leq |x|$ we have $\|x_m\|_E \to 0$ (see [18] and [22]). It is known that if $E \in$ (UM), then $E \in$ (OC).

A function $\Phi : T \times \mathcal{R} \longrightarrow [0, \infty)$ is said to be a *Musielak–Orlicz function* if $\Phi(\cdot, u)$ is measurable for each $u \in \mathcal{R}$, $\Phi(t, 0) = 0$ and $\Phi(t, \cdot)$ is convex, even, not identically equal to zero for μ-a.e. $t \in T$. We denote $(\Phi \circ x)(t) = \Phi(t, x(t))$. We will write $\Phi > 0$ if $\Phi(t, \cdot)$ vanishes only at zero for μ – a.e $t \in T$. For every Musielak–Orlicz-function Φ

we define *complementary function in the sense of Young* $\Phi^* : T \times \mathcal{R} \longrightarrow [0, \infty)$ by the formula $\Phi^*(t, v) = \sup_{u>0}\{u\,|v| - \Phi(t, u)\}$ for every $v \in \mathcal{R}$ and $t \in T$.

We say that *Musielak–Orlicz function* Φ *satisfies the* Δ_2^E*-condition* ($\Phi \in \Delta_2^E$) if there exist a constant $k \geq 2$, a set $A \in \Sigma$ of measure zero and a measurable non-negative function $h \in E$ such that

$$\Phi(t, 2u) \leq k\Phi(t, u) + h(t)$$

for every $t \in T \setminus A$ and every $u \in \mathcal{R}$ (see [11] when $E = L^1$ and [10] in general). Then $\Phi \in \Delta_2^E$ iff there exist a constant $k > 2$, a set $A \in \Sigma$ of measure zero and a measurable non-negative function $f \in L_+^0$ such that $\Phi \circ 2f \in E$ and

$$\Phi(t, 2u) \leq k\Phi(t, u)$$

for every $t \in T \setminus A$ and every $u \geq f(t)$ (see [10]).

Define on L^0 a convex modular I_Φ by

$$I_\Phi(x) = \begin{cases} \|\Phi \circ x\|_E & \text{if} \Phi \circ x \in E \\ \infty & \text{otherwise.} \end{cases}$$

By the *function lattice* E_Φ we mean

$$E_\Phi = \{x \in L^0 : I_\Phi(cx) < \infty \text{ for some } c > 0\},$$

equipped with so called *Luxemburg norm* defined as follows:

$$\|x\|_\Phi = \inf\left(\lambda > 0 : I_\Phi\left(\frac{x}{\lambda}\right) \leq 1\right).$$

We will assume in the paper that $E \in (\mathrm{FP})$, so $(E_\Phi, \|\cdot\|_\Phi)$ is a Banach space (see [9]). The space E_Φ (when $\Phi > 0$) is a special case of the Calderón–Lozanowskiĭ construction of the lattice (see [10]). As for theory of Calderón–Lozanowskiĭ space we refer to [3], [23] and [24].

If $E = L^1$, then E_Φ is the Musielak–Orlicz space equipped with the Luxemburg norm. If E is a Lorentz space Λ_ω, then E_Φ is the corresponding Musielak–Orlicz–Lorentz space $(\Lambda_\omega)_\Phi$ equipped with the Luxemburg norm (see [12, 16, 17]). If additionally $\Phi(t, u) = |u|$ for every $t \in T$, then the space $(\Lambda_\omega)_\Phi$ is the Lorentz space Λ_ω. Recall that the function $\omega : [0, \gamma) \to R_+$ with $\gamma = \mu(T)$ is said to be the weight function, if it is strictly positive, nonincreasing and locally integrable function with the respect to the Lebesgue measure ν. Then Λ_ω consists of all functions $x : [0, \gamma) \to \mathcal{R}$ measurable with respect to ν for which $\|x\| = \int_0^\gamma x^*(t)\omega(t)\mathrm{d}t < \infty$, where x^* is the decreasing rearrangement of x (see [2]). Recall that m_x denotes the distribution function of x, i.e. $m_x(\lambda) = \nu\left(\left\{t \in \left[0, \gamma\right] : |x(t)| > \lambda\right\}\right)$ for all $\lambda \geq 0$. The decreasing rearrangement function of x is denoted by x^* and is defined by $x^*(t) = \inf\{\lambda > 0 : m_x(\lambda) \leq t\}$. Denote by $S(t) = \int_0^t \omega(s)\mathrm{d}s$. The weight function is called regular if $\inf_{t>0} S(2t)/S(t) > 1$ in the case $\gamma = \infty$ and $\inf_{0<t<\gamma_0} S(2t)/S(t) > 1$ for some $\gamma_0 < \gamma/2$ in the case $\gamma < \infty$.

2. Results

It is a natural question whether a geometric property in Banach lattices can be equivalently considered only for nonnegative elements (see [13, 17]). In order to consider that problem, in case of property (β), let us introduce new notions.

DEFINITION 1

We say that a Banach lattice E has the property (β^{+}) if for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that for each element $x \in B(E_{+})$ and each sequence (x_n) in $B(E_{+})$ with $\operatorname{sep}\{x_n\} \geq \varepsilon$ there is an index k for which

$$\left\| \frac{x + x_k}{2} \right\| \leq 1 - \delta.$$

DEFINITION 2

We say that the Banach lattice $(E, \|\cdot\|_E)$ is *orthogonally uniformly convex* ($E \in (\mathrm{UC}^{\perp})$ for short), if for each $\varepsilon > 0$ there is $\delta = \delta(\varepsilon) > 0$ such that for $x, y \in B(E)$ the inequality $\max\left\{ \left\|x\chi_{A_{xy}}\right\|_E, \left\|y\chi_{A_{xy}}\right\|_E \right\} \geq \varepsilon$ implies $\|(x+y)/2\|_E \leq 1-\delta$, where $A_{xy} = \operatorname{supp} x \div \operatorname{supp} y$ and $A \div B = (A \setminus B) \cup (B \setminus A)$.

Lemma 1. (*Theorem 7 from* [13]). *Let E be any Banach function lattice. Then* $E \in (\mathrm{UM})$ *iff for any* $\varepsilon \in (0, 1)$ *there* $\eta(\varepsilon) > 0$ *such that for any* $x \in E_{+}$ *with* $\|x\|_E = 1$ *and for any* $A \in \Sigma$ *such that* $\|x\chi_A\|_E \geq \varepsilon$ *there holds* $\left\|x\chi_{T\setminus A}\right\|_E \leq 1 - \eta(\varepsilon)$.

Lemma 2. (*Lemma* 1.4 *in* [14]). *Let* $x, y \in X \setminus \{0\}$. *Denote* $\hat{x} = x / \|x\|_X$. *If* $\min\{\|x\|_X, \|y\|_X\} \geq \eta \max\{\|x\|_X, \|y\|_X\}$ *and* $\left\| \hat{x} - \hat{y} \right\|_X \geq \varepsilon$, *then*

$$\|x + y\|_X \leq \frac{1}{2}\left(1 - \eta\delta_X(\varepsilon)\right)\left(\|x\|_X + \|y\|_X\right).$$

Obviously if $E \in (\mathrm{UC})$, then $E \in (\mathrm{UC}^{\perp})$. It is known that uniformly convex Banach function lattice is uniformly monotone ([13]). Moreover, we will prove the following:

Lemma 3. *Let E be any Banach function lattice. If* $E \in (\mathrm{UC}^{\perp})$, *then* $E \in (\mathrm{UM})$.

Proof. Assume that $E \notin (\mathrm{UM})$. By Lemma 1 we conclude that there exists $\varepsilon > 0$ such that for every $n \in \mathcal{N}$ there exist $x_n \in E_{+}$ with $\|x_n\|_E = 1$ and a set $B_n \in \Sigma$ such that $\left\|x_n\chi_{B_n}\right\|_E \geq \varepsilon$ and $\left\|x_n\chi_{T\setminus B_n}\right\|_E > 1 - 1/n$. Let $u_n = x_n$ and $v_n = x_n\chi_{T\setminus B_n}$. Denote $A_n = \operatorname{supp} u_n \div \operatorname{supp} v_n$. Then $\max\left\{\left\|u_n\chi_{A_n}\right\|_E, \left\|v_n\chi_{A_n}\right\|_E\right\} \geq \varepsilon$. Moreover, since $u_n \geq v_n$, so we get $\|u_n + v_n\|_E \geq \|2v_n\|_E > 2(1 - 1/n)$. Hence $E \notin (\mathrm{UC}^{\perp})$.

Remark 1. The converse implication is not true. Let $E = L^1$. Obviously $L^1 \in (UM)$. Let $x, y \in S(L^1)$ and $\operatorname{supp} x \cap \operatorname{supp} y = \emptyset$. Then $\max\left\{\left\|x\chi_{A_{xy}}\right\|_{L^1}, \left\|y\chi_{A_{xy}}\right\|_{L^1}\right\} = 1$ and $\|(x+y)/2\|_{L^1} = 1$. Thus $L^1 \notin (\mathrm{UC}^{\perp})$.

Theorem 1. *Let E be a function Banach lattice. Then* $E \in (\beta)$ *iff* $E \in (\beta^{+})$ *and* $E \in (\mathrm{UC}^{\perp})$.

Proof. Necessity. Clearly, if $E \in (\beta)$, then $E \in (\beta^{+})$. Moreover, taking into account that (NUC) $\Rightarrow$ (H) (see [15]) and (H) $\Rightarrow$ (OC) in any Banach function lattice (see [8]), we get $(\beta) \Rightarrow$ (OC). Moreover, by Lemma 3 we conclude that $(\mathrm{UC}^{\perp}) \Rightarrow$ (OC). Hence it is enough to prove that if $E \notin (\mathrm{UC}^{\perp})$ and $E \in (\mathrm{OC})$, then $E \notin (\beta)$. Assume that $E \notin (\mathrm{UC}^{\perp})$ and $E \in (\mathrm{OC})$. Then there exists $\varepsilon > 0$ such that for every $n \in \mathcal{N}$ there exist $x_n, y_n \in B(E)$ with $\max\left\{\left\|x_n\chi_{A_{x_ny_n}}\right\|_E, \left\|y_n\chi_{A_{x_ny_n}}\right\|_E\right\} \geq \varepsilon$ and $\|x_n + y_n\|_E > 2(1 - 1/n)$. Denote $A_n = A_{x_ny_n}$. Divide the set A_n into two disjoint subsets $A_n^1 = \operatorname{supp} x_n \setminus \operatorname{supp} y_n$ and $A_n^2 = \operatorname{supp} y_n \setminus \operatorname{supp} x_n$. We divide the proof into two parts:

1. Suppose that $\left\|x_n\chi_{A_n^1}\right\|_E \geq \varepsilon$. We claim that for every $n \in \mathcal{N}$ there exists a set $B_n \subset A_n^1$ of finite measure such that $\left\|x_n\chi_{A_n^1\setminus B_n}\right\|_E < \varepsilon/2$. Fix $n \in \mathcal{N}$. Since the measure μ is σ-finite, there exists an increasing sequence of sets of finite measure $(S_k)_{k=1}^{\infty} \subset A_n^1$ such that $\bigcup_{k=1}^{\infty} S_k = A_n^1$. Denote $w_k = x_n\chi_{A_n^1\setminus S_k}$. Then $w_k \downarrow 0$ a.e.. Since E is order continuous, so $\|w_k\|_E \to 0$, what proves the claim. Then $\left\|x_n\chi_{B_n}\right\|_E \geq \varepsilon/2$ for every $n \in \mathcal{N}$. We decompose each set B_n into the family of sets $B_n\left(j, 2^k\right)$ for $j = 1, 2, \ldots, 2^k$ and $k = 1, 2, \ldots$, by the following iteration. We divide B_n into two disjoint sets $B_n(1, 2)$ and $B_n(2, 2)$ such that $\mu(B_n(1, 2)) = \mu(B_n(2, 2))$. Suppose that for fixed k the sets $B_n\left(j, 2^k\right)$ $(1 \leq j \leq 2^k)$ are already defined. To obtain sets $B_n\left(j, 2^{k+1}\right)$ $(1 \leq j \leq 2^{k+1})$ we divide every set $B_n\left(j, 2^k\right)$ $(1 \leq j \leq 2^k)$ into two disjoint sets $B_n\left(2j-1, 2^{k+1}\right)$ and $B_n\left(2j, 2^{k+1}\right)$ such that $\mu\left(B_n\left(2j-1, 2^{k+1}\right)\right) = \mu\left(B_n\left(2j, 2^{k+1}\right)\right)$. Define on the set B_n the kth Rademacher function by

$$r_k^n(t) = \begin{cases} 1 & \text{for } t \in B_n\left(j, 2^k\right) \text{ with odd } j,\ 1 \leq j \leq 2^k, \\ -1 & \text{for } t \in B_n\left(j, 2^k\right) \text{ with even } j,\ 1 \leq j \leq 2^k, \\ 0 & \text{for } t \notin B_n\left(j, 2^k\right). \end{cases}$$

Let

$$f_k^n(t) = x_n(t)\, r_k^n(t)$$

for every $n, k \in \mathcal{N}$. We will prove that for every $n \in \mathcal{N}$ we have $f_k^n \overset{w}{\to} 0$ as $k \to \infty$ in E. Recall that a *Köthe dual* E' of E is defined by

$$E' = \left\{h \in L^0 : \|h\|_{E'} = \sup\left\{\int_T |h(t)\, g(t)|\, \mathrm{d}\mu : g \in E, \|g\|_E \leq 1\right\} < \infty\right\}.$$

It is known that E' is a Banach function lattice. Moreover $E^* = E'$ iff $E \in (\mathrm{OC})$ (see [18] and [22]). Then for fixed $n \in \mathcal{N}$ and every $h^* \in E^*$ we get

$$\lim_{k\to\infty} h^*\left(f_k^n\right) = \lim_{k\to\infty} \int_T f_k^n(t)\, h(t)\, \mathrm{d}\mu = \lim_{k\to\infty} \int_{B_n} r_k^n(t)\, x_n(t)\, h(t)\, \mathrm{d}\mu = 0$$

since $x_n(t)\, h(t)$ is real integrable function. It follows by the fact that the set of simple functions is dense in L^1 and for every simple function b defined on B_n there holds $\lim_{k\to\infty} \int_{B_n} r_k^n(t)\, b(t)\, \mathrm{d}\mu = 0$. Therefore for every $n \in \mathcal{N}$ we have $f_k^n \overset{w}{\to} 0$ as $k \to \infty$ in E. Moreover $\left\|f_k^n\right\| \geq \varepsilon/2$ for every $n, k \in \mathcal{N}$. Then, applying Hahn–Banach theorem, it is easy to prove that for every $n \in \mathcal{N}$ there exists a subsequence $\left(g_k^n\right)_{k=1}^{\infty} \subset \left(f_k^n\right)_{k=1}^{\infty}$ such that $\operatorname{sep}\left\{g_k^n\right\}_E \geq \varepsilon/4$. For every $n \in \mathcal{N}$ let

$$h_k^n = g_k^n + x_n\chi_{\operatorname{supp} x_n\setminus B_n},\ k = 1, 2, \ldots.$$

Then for every $n \in \mathcal{N}$ we get $\operatorname{sep}\{h_k^n\}_E \geq \varepsilon/4$. On the other hand for every $n \in \mathcal{N}$

$$\left\|y_n + h_k^n\right\|_E = \|y_n + x_n\|_E > 2(1 - 1/n)$$

for all $k \in \mathcal{N}$. It means that $E \notin (\beta)$.

2. If $\left\|y_n \chi_{A_n^2}\right\|_E \geq \varepsilon$, then the proof is analogous.

Sufficiency. Take $\varepsilon > 0$. Let $x \in B(E)$. Take $(x_n)_{n=1}^{\infty} \subset B(E)$ with $\operatorname{sep}\{x_n\}_E \geq \varepsilon$. Denote by x^+ and x^- the positive and negative part of x, respectively. We will show that there exists a subsequence $(z_n)_{n=1}^{\infty} \subset (x_n)_{n=1}^{\infty}$ such that $\operatorname{sep}\{z_n^+\} \geq \varepsilon/2$ or $\operatorname{sep}\{z_n^-\} \geq \varepsilon/2$. For every $n \neq m$ we have $\left\|x_n^+ - x_m^+\right\|_E \geq \varepsilon/2$ or $\left\|x_n^- - x_m^-\right\|_E \geq \varepsilon/2$.

1. Consider the element x_1 and the sequence $(x_n)_{n=2}^{\infty}$. Then there exists a subsequence $\left(x_n^{(1)}\right)_{n=1}^{\infty} \subset (x_n)_{n=2}^{\infty}$ such that

   $$\left\|x_1^+ - x_n^{(1)+}\right\|_E \geq \varepsilon/2 \text{ for every } n \in \mathcal{N} \text{ or } \left\|x_1^- - x_n^{(1)-}\right\|_E \geq \varepsilon/2 \text{ for every } n \in \mathcal{N}.$$

   Denote $y_1^{(1)} = x_1$ and $y_{n+1}^{(1)} = x_n^{(1)}$ for every $n \in \mathcal{N}$.

2. Consider the element $x_1^{(1)}$ and the sequence $\left(x_n^{(1)}\right)_{n=2}^{\infty}$. Then there exists a subsequence $\left(x_n^{(2)}\right)_{n=1}^{\infty} \subset \left(x_n^{(1)}\right)_{n=2}^{\infty}$ such that

   $$\left\|x_1^{(1)+} - x_n^{(2)+}\right\|_E \geq \varepsilon/2 \text{ for every } n \in \mathcal{N} \text{ or } \left\|x_1^{(1)-} - x_n^{(2)-}\right\|_E \geq \varepsilon/2 \text{ for every } n \in \mathcal{N}.$$

Denote $y_1^{(2)} = x_1^{(1)}$ and $y_{n+1}^{(2)} = x_n^{(2)}$ for every $n \in \mathcal{N}$.

Taking the next steps analogously we conclude that there exists a sequence $(j_k)_{k=1}^{\infty}$ of natural numbers and the sequence of subsequences $\left(y_n^{(j_k)}\right)_{n=1}^{\infty}$, $k = 1, 2, \ldots$ such that

$$\left(y_n^{(j_1)}\right)_{n=1}^{\infty} \supset \left(y_n^{(j_2)}\right)_{n=1}^{\infty} \supset \ldots$$

and

$$\left\|y_1^{(j_k)+} - y_n^{(j_k)+}\right\|_E \geq \varepsilon/2 \quad \text{for all } k, n \in \mathcal{N}, n \geq 2$$

or

$$\left\|y_1^{(j_k)-} - y_n^{(j_k)-}\right\|_X \geq \varepsilon/2 \quad \text{for all } k, n \in \mathcal{N}, n \geq 2.$$

Define $z_n = y_1^{(j_n)}$ for every $n \in \mathcal{N}$. The sequence (z_n) satisfies the required condition. Denote still this subsequence (z_n) by (x_n). Let $\operatorname{sep}\{x_n^+\} \geq \varepsilon/2$. Denote by $p(\cdot)$ the modulus of the uniform monotonicity of E, by $\delta_E(\cdot)$ the function $\delta(\cdot)$ given in Definition 1 and by $\delta_E^{\perp}(\cdot)$ the function $\delta(\cdot)$ given in Definition 2. We denote some constants

$$\begin{array}{ll} \delta_1 = \delta_E^{\perp}(\varepsilon/16) > 0, & 0 < \alpha < \min\{\delta_1/8, \varepsilon/144\}, \\ 0 < b < \alpha, & \delta_2 = \delta_E(\alpha b) > 0, \\ p_1 = p(4\alpha b) > 0, & p_2 = p\left(\alpha^2 b^2/2\right) > 0. \end{array} \tag{1}$$

For every $n \neq m$ we have $\||x_n| - |x_m|\|_E \geq \alpha b$ or $\||x_n| - |x_m|\|_E < \alpha b$. Hence, analogously as in the previous part of the proof, we may find a subsequence $(y_n)_{n=1}^{\infty} \subset (x_n)_{n=1}^{\infty}$ such that

$$\||y_n| - |y_m|\|_E \geq \alpha b \text{ for all } n \neq m \text{ or } \||y_n| - |y_m|\|_E < \alpha b \text{ for all } n \neq m.$$

Denote still (y_n) by (x_n). We consider two cases:

I. Assume that $\||x_n| - |x_m|\|_E \geq \alpha b$ for every $n \neq m$. Then $\operatorname{sep}\{|x_n|\} \geq \alpha b$. Denote $y_n = |x_n|$ and $y = |x|$. Hence $y_n \in B(E_+)$ and $\operatorname{sep}\{y_n\}_E \geq \alpha b$. Basing on property (β^+) we find a number $k \in \mathcal{N}$ such that $\|(y + y_k)/2\|_E \leq 1 - \delta_2$, where δ_2 is defined in (1). Consequently $\|x + x_k\|_E \leq \|y + y_k\|_E \leq 2(1 - \delta_2)$.

II. Suppose that

$$\||x_n| - |x_m|\|_E < \alpha b \text{ for every } n \neq m. \tag{2}$$

For every $n \neq m$ denote

$$A_{nm}^1 = \operatorname{supp} x_n^+ \cap \operatorname{supp} x_m^+ \text{ and } A_{nm}^2 = \operatorname{supp} x_n^+ \div \operatorname{supp} x_m^+,$$

where $A \div B = (A \setminus B) \cup (B \setminus A)$. Since $\operatorname{sep}\{x_n^+\} \geq \varepsilon/2$, then for every $n \neq m$ we get $\|(x_n^+ - x_m^+)\chi_{A_{nm}^1}\|_E \geq \varepsilon/4$ or $\|(x_n^+ - x_m^+)\chi_{A_{nm}^2}\|_E \geq \varepsilon/4$. Suppose that $\|(x_n^+ - x_m^+)\chi_{A_{nm}^1}\|_E \geq \varepsilon/4$ for some $n \neq m$. Then

$$\||x_n| - |x_m|\|_E \geq \left\|(|x_n| - |x_m|)\chi_{A_{nm}^1}\right\|_E = \left\|(x_n^+ - x_m^+)\chi_{A_{nm}^1}\right\|_E \geq \varepsilon/4 > \alpha b,$$

which is a contradiction with (2). Hence

$$\left\|(x_n^+ - x_m^+)\chi_{A_{nm}^2}\right\|_E \geq \varepsilon/4 \text{ for every } n \neq m. \tag{3}$$

Decompose the set A_{nm}^2 into two disjoint subsets

$$A_{nm}^{21} = \left\{t \in A_{nm}^2 : \operatorname{sgn}(x_n(t)x_m(t)) = -1\right\}, \quad A_{nm}^{22} = A_{nm}^2 \setminus A_{nm}^{21}.$$

Notice that for every $t \in A_{nm}^{22}$ we have $\operatorname{sgn}(x_n(t)x_m(t)) = 0$. First we will show that

$$\left\|(x_n^+ - x_m^+)\chi_{A_{nm}^{21}}\right\|_E \geq \varepsilon/4 - 2\alpha b \text{ for every } n \neq m. \tag{4}$$

If $\max\left\{\|x_n\chi_{A_{nm}^{22}}\|_E, \|x_m\chi_{A_{nm}^{22}}\|_E\right\} \geq \alpha b$, then $\||x_n| - |x_m|\|_E \geq \alpha b$, which is a contradiction with (2). Hence $\max\left\{\|x_n\chi_{A_{nm}^{22}}\|_E, \|x_m\chi_{A_{nm}^{22}}\|_E\right\} < \alpha b$. Suppose that (4) is not true. Then, in view of (3), for some $n \neq m$ we get

$$\varepsilon/4 \leq \left\|(x_n^+ - x_m^+)\chi_{A_{nm}^2}\right\|_E \leq \left\|(x_n^+ - x_m^+)\chi_{A_{nm}^{21}}\right\|_E + 2\alpha b < \varepsilon/4,$$

which is a contradiction, so the inequality (4) holds. Decompose the set A_{nm}^{21} into two disjoint subsets $A_{nm}^{211} = \operatorname{supp} x_n^+ \setminus \operatorname{supp} x_m^+$ and $A_{nm}^{212} = \operatorname{supp} x_m^+ \setminus \operatorname{supp} x_n^+$. Consequently

$$\begin{aligned}\alpha b &> \||x_n| - |x_m|\|_E \geq \left\|(|x_n| - |x_m|)\chi_{A_{nm}^{21}}\right\|_E \\ &= \left\|(x_n^+ - x_m^-)\chi_{A_{nm}^{211}} + (x_m^+ - x_n^-)\chi_{A_{nm}^{212}}\right\|_E\end{aligned}$$

$$\geq \max\left\{\left\|\left(x_n^+ - x_m^-\right)\chi_{A_{nm}^{211}}\right\|_E, \left\|\left(x_m^+ - x_n^-\right)\chi_{A_{nm}^{212}}\right\|_E\right\}$$

$$\geq \max\left\{\left|\left\|x_n^+\chi_{A_{nm}^{211}}\right\|_E - \left\|x_m^-\chi_{A_{nm}^{211}}\right\|_E\right|, \left|\left\|x_m^+\chi_{A_{nm}^{212}}\right\|_E - \left\|x_n^-\chi_{A_{nm}^{212}}\right\|_E\right|\right\}$$

for every $n \neq m$. By (4) we get $\max\left\{\left\|x_n^+\chi_{A_{nm}^{211}}\right\|_E, \left\|x_m^+\chi_{A_{nm}^{212}}\right\|_E\right\} \geq \varepsilon/8 - \alpha b$. If $\left\|x_n^+\chi_{A_{nm}^{211}}\right\|_E \geq \varepsilon/8 - \alpha b$, then $\left\|x_m^-\chi_{A_{nm}^{211}}\right\|_E \geq \varepsilon/8 - 2\alpha b$. If $\left\|x_m^+\chi_{A_{nm}^{212}}\right\|_E \geq \varepsilon/8 - \alpha b$, then $\left\|x_n^-\chi_{A_{nm}^{212}}\right\|_E \geq \varepsilon/8 - 2\alpha b$. Hence

$$\left\|x_i\chi_{A_{nm}^{21}}\right\|_E \geq \varepsilon/8 - 2\alpha b \ \text{ for } i = n, m. \tag{5}$$

Denote $r \wedge s = \min\{r, s\}$ and $r \vee s = \max\{r, s\}$ for $r, s \in \mathcal{R}$. For every $n \in \mathcal{N}$ define

$$\begin{array}{ll}
B_n = \{t \in \operatorname{supp} x : |x(t)| \wedge |x_n(t)| \geq b\,(|x(t)| \vee |x_n(t)|)\}, & C_n = \operatorname{supp} x \setminus B_n, \\
B_n^1 = \{t \in B_n : \operatorname{sgn}(x(t)\,x_n(t)) = -1\}, & B_n^2 = B_n \setminus B_n^1, \\
C_n^1 = \{t \in C_n : |x(t)| = |x(t)| \wedge |x_n(t)|\}, & C_n^2 = C_n \setminus C_n^1
\end{array}$$

and for every $k \neq n$ let

$$\begin{array}{ll}
D_{nk}^1 = \left\{t \in B_n^2 : |x_k(t)| \wedge |x_n(t)| \geq \alpha b\,(|x_k(t)| \vee |x_n(t)|)\right\}, & D_{nk}^2 = B_n^2 \setminus D_{nk}^1, \\
E_{nk}^1 = \left\{t \in D_{nk}^1 : \operatorname{sgn}(x_k(t)\,x_n(t)) = -1\right\}, & E_{nk}^2 = D_{nk}^1 \setminus E_{nk}^1.
\end{array}$$

II.1. Suppose that $\left\|x\chi_{B_n^1}\right\|_E \geq 8\alpha$ for some $n \in \mathcal{N}$. Denote by $\delta_{\mathcal{R}}(\cdot)$ the modulus of convexity of $\mathcal{R}$. Note that $\delta_{\mathcal{R}}(2) = 1$. Applying Lemma 2 we get

$$\left|(x + x_n)\,\chi_{B_n^1}\right| \leq (1 - b)\,(|x| + |x_n|)\,\chi_{B_n^1}.$$

Consequently

$$\frac{|x + x_n|}{2} \leq \frac{|x| + |x_n|}{2} - \frac{b}{2}\,(|x| + |x_n|)\,\chi_{B_n^1}.$$

Hence, applying the uniform monotonicity of E, we get $\|(x + x_n)/2\|_E \leq 1 - p_1$, where p_1 is defined in (1).

II.2. Let

$$\left\|x\chi_{B_n^1}\right\|_E < 8\alpha \text{ for some } n \in \mathcal{N}. \tag{6}$$

Note that if $\left\|x\chi_{C_n^1}\right\|_E \geq \alpha$, then $\|x_n\|_E \geq \left\|x_n\chi_{C_n^1}\right\|_E \geq \frac{1}{b}\alpha > 1$. Hence

$$\left\|x\chi_{C_n^1}\right\|_E < \alpha. \tag{7}$$

Furthermore $\left\|x_n\chi_{C_n^2}\right\|_E < b < \alpha$. Consequently if $\left\|x_k\chi_{C_n^2}\right\|_E \geq 2\alpha$ for some $k \neq n$, then $\||x_n| - |x_k|\| \geq \alpha > \alpha b$, but this contradicts (2). Thus

$$\left\|x_k\chi_{C_n^2}\right\|_E < 2\alpha \text{ for every } k \neq n. \tag{8}$$

Moreover we will show that

$$\left\| x_k \chi_{D^2_{nk}} \right\|_E < 4\alpha b \text{ for every } k \neq n. \tag{9}$$

Suppose conversely that $\left\| x_k \chi_{D^2_{nk}} \right\|_E \geq 4\alpha b$ for some $k \neq n$ and let

$$D^{21}_{nk} = \left\{ t \in D^2_{nk} : |x_k(t)| = |x_k(t)| \wedge |x_n(t)| \right\}, \quad D^{22}_{nk} = D^2_{nk} \setminus D^{21}_{nk}.$$

If $\left\| x_k \chi_{D^{21}_{nk}} \right\|_E \geq 2\alpha b$, then $\left\| x_n \chi_{D^{21}_{nk}} \right\|_E \geq 2$. But $x_n \in B(E)$. Hence $\left\| x_k \chi_{D^{22}_{nk}} \right\|_E \geq 2\alpha b$. But $\left\| x_n \chi_{D^{22}_{nk}} \right\|_E < \alpha b$. Consequently $\| |x_n| - |x_k| \| \geq \alpha b$, which is a contradiction with (2), so (9) is proved. We divide the proof into two parts:

a. $\left\| x_k \chi_{E^1_{nk}} \right\|_E \geq \alpha$ for some $k \neq n$. Note that for every $t \in E^1_{nk}$ we have

$$|x_k(t)| \wedge |x(t)| \geq \alpha b^2 \left(|x_k(t)| \vee |x(t)| \right) \text{ and } \operatorname{sgn}\left(x_k(t)\, x(t) \right) = -1.$$

Since $\delta_{\mathcal{R}}(2) = 1$, so applying Lemma 2 we get

$$\left| (x + x_k)\, \chi_{E^1_{nk}} \right| \leq \left(1 - \alpha b^2 \right) \left(|x| + |x_k| \right) \chi_{E^1_{nk}}.$$

Then

$$\frac{|x + x_k|}{2} \leq \frac{|x| + |x_k|}{2} - \frac{\alpha b^2}{2} \left(|x| + |x_k| \right) \chi_{E^1_{nk}}.$$

Hence, similarly as in case II.1, we conclude that $\| (x + x_k)/2 \|_E \leq 1 - p_2$, where p_2 is defined in (1).

b. $\left\| x_k \chi_{E^1_{nk}} \right\|_E < \alpha$ for every $k \neq n$. Then, by (8) and (9), we get

$$\left\| x_k \chi_{C^2_n \cup D^2_{nk} \cup E^1_{nk}} \right\|_E < 3\alpha + 4\alpha b < 7\alpha \text{ for every } k \neq n. \tag{10}$$

Notice that $A^{21}_{nk} \cap E^2_{nk} = \emptyset$ for every $k \neq n$. Furthermore the inequality (5) yields $\left\| x_k \chi_{A^{21}_{nk}} \right\|_E \geq \varepsilon/8 - 2\alpha b$ for every $k \neq n$. Consequently, by (1) and (10), we obtain

$$\left\| x_k \chi_{A^{21}_{nk} \setminus (B^2_n \cup C^2_n)} \right\|_E = \left\| x_k \chi_{A^{21}_{nk} \setminus (D^2_{nk} \cup E^1_{nk} \cup E^2_{nk} \cup C^2_n)} \right\|_E \geq \frac{\varepsilon}{8} - 9\alpha \geq \frac{\varepsilon}{16} \tag{11}$$

for every $k \neq n$. Let $z = |x|\, \chi_{B^2_n \cup C^2_n}$. Denote $T_k = \operatorname{supp} z \div \operatorname{supp} x_k$. Then, by (11), we get $\| z \chi_{T_k} \|_E \vee \| x_k \chi_{T_k} \|_E \geq \varepsilon/16$ for every $k \in \mathcal{N}$. Since $E \in \left(\mathrm{UC}^{\perp} \right)$, then $\| z + x_k \|_E \leq 2(1 - \delta_1)$ for every $k \in \mathcal{N}$, where δ_1 is defined in (1). Thus, by (1), (6) and (7), we obtain

$$\left\| \frac{x + x_k}{2} \right\|_E = \left\| \frac{x \chi_{B^2_n \cup C^2_n} + x \chi_{T \setminus (B^2_n \cup C^2_n)} + x_k}{2} \right\|_E$$

$$\leq \frac{9\alpha}{2} + \left\| \frac{z + x_k}{2} \right\|_E \leq 5\alpha + 1 - \delta_1 \leq 1 - 3\delta_1/8.$$

Combining all of the cases we get $\| (x + x_k)/2 \|_E \leq 1 - \lambda$ for some $k \in \mathcal{N}$, where $\lambda = \min\{\delta_2, p_1, p_2, 3\delta_1/8\}$, which finishes the proof. □

An immediate consequence of Lemma 3 and Theorem 1 is the following.

COROLLARY 1

Let E be a Banach function lattice. If $E \in (\beta)$, then $E \in$ (UM).

Here we will find some necessary and sufficient conditions for the *property* (β) in the space E_Φ. We will need the following notion. We say that Musielak–Orlicz function Φ satisfies *the uniform Δ_2^E-condition* (we write $\Phi \in \Delta_2$ for short) if there exist a constant $k \geq 2$ and a set $A \in \Sigma$ of measure zero such that $\Phi(t, 2u) \leq k\Phi(t, u)$ for every $t \in T \setminus A$ and every $u \in \mathcal{R}$.

For $t \in T$ the function $\Phi(t, \cdot)$ is *strictly convex* if $\Phi(t, (u+v)/2) < (\Phi(t, u) + \Phi(t, v))/2$ for all $u, v \in \mathcal{R}$, $u \neq v$.

Lemma 4. *Suppose that $\Phi^* \in \Delta_2^E$. Then there exist a number $\xi > 1$ and nonnegative function f^* with $\Phi \circ 2f^* \in E$ such that*

$$\Phi\left(t, \frac{\xi}{2}u\right) \leq \frac{1}{2\xi}\Phi(t, u)$$

for μ-a.e. $t \in T$ and $u \geq f^*(t)$.

Proof. Lemma 4 for $E = L^1$ was proved in [1], the proof in general case is similar.

The proof of the next lemma is similar to the proof of Theorem 7 in [9].

Lemma 5. *If $E \in (\mathrm{UC}^\perp)$, $\Phi \in \Delta_2^E$ and $\Phi > 0$, then $E_\Phi \in (\mathrm{UC}^\perp)$.*

Theorem 2. *The following assertions are true:*

(i) *If $E_\Phi \in (\beta)$, then $\Phi \in \Delta_2^E$.*

(ii) *Let $E \in$ (UM), $\Phi \in \Delta_2^E$, $\Phi > 0$ and $\Phi^* \in \Delta_2^E$. Denote by f^* the function from Lemma 4. If for μ-a.e. $t \in T$, $\Phi(t, \cdot)$ is strictly convex function on the interval $[0, f^*(t)]$, then $E_\Phi \in (\beta)$.*

(iii) *Assume that $E \in$ (UM), $\Phi \in \Delta_2^E$, $\Phi > 0$ and $\Phi^* \in \Delta_2$. Then $E_\Phi \in (\beta)$.*

(iv) *If $E \in (\beta)$, $\Phi \in \Delta_2^E$ and $\Phi > 0$, then $E_\Phi \in (\beta)$.*

Proof. (i) If $\Phi \notin \Delta_2^E$, then E_Φ contains an order isomorphically isometric copy of l^∞ (see [10]). But $(\beta) \Rightarrow$ (OC) (see the proof of necessity of Theorem 1). Because $l^\infty \notin$ (OC), then $l^\infty \notin (\beta)$ and so $E_\Phi \notin (\beta)$.

(ii) Let $\varepsilon > 0$ be arbitrary. Take $x, x_n \in B(E_\Phi)$ with $\operatorname{sep}\{x_n\} \geq \varepsilon$. There exists an index k with $\|x - x_k\|_\Phi \geq \varepsilon/2$. Denote $\mu = \varepsilon/2$. Then the definition of Luxemburg norm yields

$$\left\|\Phi \circ \left(\frac{x - x_k}{\mu}\right)\right\|_E \geq 1. \tag{12}$$

By the assumption that $\Phi^* \in \Delta_2^E$ and Φ is strictly convex function on the interval $[0, f^*(t)]$ for μ-a.e. $t \in T$, applying Lemma 4, it is easy to show that there exist numbers $\lambda \in (0, 1)$,

$\delta \in (0,1)$ and a nonnegative function f_1 with $\left\|\Phi \circ \frac{2}{\mu} f_1\right\|_E \le \frac{\lambda}{8}$ such that

$$\Phi\left(t, \frac{u+v}{2}\right) \le \frac{1-\delta}{2}\left(\Phi(t,u)+\Phi(t,v)\right) \tag{13}$$

for μ-a.e. $t \in T$ and $u, v \ge 0$ and satisfying conditions $\max(u,v) \ge f_1(t)$ and $|u-v| \ge \frac{\mu(1-\lambda)}{2}|u+v|$ (see [1] and [6]). Moreover, $E \in (\mathrm{UM})$, so $E \in (\mathrm{OC})$. Then, by $\Phi > 0$, so we conclude that, there exist a number $k \ge 2$ and nonnegative function f_2 with $\left\|\Phi \circ \frac{2}{\mu} f_2\right\|_E \le \frac{\lambda}{8}$ and

$$\Phi(t, 2u) \le k\Phi(t,u) \tag{14}$$

for μ-a.e. $t \in T$ and $u \ge f_2(t)$ (see Lemma 2 in [10]). Define $f(t) = \max\{f_1(t), f_2(t)\}$. Then $\left\|\Phi \circ \frac{2}{\mu} f\right\|_E \le \frac{\lambda}{4}$. Denote

$$\begin{aligned} A &= \left\{t \in T : |x(t) - x_k(t)| \ge \tfrac{1}{2}\mu(1-\lambda)\,|x(t) + x_k(t)|\right\}, \\ B &= \{t \in T : |x(t) - x_k(t)| \ge 2f(t)\}, \\ C &= A \cap B. \end{aligned}$$

Then

$$\begin{aligned} &\left\|\Phi \circ \left(\tfrac{x-x_k}{\mu}\right)\chi_{T\setminus A}\right\|_E < \tfrac{1}{2}(1-\lambda)\left\{\left\|\Phi \circ x\chi_{T\setminus A}\right\|_E \right. \\ &\quad \left. + \left\|\Phi \circ x_k\chi_{T\setminus A}\right\|_E\right\} \le 1-\lambda. \end{aligned}$$

Moreover

$$\left\|\Phi \circ \left(\frac{x-x_k}{\mu}\right)\chi_{T\setminus B}\right\|_E < \left\|\Phi \circ \left(\frac{2}{\mu}f\right)\chi_{T\setminus B}\right\|_E \le \frac{\lambda}{4}.$$

Consequently

$$\begin{aligned} &\left\|\Phi \circ \left(\tfrac{x-x_k}{\mu}\right)\chi_{T\setminus C}\right\|_E \le \left\|\Phi \circ \left(\tfrac{x-x_k}{\mu}\right)\chi_{T\setminus A}\right\|_E \\ &\quad + \left\|\Phi \circ \left(\tfrac{x-x_k}{\mu}\right)\chi_{T\setminus B}\right\|_E \le 1 - \tfrac{3\lambda}{4}. \end{aligned}$$

By (12) we get $\left\|\Phi \circ \left(\frac{x-x_k}{\mu}\right)\chi_C\right\|_E \ge \frac{3\lambda}{4}$. Take a natural number m with $\frac{2}{\mu} \le 2^m$. Applying the convexity of Φ and using (14), we obtain

$$\begin{aligned} \frac{3\lambda}{4} &\le \left\|\Phi \circ \left(\frac{x-x_k}{\mu}\right)\chi_C\right\|_E = \left\|\Phi \circ \frac{2}{\mu}\left(\frac{x-x_k}{2}\right)\chi_C\right\|_E \le \left\|\Phi \circ 2^m\left(\frac{x-x_k}{2}\right)\chi_C\right\|_E \\ &\le k^m\left\|\Phi \circ \left(\frac{x-x_k}{2}\right)\chi_C\right\|_E \le \frac{k^m}{2}\left(\|\Phi \circ x\chi_C + \Phi \circ x_k\chi_C\|_E\right). \end{aligned}$$

So

$$\|\Phi \circ x\chi_C + \Phi \circ x_k\chi_C\|_E \ge \frac{3\lambda}{2k^m}. \tag{15}$$

Furthermore $2f(t) \le |x(t) - x_k(t)| \le 2\max\{|x(t)|, |x_k(t)|\}$ for every $t \in C$. Then the inequality (13) yields

$$\Phi \circ \left(\frac{x+x_k}{2}\right) \le \frac{1}{2}(\Phi \circ x + \Phi \circ x_k) - \frac{\delta}{2}(\Phi \circ x + \Phi \circ x_k)\chi_C. \tag{16}$$

Denote by $p(\cdot)$ the modulus of uniform monotonicity of the Banach lattice E. From (15) and (16) we conclude

$$\left\| \Phi \circ \left(\frac{x + x_k}{2} \right) \right\|_E \le 1 - p\left(\frac{3\lambda\delta}{4k^m} \right).$$

Finally we get $\|(x + x_k)/2\|_\Phi \le 1 - q$, where $q \in (0, 1)$ depends only on p (see Lemma 3 in [10]).

(iii) The proof is analogous as in (ii).

(iv) Since $E \in (\beta)$, then by Theorem 1 we get $E \in \left(\mathrm{UC}^\perp\right)$. Furthermore $\Phi \in \Delta_2^E$ and $\Phi > 0$. By Lemma 5 we conclude that $E_\Phi \in \left(\mathrm{UC}^\perp\right)$. Basing on Theorem 1, it is enough to show that $E_\Phi \in (\beta^+)$. Denote by E_Φ^+ the positive cone of E_Φ. Let $\varepsilon \in (0, 1)$ and $x \in B(E_\Phi^+)$. Take $(x_n)_{n=1}^\infty \subset B(E_\Phi^+)$ such that $\operatorname{sep}\{x_n\} \ge \varepsilon$. Since $E \in$ (FP), $I_\Phi(\cdot)$ is left continuous. So, by the definition of the Luxemburg norm, we get $\|\Phi \circ x\|_E \le 1$ and $\|\Phi \circ x_n\|_E \le 1$. Moreover $(\beta) \Rightarrow (\mathrm{OC})$. Hence the assumptions that $E \in (\mathrm{OC})$, $\Phi \in \Delta_2^E$ and $\Phi > 0$ imply that there exists a number $\sigma(\varepsilon) \in (0, 1)$ such that $\|\Phi \circ (x_n - x_m)\|_E \ge \sigma(\varepsilon)$ for every natural $n \ne m$ (see Lemma 6 in [10]). The function Φ is superadditive on $\mathcal{R}_+$, so $|\Phi \circ (x_n - x_m)| \le |\Phi \circ x_n - \Phi \circ x_m|$, and consequently $\|\Phi \circ x_n - \Phi \circ x_m\|_E \ge \sigma(\varepsilon)$ for every $n \ne m$. By the property (β) of E we conclude that there exist a number $\delta = \delta(\sigma(\varepsilon)) > 0$ and a number $k \in \mathcal{N}$ such that $\|\Phi \circ x + \Phi \circ x_k\|_E \le 2(1 - \delta)$. Furthermore

$$\left\| \Phi \circ \left(\frac{x + x_k}{2} \right) \right\|_E \le \left\| \frac{\Phi \circ x + \Phi \circ x_k}{2} \right\|_E \le 1 - \delta.$$

Finally, (see Lemma 3 in [10]), there exists a number $\gamma = \gamma(\delta) \in (0, 1)$ such that $\|(x + x_k)/2\|_\Phi \le 1 - \gamma$.

Remark 2. The condition $E \in (\beta)$ is not necessary for $E_\Phi \in (\beta)$. It is sufficient to take $E = L^1$ over the finite measure space (T, Σ, μ) and the function Φ which does not depend on the parameter t. Then $L^\Phi \in (\beta)$ iff $\Phi \in \Delta_2$ and Φ is uniformly convex on the interval $[u_0, \infty]$ for every $u_0 > 0$ (see [7]). But $L^1 \notin (\beta)$, because it is an Orlicz space generated by the function $\Phi(u) = |u|$ which is not uniformly convex.

Now we will assume that Φ does not depend on the parameter t. The function Φ is *strictly convex* if $\Phi((u + v)/2) < (\Phi(u) + \Phi(v))/2$ for all $u, v \in \mathcal{R}$, $u \ne v$. The function Φ is *uniformly convex* (*uniformly convex for large arguments*), if for any $a > 0$ ($a > 0$ and $u_0 > 0$) there exists $\delta(a) > 0$ ($\delta = \delta(a, u_0) > 0$) such that $\Phi((u + au)/2) \le (1 - \delta)(\Phi(u) + \Phi(au))/2$ holds true for every $u \ge 0$ (for every $u \ge u_0$).

The implication (UC) $\Rightarrow$ (β) can be reversed in Orlicz function spaces over the finite measure space. It was shown for both Luxemburg and Orlicz norm ([7]). Here we will extend this result to the case of Orlicz–Lorentz spaces over the finite and infinite measure space.

Theorem 3. *Let $E = \Lambda_\omega$ with $\gamma = \infty$ and assume that Φ does not depend on t. Then the following statements are equivalent:*

(a) $(\Lambda_\omega)_\Phi \in (\mathrm{UC})$.

(b) $(\Lambda_\omega)_\Phi \in (\beta)$.

(c) *Φ is uniformly convex, Φ satisfies the Δ_2-condition for all arguments and the function ω is regular.*

Proof. The implication (UC) $\Rightarrow$ (β) holds in any Banach space. It is also true that (c) $\Leftrightarrow$ (a) (see [17]). Note that if $\gamma = \infty$ and w is regular, then $\int_0^\infty \omega(t)\mathrm{d}t = \infty$. It is enough to prove the implication (b) $\Rightarrow$ (c). Assume then that $(\Lambda_\omega)_\Phi \in (\beta)$. First we will show that ω is regular.

Take any $t > 0$. Since ω is locally integrable, there exists a number $a_t > 0$ such that

$$\Phi(a_t)\int_0^{2t} \omega(s)\mathrm{d}s = \Phi(a_t)S(2t) = 1. \tag{17}$$

Divide the interval $[0, 2t]$ into two intervals $G_1^1 = [0, t]$ and $G_2^1 = [t, 2t]$. Suppose that the sequence of intervals

$$\left(G_1^{n-1} = [0, 2t/2^{n-1}], G_2^{n-1} = [2t/2^{n-1}, 2t/2^{n-2}], ..., G_{2^{n-1}}^{n-1} = [2t - 2t/2^{n-1}, 2t]\right),$$

$n > 2$, is already defined. We divide each set $G_i^{n-1} = [(i-1)2t/2^{n-1}, i2t/2^{n-1}], i = 1, 2, ..., 2^{n-1}$ into two subsets G_{2i-1}^n, G_{2i}^n such that

$$G_{2i-1}^n = [(i-1)2t/2^{n-1}, (i-1)2t/2^{n-1} + t/2^{n-1}]$$

and

$$G_{2i}^n = [(i-1)2t/2^{n-1} + t/2^{n-1}, i2t/2^{n-1}], \ (i = 1, 2, ..., 2^{n-1}).$$

In such a way, we obtain a partition $\left(G_1^n, G_2^n, ..., G_{2^n}^n\right)$, $n = 1, 2, ...$ of the interval $[0, 2t]$ such that $\nu(G_i^n) = 2^{-n+1}t$, $(n = 1, 2, ..., \ i = 1, 2, ..., 2^n)$ and ν denotes the Lebesgue measure. Define

$$x_t = a_t\chi_{[0,2t]} \text{ and } x_{n,t} = a_t\chi_{E_{1,n}} - a_t\chi_{E_{2,n}},$$

where $E_{1,n} = \bigcup_{k=1}^{2^{n-1}} G_{2k-1}^n$, $E_{2,n} = \bigcup_{k=1}^{2^{n-1}} G_{2k}^n$, $(n = 1, 2, ...)$. We get $I_\Phi(x_t) = \Phi(a_t)\int_0^{2t} \omega(s)\mathrm{d}s = 1$. Moreover $x_{n,t}^* = a_t\chi_{[0,2t]}$ for every $n \in \mathcal{N}$. Consequently $I_\Phi(x_{n,t}) = 1$. Furthermore $(x_{n,t} - x_{m,t})^* = 2a_t\chi_{[0,t]}$ for every $n, m \in \mathcal{N}, n \neq m$. Hence $I_\Phi(x_{n,t} - x_{m,t}) = \Phi(2a_t)\int_0^t \omega(s)\mathrm{d}s$. Taking into account that Φ is convex and ω is nonincreasing function, in view of (17), we get

$$\Phi(2a_t)\int_0^t \omega(s)\mathrm{d}s \geq 2\Phi(a_t)\frac{1}{2}\int_0^{2t} \omega(s)\mathrm{d}s = 1.$$

for every $t > 0$. Then

$$\inf_{t>0} \Phi(2a_t)\int_0^t \omega(s)\mathrm{d}s \geq 1. \tag{18}$$

Thus $\|x_{n,t} - x_{m,t}\|_\Phi \geq 1$ for every $t > 0$ and $n, m \in \mathcal{N}, n \neq m$. We have constructed for every $t > 0$ an element $x_t \in S(\Lambda_{\omega,\Phi})$ and a sequence $(x_{n,t})_{n=1}^\infty \in S(\Lambda_{\omega,\Phi})$ with $\mathrm{sep}\{x_{n,t}\} \geq 1$. By the property ($\beta$) of $(\Lambda_\omega)_\Phi$, there exists a number $\delta = \delta(1) > 0$ and an index k for which

$$\left\|\frac{x_t + x_{k,t}}{2}\right\|_\Phi \leq 1 - \delta. \tag{19}$$

Notice that $\left(\frac{x_t+x_{n,t}}{2}\right)^* = a_t\chi_{[0,t]}$ for every $n \in \mathcal{N}$. Thus, in view of (19) we get

$$\begin{aligned}\Phi(a_t)\textstyle\int_0^t \omega(s)\mathrm{d}s &= I_\Phi\left(\frac{x_t+x_{k,t}}{2}\right) \le \left\|\frac{x_t+x_{k,t}}{2}\right\|_\Phi \\ &\le 1-\delta = (1-\delta)\Phi(a_t)\textstyle\int_0^{2t}\omega(s)\mathrm{d}s.\end{aligned}$$

This shows the regularity of the weight ω.

By the assumption $(\Lambda_\omega)_\Phi \in (\beta)$. Then Φ satisfies the suitable Δ_2^E condition (Theorem 2(i) in the case when Φ does not depend on the parameter was proved in [12]), i.e. there exists a number $k > 0$ such that for every $u \in \mathcal{R}$ we have $\Phi(2u) \le k\Phi(u)$ (see also [16]). We get in particular that $\Phi > 0$. Moreover, by Theorem 1.13 in [5], for every $l > 1$ there exists $k_l > 1$ such that for every $u \in \mathcal{R}$ we have

$$\Phi(lu) \le k_l\Phi(u). \tag{20}$$

Now we will show that Φ is uniformly convex. At first we will prove that Φ must be strictly convex. Suppose conversely that Φ is affine on the interval $[u, v]$. The weight function ω is locally integrable, so there exists a number $a > 0$ such that $0 < M_0 = \Phi(v)\int_0^a \omega(t)\mathrm{d}t < 1$. Moreover, if we define $\mu_\omega(A) = \int_A \omega(t)\mathrm{d}t$, then we conclude that μ_ω is non-atomic. By the Lapunov's theorem $\{\mu_\omega(A) : A \text{ is Lebesgue measurable}\} = [0, \infty)$. Consequently for every $\lambda > 0$ there exists a number $\gamma > 0$ such that $\int_a^\lambda \omega(t)\mathrm{d}t = \int_\lambda^{\frac{\lambda+\gamma}{2}} \omega(t)\mathrm{d}t$. Take numbers $\gamma, \lambda > 0$ satisfying

(i) $M_1 = \Phi(v)\int_a^\lambda \omega(t)\mathrm{d}t + \Phi(u)\int_\lambda^\gamma \omega(t)\mathrm{d}t \le 1 - M_0$,

(ii) $\int_a^\lambda \omega(t)\mathrm{d}t = \int_\lambda^{\frac{\lambda+\gamma}{2}} \omega(t)\mathrm{d}t$.

Then we find a number $c \ge v$ with $\Phi(c)\int_0^a \omega(t)\mathrm{d}t + M_1 = 1$. Define a partition $\left(G_1^n, G_2^n, ..., G_{2^n}^n\right), n = 1, 2, ...$ of the interval $[\lambda, \gamma]$ in the same way as in the previous part of the proof. Let

$$x = c\chi_{[0,a]} + v\chi_{[a,\lambda]} + u\chi_{[\lambda,\gamma]} \text{ and } x_n = c\chi_{[0,a]} + \frac{u+v}{2}\chi_{[a,\lambda]} + \frac{u+v}{2}\chi_{E_{1,n}} + u\chi_{E_{2,n}},$$

where $E_{1,n} = \bigcup_{k=1}^{2^{n-1}} G_{2k-1}^n$, $E_{2,n} = \bigcup_{k=1}^{2^{n-1}} G_{2k}^n$, $(n = 1, 2, ...)$. We get $I_\Phi(x) = 1$. Moreover

$$x_n^* = c\chi_{[0,a]} + \frac{u+v}{2}\chi_{[a,(\lambda+\gamma)/2]} + u\chi_{[(\lambda+\gamma)/2,\gamma]}.$$

Then, by (ii) and the linearity of the function Φ on the interval $[u, v]$, we conclude

$$I_\Phi(x) - I_\Phi(x_n) = \frac{1}{2}(\Phi(v) - \Phi(u))\left(\int_a^\lambda \omega(t)\mathrm{d}t - \int_\lambda^{(\lambda+\gamma)/2} \omega(t)\mathrm{d}t\right) = 0.$$

Furthermore $(x_n - x_m)^* = (v-u)/2\chi_{[0,(\gamma-\lambda)/2]}$ for every $n, m \in \mathcal{N}, n \ne m$. But $\left\|(v-u)/2\chi_{[0,(\gamma-\lambda)/2]}(\cdot)\right\|_\Phi = q$ for some $q > 0$. We have defined an element $x \in S(\Lambda_{\omega,\Phi})$ and a sequence $(x_n)_{n=1}^\infty \in S(\Lambda_{\omega,\Phi})$ with $\mathrm{sep}\{x_n\} \ge q$. On the other hand

$$\begin{aligned}\left(\frac{x+x_n}{2}\right)^* &= c\chi_{[0,a]} + \left(\frac{u+3v}{4}\right)\chi_{[a,\lambda]} \\ &+ \left(\frac{3u+v}{4}\right)\chi_{[\lambda,(\lambda+\gamma)/2]} + u\chi_{[(\lambda+\gamma)/2,\gamma]}\end{aligned}$$

for every $n \in \mathcal{N}$. Applying the fact that Φ is affine on the interval $[u, v]$, by (ii), we get

$$I_\Phi\left(\frac{x+x_n}{2}\right) = \Phi(c)\int_0^\gamma \omega(t)\mathrm{d}t + \left(\frac{\Phi(u)+3\Phi(v)}{4}\right)\int_a^\lambda \omega(t)\mathrm{d}t$$
$$+\left(\frac{3\Phi(u)+\Phi(v)}{4}\right)\int_\lambda^{(\lambda+\gamma)/2} \omega(t)\mathrm{d}t + \Phi(u)\int_{(\lambda+\gamma)/2}^{\gamma} \omega(t)\mathrm{d}t = 1$$

for every $n \in \mathcal{N}$. Hence $\left\|\frac{x+x_n}{2}\right\| = 1$ for every $n \in \mathcal{N}$. But this is a contradiction with the property (β).

To finish the proof it is enough to show that Φ is uniformly convex. Suppose that this is not true i.e. there exists a sequence u_k of positive numbers and a constant $b \in (0, 1)$ such that

$$\Phi\left(\frac{u_k+bu_k}{2}\right) > \left(1-\frac{1}{k}\right)\frac{1}{2}\left(\Phi\left(u_k\right)+\Phi\left(bu_k\right)\right).$$

Notice that $\frac{u_k+bu_k}{2} = \frac{2}{3}\left(\frac{u_k+3bu_k}{4}\right)+\frac{1}{3}u_k$ and $\frac{u_k+bu_k}{2} = \frac{2}{3}\left(\frac{3u_k+bu_k}{4}\right)+\frac{1}{3}bu_k$. Consequently, applying the convexity of Φ, it is easy to prove that

$$\Phi\left(\frac{u_k+3bu_k}{4}\right) > \left(1-\frac{4}{k}\right)\left(\frac{1}{4}\Phi(u_k)+\frac{3}{4}\Phi(bu_k)\right) \tag{21}$$

and

$$\Phi\left(\frac{3u_k+bu_k}{4}\right) > \left(1-\frac{4}{k}\right)\left(\frac{3}{4}\Phi(u_k)+\frac{1}{4}\Phi(bu_k)\right). \tag{22}$$

If there is a subsequence of (u_k) approaching a number $u > 0$, then Φ is affine on the interval $[bu, u]$ and thus Φ is not strictly convex. Consequently, without loss of generality, we assume that $u_k \to 0$ or $u_k \to \infty$. The proof will be done only for the case $u_k \to 0$, in another case it is analogous. The weight function ω is locally integrable, so there exists a number $a > 0$ such that $0 < M_0 = \Phi(u_1)\int_0^a \omega(t)\mathrm{d}t < 1/2$. Then, similarly as in the proof of the strict convexity of Φ, for every $k \in N$ there exist numbers $\lambda_k > 0$ and $\gamma_k = \gamma_k(\lambda_k) > 0$ satisfying

$$\int_a^{\lambda_k} \omega(t)\mathrm{d}t = \int_{\lambda_k}^{\frac{\lambda_k+\gamma_k}{2}} \omega(t)\mathrm{d}t, \tag{23}$$

$$M_0 < M_k \leq 1 - M_0, \tag{24}$$

where $M_k = \Phi(u_k)\int_a^{\lambda_k} \omega(t)\mathrm{d}t + \Phi(bu_k)\int_{\lambda_k}^{\gamma_k} \omega(t)\mathrm{d}t$. Then take a sequence $c_k \geq u_k$ with

$$\Phi(c_k)\int_0^a \omega(t)\mathrm{d}t + M_k = 1. \tag{25}$$

Then there exists a number $p > 0$ such that

$$\Phi\left(u_k\left(1-b\right)/2\right) S\left((\gamma_k-\lambda_k)/2\right) \geq p \tag{26}$$

for all $k \in \mathcal{N}$. Indeed, suppose conversely that $\Phi\left(u_k\left(1-b\right)/2\right) S\left((\gamma_k-\lambda_k)/2\right) \to 0$. But $\Phi \in \Delta_2$. Putting $l = \frac{2}{1-b}$ in inequality (20) and denoting $\beta_b = \frac{1}{k_l}$, $v = \frac{2}{1-b}u$, we get

$\Phi((1-b)v/2) \geq \beta_b \Phi(v)$ for every $v \in \mathcal{R}$. Thus $\Phi(u_k) S((\gamma_k - \lambda_k)/2) \to 0$. Moreover, by (23) and (24), taking into account that ω is nonincreasing and Φ is a convex function we obtain

$$\begin{aligned} M_0 &< \Phi(u_k) \int_a^{\lambda_k} \omega(t) \mathrm{d}t + \Phi(bu_k) \int_{\lambda_k}^{\gamma_k} \omega(t) \mathrm{d}t \\ &\leq \Phi(u_k) S((\gamma_k - \lambda_k)/2) + b\Phi(u_k) S(\gamma_k - \lambda_k) \\ &\leq \Phi(u_k) S((\gamma_k - \lambda_k)/2) + b\Phi(u_k) 2S((\gamma_k - \lambda_k)/2) \to 0, \end{aligned}$$

but this is a contradiction, so (26) is true. For every $k \in \mathcal{N}$ let $\left(G_1^{k,n}, G_2^{k,n}, \ldots, G_{2^n}^{k,n}\right)$, $n = 1, 2, \ldots$ be a partition of the interval $[\lambda_k, \gamma_k]$ constructed in the same way as in the previous part of the proof. Define for every $k \in \mathcal{N}$ an element $x^k \in S(\Lambda_{\omega,\Phi})$ and a sequence $(x_n^k)_{n=1}^{\infty} \in S(\Lambda_{\omega,\Phi})$ by

$$x^k = c_k \chi_{[0,a]} + u_k \chi_{[a,\lambda_k]} + bu_k \chi_{[\lambda_k, \gamma_k]}$$

and

$$x_n^k = c_k \chi_{[0,a]} + \frac{bu_k + u_k}{2} \chi_{[a,\lambda_k]} + \frac{bu_k + u_k}{2} \chi_{E_{1,n}^k} + bu_k \chi_{E_{2,n}^k},$$

where $E_{1,n}^k = \bigcup_{i=1}^{2^{n-1}} G_{2i-1}^{k,n}$, $E_{2,n}^k = \bigcup_{i=1}^{2^{n-1}} G_{2i}^{k,n}$, $(n = 1, 2, \ldots)$. Then, by (26), we get $I_\Phi(x_n^k - x_m^k) \geq p$ for every $k \in \mathcal{N}$ and $n \neq m$. So there is a number $q > 0$ such that $\|x_n^k - x_m^k\|_\Phi \geq q$ for all $k \in \mathcal{N}$ and $n \neq m$. Moreover, by (21), (22), (23) and (25), we get $I_\Phi\left(\left(x^k + x_n^k\right)/2\right) > 1 - 4/k$ for every $n \in \mathcal{N}$. The Δ_2-condition implies that there exists a sequence $(\sigma_k)_{k=1}^{\infty} \subset \mathcal{R}$ with $\lim_{k\to\infty} \sigma_k = 0$ such that $\left\|\left(x_n^k + x^k\right)/2\right\|_\Phi \geq 1 - \sigma_k$, $n = 1, 2, \ldots$ (see [10]). This contradiction shows that the property (β) implies that Φ is uniformly convex on the whole real line. □

For γ finite one can prove in the similar way as Theorem 3 the following:

Theorem 4. *Let $E = \Lambda_\omega$ with $\gamma < \infty$ and assume that Φ does not depend on t. Then the following statements are equivalent:*

(a) $(\Lambda_\omega)_\Phi \in (UC)$
(b) $(\Lambda_\omega)_\Phi \in (\beta)$
(c) *Φ is uniformly convex for large arguments, Φ satisfies the Δ_2-condition for large arguments and the function ω is regular.*

Taking $\omega \equiv 1$ in Theorems 3 and 4 we get the following characterization for Orlicz spaces equipped with the Luxemburg norm over finite or infinite measure space (see [7] for the finite measure space).

COROLLARY 2

Let Φ be an Orlicz function and let L_Φ be the Orlicz function space over the finite or infinite measure space. Then the following statements are equivalent:

(a) $L_\Phi \in$(UC).
(b) $L_\Phi \in(\beta)$.

Using Theorems 3 and 4 for $\Phi(u) = |u|$ we get immediately

COROLLARY 3

Let $\gamma < \infty$ *or* $\gamma = \infty$. *The Lorentz space* Λ_ω *does not have the property* (β).

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On oscillation and asymptotic behaviour of solutions of forced first order neutral differential equations

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Abstract. In this paper, sufficient conditions have been obtained under which every solution of

$$\Big[y(t) \pm y(t-\tau)\Big]' \pm Q(t)\,G(y(t-\sigma)) = f(t), \quad t \geq 0,$$

oscillates or tends to zero or to ±∞ as $t \to \infty$. Usually these conditions are stronger than

$$\int_0^\infty Q(t)\,\mathrm{d}t = \infty. \tag{$*$}$$

An example is given to show that the condition (*) is not enough to arrive at the above conclusion. Existence of a positive (or negative) solution of

$$\Big[y(t) - y(t-\tau)\Big]' + Q(t)\,G(y(t-\sigma)) = f(t)$$

is considered.

Keywords. Oscillation; nonoscillation; neutral equations; asymptotic behaviour.

1. Introduction

In a recent paper [8], the authors have obtained necessary and sufficient conditions so that every solution of

$$[y(t) - p\,y(t-\tau)]' + Q(t)\,G(y(t-\sigma)) = f(t)$$

oscillates or tends to zero as $t \to \infty$ on various ranges of p, where $G \in C(R, R)$, $Q \in C([0,\infty), [0,\infty))$, $f \in C([0,\infty), R)$, $\tau \geq 0$ and $\sigma \geq 0$. They have studied the similar problem in [9] for equations of the form

$$[y(t) - p(t)\,y(t-\tau)]' \pm Q(t)\,G(y(t-\sigma)) = f(t)$$

for different ranges of $p \in C([0,\infty), R)$, where f, G, Q, τ and σ are same as above. In these results, the primary assumption is

$$\int_0^\infty Q(t)\,\mathrm{d}t = \infty. \tag{1}$$

However, these results don't hold good for the critical case $p(t) \equiv 1$ or $p(t) \equiv -1$. In this paper, an attempt is made to study oscillatory and asymptotic behaviour of solutions of equations of the form

$$\left[y(t) \pm y(t-\tau)\right]' \pm Q(t)\, G(y(t-\sigma)) = f(t), \tag{2}$$

where $xG(x) > 0$ for $x \neq 0$ and G is nondecreasing. We assume that

$$\int_0^\infty |f(t)|\,\mathrm{d}t < \infty.$$

In most of our results, the assumptions are stronger than (1). It seems that it is possible to obtain an example of a neutral differential equation in the critical case such that (1) holds but the equation admits a nonoscillatory solution which does not tend to zero as $t \to \infty$. A similar example is obtained in the discrete case by Yu and Wang [13].

Several open problems are stated in [2] (see 6.12.9 and 6.12.10, pp. 161) for equations of the type

$$\left[y(t) \pm y(t-\tau)\right]' + Q(t)\, y(t-\sigma) = 0.$$

In a recent paper [10], Piao has solved one open problem with an extra condition. Indeed, he showed that every nonoscillatory solution of

$$\left[y(t) + y(t-\tau)\right]' + Q(t)\, y(t-\sigma) = 0$$

tends to zero as $t \to \infty$ if (1) holds and $Q(t + \tau/n) \leq Q(t)$ for $t \in [0, \infty)$ where n is any fixed positive integer. However, Ladas and Sficas [6] have shown that every solution of

$$\left[y(t) - y(t-\tau)\right]' + Q(t)\, y(t-\sigma) = 0 \tag{3}$$

oscillates if (1) holds. Chuanxi and Ladas [1] posed the open problem that whether (1) is a necessary condition for the oscillation of all solutions of (3). In other words, whether

$$\int_0^\infty Q(t)\,\mathrm{d}t < \infty$$

implies that (3) admits a nonoscillatory solution. Liu *et al* [7] (see also [11]) gave an example to show that the open problem is not true. They have shown that a stronger condition, viz,

$$\int_0^\infty t\, Q(t)\,\mathrm{d}t < \infty$$

implies that (3) admits a bounded nonoscillatory solution.

By a solution of eq. (2) on $[T, \infty)$, $T \geq 0$, we mean a function $y \in C([T - r, \infty), R)$ such that $y(t) \pm y(t-\tau)$ is continuously differentiable and (2) is satisfied identically for $t \geq T$, where $r = \max\{\tau, \sigma\}$ and T is depending on y. Such a solution of (2) is said to be scillatory if it has arbitrarily large zeros; otherwise, it is called nonoscillatory.

2. Sufficient conditions

In this section we obtain sufficient conditions so that every solution of (2) oscillates or tends to zero or to ± ∞ as $t \to \infty$.

Theorem 2.1. *Suppose that*

$$G(x) + G(y) \geq \alpha G(x + y), \quad x > 0, y > 0$$

and

$$G(x) + G(y) \leq \beta G(x + y), \quad x < 0, y < 0, \tag{H$_1$}$$

where $\alpha > 0$ *and* $\beta > 0$ *are constants. If*

$$\int_{\tau}^{\infty} Q^*(t)\,\mathrm{d}t = \infty, \tag{H$_2$}$$

where $Q^*(t) = \min\{Q(t), Q(t - \tau)\}$, *then every solution of*

$$[y(t) + y(t - \tau)]' + Q(t)\, G(y(t - \sigma)) = f(t) \tag{4}$$

oscillates or tends to zero as $t \to \infty$.

Proof. Let $y(t)$ be a nonoscillatory solution of (4) on $[T_y, \infty)$, $T_y \geq 0$. Hence there exists a $t_0 > T_y$ such that $y(t) > 0$ or < 0 for $t \geq t_0$. Let $y(t) > 0$ for $t \geq t_0$. Setting

$$z(t) = y(t) + y(t - \tau)$$

and

$$w(t) = z(t) - F(t), \quad F(t) = \int_0^t f(s)\,\mathrm{d}s, \tag{5}$$

for $t \geq t_1 > t_0 + r$, we obtain $z(t) > 0$ and

$$w'(t) = -Q(t)G(y(t - \sigma)) \leq 0 \tag{6}$$

for $t \geq t_1$. Hence $w(t) > 0$ or < 0 for $t \geq t_2 > t_1$. If $w(t) > 0$ for $t \geq t_2$, then $\lim_{t\to\infty} w(t)$ exists. If $w(t) < 0$ for $t \geq t_2$, then $0 < y(t) < z(t) < F(t)$ implies that $y(t)$ is bounded and hence $w(t)$ is bounded. Thus $\lim_{t\to\infty} w(t)$ exists. In either case $\lim_{t\to\infty} z(t)$ exists. We claim that $\lim_{t\to\infty} z(t) = 0$. If not, then $z(t) > \lambda > 0$ for $t \geq t_3 > t_2$. From (4) we obtain, for $t \geq t_4 > t_3 + \sigma + \tau$,

$$\begin{aligned}
f(t) + f(t - \tau) &= z'(t) + z'(t - \tau) + Q(t)G(y(t - \sigma)) + Q(t - \tau)G(y(t - \tau - \sigma)) \\
&\geq z'(t) + z'(t - \tau) + Q^*(t)(G(y(t - \sigma)) + G(y(t - \tau - \sigma))) \\
&\geq z'(t) + z'(t - \tau) + \alpha Q^*(t)G(y(t - \sigma) + y(t - \tau - \sigma)) \\
&= z'(t) + z'(t - \tau) + \alpha Q^*(t)G(z(t - \sigma)) \\
&\geq z'(t) + z'(t - \tau) + \alpha Q^*(t)G(\lambda).
\end{aligned}$$

Hence,

$$z(t) < z(t) + z(t-\tau) \leq z(t_4) + z(t_4 - \tau)$$
$$-\alpha\, G(\lambda) \int_{t_4}^{t} Q^*(s)\,\mathrm{d}s + \int_{t_4}^{t} f(s)\,\mathrm{d}s + \int_{t_4}^{t} f(s-\tau)\,\mathrm{d}s$$

implies that $z(t) < 0$ for large t, a contradiction. Hence the claim holds. Consequently, $\lim_{t\to\infty} y(t) = 0$. Similarly, when $y(t) < 0$ for $t \geq t_0$, we obtain $\lim_{t\to\infty} y(t) = 0$. Thus the theorem is proved.

Remark. Clearly, (H_2) implies (1).

Remark. If $G(u) = u^{\gamma}$, where $\gamma > 0$ is a ratio of odd integers, then (H_1) is satisfied due to well-known inequalities

$$\left(|a| + |b|\right)^p \leq |a|^p + |b|^p,\ 0 < p \leq 1,$$
$$\left(|a| + |b|\right)^p \leq 2^{p-1}\left(|a|^p + |b|^p\right),\ p \geq 1,$$

where a and b are any two real numbers. If $G(u) = |u|^{\gamma} \operatorname{sgn} u$, where $\gamma > 0$, then (H_1) is also satisfied.

Remark. Clearly, (1) and $Q\left(t + \frac{\tau}{n}\right) \leq Q(t)$ for $t \in [0, \infty)$, where n is any fixed positive integer, imply (H_2) because $Q(t) \geq Q(t + \tau)$, $t \in [0, \infty)$. Hence Theorem 2.1 may be regarded as an improvement and generalization of the work in [10].

Theorem 2.2. *If* (H_3) *holds, then every solution of* (4) *oscillates or tends to zero as* $t \to \infty$, *where* (H_3) *is stated as follows*:
(H_3) *For every sequence* $\langle\sigma_i\rangle \subset (0, \infty)$, $\sigma_i \to \infty$ *as* $i \to \infty$; *and for every* $\eta > 0$, *such that the intervals* $(\sigma_i - y, \sigma_i + y)$, $i = 1, 2, \ldots$, *and nonoverlapping,*

$$\sum_{i=0}^{\infty} \int_{\sigma_i - \eta}^{\sigma_i + \eta} Q(t)\,\mathrm{d}t = \infty.$$

Proof. If $y(t)$ is a nonoscillatory solution of (4) on $[T_y, \infty)$, $T_y \geq 0$, then $y(t) > 0$ or < 0 for $t \geq T_0 > T_y$. Let $y(t) > 0$, $t \geq T_0$. Setting $z(t)$ and $w(t)$ as in (5) for $t \geq T_1 > T_0 + r$, we obtain (6). Hence $w(t) > 0$ or < 0 for $t \geq T_2 > T_1$. Proceeding as in Theorem 2.1, we show that $\lim_{t\to\infty} w(t)$ and $\lim_{t\to\infty} z(t)$ exist. Since $y(t) < z(t)$, then $\limsup_{t\to\infty} y(t)$ exists. We claim that $\limsup_{t\to\infty} y(t) = 0$. If not, then $\limsup_{t\to\infty} y(t) = \alpha$, $0 < \alpha < \infty$. Hence there exists a sequence $\langle t_n\rangle \subset [T, \infty)$, $T \geq T_2$, such that $t_n \to \infty$ as $n \to \infty$ and $y(t_n) \to \alpha$ as $n \to \infty$. Thus, for large $N_1 > 0$, $y(t_n) > \beta > 0$ if $n \geq N_1$. Since $y(t)$ is continuous at t_n, then there exists $\delta_n > 0$ such that $y(t) > \beta$ for $t \in (t_n - \delta_n,\ t_n + \delta_n)$ and $\liminf_{n\to\infty} \delta_n > 0$. Hence $\delta_n > \delta > 0$ for $n \geq N_2$. Choosing $N = \max\{N_1,\ N_2\}$, we obtain

$$\begin{aligned}
&\int_T^\infty Q(t)\, G(y(t-\sigma))\, \mathrm{d}t \\
&\geq \sum_{n=N}^{\infty} \int_{t_n-\delta_n+\sigma}^{t_n+\delta_n+\sigma} Q(t)\, G(y(t-\sigma))\, \mathrm{d}t \\
&\geq G(\beta) \sum_{n=N}^{\infty} \int_{t_n-\delta_n+\sigma}^{t_n+\delta_n+\sigma} Q(t)\, \mathrm{d}t \\
&\geq G(\beta) \sum_{n=N}^{\infty} \int_{t_n-\delta+\sigma}^{t_n+\delta+\sigma} Q(t))\, \mathrm{d}t
\end{aligned}$$

which implies that

$$\int_T^\infty Q(t)\, G(y(t-\sigma))\, \mathrm{d}t = \infty$$

by (H_3). However, integrating (6) we obtain

$$\int_T^t Q(s)\, G(y(s-\sigma))\, \mathrm{d}s = w(T) - w(t).$$

Thus

$$\int_T^\infty Q(t)\, G(y(t-\sigma))\, \mathrm{d}t < \infty,$$

a contradiction. Hence our claim holds. Consequently, $\lim_{t\to\infty} y(t) = 0$. The proof is similar for $y(t) < 0,\ t \geq T_0$. This completes the proof of the theorem.

Remark. Clearly, (H_3) implies (1). From the following example it is clear that (1) does not imply (H_3).

Remark. Theorem 2.2 holds if we assume that

$$-\infty < \liminf_{t\to\infty} F(t) < \limsup_{t\to\infty} F(t) < \infty$$

instead of

$$\int_0^\infty |f(t)| < \infty,$$

where $F(t)$ is given by (5). In the following we give an example to show that the condition (1) is not enough to arrive at the conclusion of Theorem 2.

Example. Consider

$$[y(t) + y(t-1)]' + Q(t)y(t-1) = h'(t) + h'(t-1),\ t \geq 1,$$

where

$$Q(t) = \left(e^2 + e\right)\left[e^{t+1}h(t-1) + e^2\right]^{-1} > 0,\ t \geq 1,$$

and $h \in C^1\,([0,\infty),\ [0,\infty))$ defined by

$$h(t) = \begin{cases} 0, & t \in [0,1] \\ (t-1)^2(2-t)^2, & t \in [1,2] \end{cases}$$

and extended to ∞ by the periodicity $h(t) = h(t+2)$, $t \geq 0$. Clearly, $y(t) = h(t) + e^{-t}$ is a positive solution of the equation with $\limsup_{t\to\infty} y(t) = \limsup_{t\to\infty} h(t) = \frac{1}{16}$. Further,

$$\int_1^\infty Q(t)\,\mathrm{d}t > \sum_{n=1}^\infty \int_{2n-1}^{2n} Q(t)\,\mathrm{d}t = \sum_{n=1}^\infty \frac{\left(e^2+e\right)}{e^2} = \infty.$$

Thus (1) holds but the equation admits a nonoscillatory solution which does not tend to zero as $t \to \infty$. This suggests that stronger conditions are needed to show that every nonoscillatory solution of (4) tends to zero as $t \to \infty$.

Example. Consider

$$[y(t) + y(t-\pi)]' + (t-\pi)^{-1/2}\, y(t-\pi) = f(t),\ t \geq 2\pi$$

where

$$f(t) = \frac{\cos t}{t^2} + \frac{2\sin t}{(t-\pi)^3} - \frac{2\sin t}{t^3} - \frac{\cos t}{(t-\pi)^2} - \frac{\sin t}{(t-\pi)^{5/2}}.$$

Since

$$Q^*(t) = \min\left\{Q(t), Q(t-\pi)\right\} = \min\left\{\frac{1}{\sqrt{t-\pi}}, \frac{1}{\sqrt{t-2\pi}}\right\} = \frac{1}{\sqrt{t-\pi}},$$

then

$$\int_{2\pi}^\infty Q^*(t)\mathrm{d}t = \infty.$$

From Theorem 2.1 it follows that every solution of the equation oscillates or tends to zero as $t \to \infty$. In particular, $y(t) = \sin t/t^2$ is such a solution of the equation. We may note that Theorem 2.2 fails to hold for this equation because

$$\begin{aligned}\sum_{i=0}^\infty \int_{\sigma_i-\eta}^{\sigma_i+\eta} Q(t)\,\mathrm{d}t &= 2\sum_{i=0}^\infty \left[(\sigma_i+\eta-\pi)^{1/2} - (\sigma_i-\eta-\pi)^{1/2}\right] \\ &= 4\eta\sum_{i=0}^\infty \left[(\sigma_i+\eta-\pi)^{1/2} + (\sigma_i-\eta-\pi)^{1/2}\right]^{-1} \\ &< \infty\end{aligned}$$

for a sequence $\langle \sigma_i \rangle \equiv \langle i^4 \rangle \subset [2\pi, \infty)$.

Example. Consider

$$\left[y(t) + y(t-1)\right]' + y(t-1)\exp(y(t-1)) = f(t),\ t \geq 1,$$

where

$$f(t) = -e^{-t} - e^{-t+1} + e^{-t+1}\, e^{e^{-t+1}},\, G(u) = ue^u.$$

Since $Q(t) \equiv 1$, then (H_3) holds trivially. Thus every nonoscillatory solution of the equation tends to zero as $t \to \infty$ by Theorem 2.2. In particular, $y(t) = e^{-t}$ is such a solution. However, Theorem 2.1 cannot be applied to this equation because

$$G(u+v) = (u+v)e^{u+v} > ue^u + ve^v = G(u) + G(v)$$

for $u > 0$ and $v > 0$ and hence (H_1) fails to hold.

Theorem 2.3. *Every unbounded solution of* (4) *oscillates. In other words, every nonoscillatory solution of* (4) *is bounded.*

Proof. Let $y(t)$ be an unbounded nonoscillatory solution of (4). Let $y(t) > 0$ for $t \geq t_0 > 0$. The case $y(t) < 0$ for $t \geq t_0 > 0$ may be dealt with similarly. Setting $z(t)$ and $w(t)$ as in (5) we obtain (6). If $w(t) > 0$ for large t, then $z(t)$ is bounded and hence $y(t)$ is bounded, a contradiction. If $w(t) < 0$ for large t and is bounded, then $z(t)$ is bounded and hence $y(t)$ is bounded, a contradiction. Thus $w(t) < 0$ for large t is unbounded. Consequently, $\lim\limits_{t\to\infty} w(t) = -\infty$ which implies that $z(t) < 0$ for large t, a contradiction. Hence the theorem is proved.

Theorem 2.4. *If* (1) *holds, then every solution of*

$$\left[y(t) - y(t-\tau)\right]' + Q(t)G(y(t-\sigma)) = 0 \tag{7}$$

oscillates.

Proof. If possible, let $y(t)$ be a nonoscillatory solution of (7) on $[T_y, \infty)$. Without any loss of generality, we may assume that $y(t) > 0$ for $t \geq t_0 > T_y$. Setting $z(t) = y(t) - y(t-\tau)$ for $t \geq t_1 > t_0 + r$, we obtain

$$z'(t) = -Q(t)G(y(t-\sigma)) \leq 0.$$

Hence $z(t) > 0$ or < 0 for $t \geq t_2 > t_1$. If $z(t) > 0,\ t \geq t_2$, then

$$\int\limits_{t_2}^{\infty} Q(t)G(y(t-\sigma))\,\mathrm{d}t < z(t_2) < \infty.$$

On the other hand, $z(t) > 0$ for $t \geq t_2$ implies that $y(t) > y(t-\tau)$ and hence $\liminf\limits_{t\to\infty} y(t) > 0$. Thus $y(t) > \alpha > 0$ for $t \geq t_3 > t_2$. Then

$$\int\limits_{t_3+\sigma}^{\infty} Q(t)G(y(t-\sigma))\,\mathrm{d}t > G(\alpha) \int\limits_{t_3+\sigma}^{\infty} Q(t)\,\mathrm{d}t$$

implies that

$$\int_{t_3+\sigma}^{\infty} Q(t)G(y(t-\sigma))\,\mathrm{d}t = \infty,$$

a contradiction. Therefore, $z(t) < 0$ for $t \geq t_2$, that is, $y(t) < y(t-\tau),\ t \geq t_2$. Then $y(t)$ is bounded and hence $\liminf_{t\to\infty} y(t)$ and $\lim_{t\to\infty} z(t)$ exist. From Lemma 1.5.1 of [2] it follows that $\lim_{t\to\infty} z(t) = 0$, a contradiction because $z(t) < 0$ and monotonic decreasing. Hence the theorem is proved.

Remark. Theorem 2.4 generalizes Theorem 6.4.1 due to Gyori and Ladas [2].

Remark. In [7], an example is given to show that the condition (1) is not necessary for oscillation of all solutions of (7). They have proved that every bounded solution of (7) oscillates if and only if

$$\int_{0}^{\infty} t\,Q(t)\,\mathrm{d}t = \infty. \tag{H$_4$}$$

We may note that (1) is stronger than (H_4).

Theorem 2.5. *If* (H_3) *holds, then every solution of*

$$\left[y(t) - y(t-\tau)\right]' + Q(t)G(y(t-\sigma)) = f(t) \tag{8}$$

oscillates or tends to zero as $t \to \infty$.

Proof. Let $y(t)$ be a solution of (8) on $[T_y, \infty),\ T_y \geq 0$. If $y(t)$ oscillates, then there is nothing to prove. Let $y(t)$ be nonoscillatory. Hence $y(t) > 0$ or < 0 for $t \geq T_0 > T_y$. Let $y(t) > 0$ for $t \geq T_0$. Setting

$$z(t) = y(t) - y(t-\tau)$$

and

$$w(t) = z(t) - F(t),\ F(t) = \int_{0}^{t} f(s)\,\mathrm{d}s,$$

for $t \geq T_1 > T_0 + r$, we obtain

$$w'(t) = -Q(t)\,G(y(t-\sigma)) \leq 0.$$

If $w(t) > 0$ for $t \geq T_2 > T_1$, then $\lim_{t\to\infty} w(t)$ exists. If $w(t) < 0$ for $t \geq T_2$ is unbounded, then $\lim_{t\to\infty} w(t) = -\infty$ and hence $z(t) < 0$ for large t, that is, $y(t) < y(t-\tau)$ for large t. Thus $y(t)$ is bounded, which implies that $w(t)$ is bounded, a contradiction. Hence $w(t) < 0$ for $t \geq T_2$ is bounded. Then $\lim_{t\to\infty} w(t)$ exists. We claim that $\limsup_{t\to\infty} y(t) = 0$. If not, then $\limsup_{t\to\infty} y(t) = \alpha,\ 0 < \alpha \leq \infty$. There exists a sequence $\langle t_n\rangle \subset [T_2, \infty)$ such that

$t_n \to \infty$ and $y(t_n) \to \alpha$ as $n \to \infty$. Hence $y(t_n) > \beta > 0$ for $n \geq N_1 > 0$. Since $y(t)$ is continuous at t_n, there exists $\delta_n > 0$ such that $y(t) > \beta$ for $t \in (t_n - \delta_n, t_n + \delta_n)$ and $\liminf_{n\to\infty} \delta_n > 0$. Then $\delta_n > \delta > 0$ for $n \geq N_2 > 0$. Choosing $N = \max\{N_1, N_2\}$ and then proceeding as in the proof of Theorem 2.2, we arrive at a contradiction due to (H_3). Hence our claim holds. Thus $\lim_{t\to\infty} y(t) = 0$. Similarly, we may show that $\lim_{t\to\infty} y(t) = 0$ when $y(t) < 0$ for $t \geq T_0$. This completes the proof of the theorem.

Theorem 2.6. *Suppose that* (H_1) *and* (H_2) *hold. If* $y(t)$ *is a solution of*

$$\left[y(t) + y(t-\tau)\right]' - Q(t)G(y(t-\sigma)) = f(t), \tag{9}$$

then $y(t)$ *oscillates or tends to zero as* $t \to \infty$ *or* $\limsup_{t\to\infty} |y(t)| = +\infty$.

Proof. If possible, let $y(t)$ be nonoscillatory. Hence there exists $t_0 > 0$ such that $y(t) > 0$ or < 0 for $t \geq t_0$. Let $y(t) > 0$ for $t \geq t_0$. Setting $z(t)$ and $w(t)$ as in (5), we obtain $z(t) > 0$ and

$$w'(t) = Q(t)G(y(t-\sigma)) \geq 0$$

for $t \geq t_1 > t_0 + r$. If $w(t) < 0$ for $t \geq t_2 > t_1$, then $\lim_{t\to\infty} w(t)$ exists and hence $\lim_{t\to\infty} z(t)$ exists. If $w(t) > 0$ for $t \geq t_2$ is bounded, then $\lim_{t\to\infty} w(t)$ and $\lim_{t\to\infty} z(t)$ exist. We claim that $\lim_{t\to\infty} z(t) = 0$. If not, then $z(t) > \lambda > 0$ for $t \geq t_3 > t_2$. Using (9) and (H_1) we may write, for $t \geq t_4 > t_3 + r$,

$$\begin{aligned} f(t) + f(t-\tau) &\leq z'(t) + z'(t-\tau) - Q^*(t)(G(y(t-\sigma)) + G(y(t-\tau-\sigma))) \\ &\leq z'(t) + z'(t-\tau) - \alpha Q^*(t)G(z(t-\sigma)) \\ &\leq z'(t) + z'(t-\tau) - \alpha G(\lambda)Q^*(t). \end{aligned}$$

This implies, due to (H_2), that $\lim_{t\to\infty} z(t) = \infty$, a contradiction. Hence our claim holds. Since $z(t) > y(t)$, then $\lim_{t\to\infty} y(t) = 0$. If $w(t) > 0$, $t \geq t_2$, is unbounded, then $\lim_{t\to\infty} w(t) = +\infty$. Hence $y(t)$ is unbounded. Similarly, if $y(t) < 0$ for $t \geq t_0$, then $\lim_{t\to\infty} y(t) = 0$ or $y(t)$ is unbounded. Thus $\lim_{t\to\infty} y(t) = 0$ or $\limsup_{t\to\infty} |y(t)| = +\infty$. This completes the proof of the theorem.

COROLLARY 2.7

If (H_1) *and* (H_2) *hold, then every bounded solution of* (9) *oscillates or tends to zero as* $t \to \infty$.

This follow from Theorem 2.6.

Theorem 2.8. *Let* (H_3) *hold. If* $y(t)$ *is a solution of* (9), *then it oscillates or tends to zero as* $t \to \infty$ *or* $\limsup_{t\to\infty} |y(t)| = +\infty$.

The proof is similar to that of Theorem 2.2.

COROLLARY 2.9

If (H_3) *holds, then every bounded solution of* (9) *oscillates or tends to zero as* $t \to \infty$.

Theorem 2.10. *Suppose that* (H$_3$) *holds. If $y(t)$ is a solution of*

$$[y(t) - y(t-\tau)]' - Q(t)\, G(y(t-\sigma)) = f(t), \tag{10}$$

then $y(t)$ oscillates or tends to zero or $|y(t)| \to +\infty$ as $t \to \infty$.

The proof is similar to that of Theorem 2.5.

COROLLARY 2.11

If (H$_3$) *holds, then every bounded solution of* (10) *oscillates or tends to zero as $t \to \infty$.*

Remark. Some of our results partially answer the open problems stated in 6.12.9 and 6.12.10 [2].

3. Existence of nonoscillatory solutions

In this section we obtain necessary and sufficient conditions for the existence of a bounded positive/negative solution of the eq. (8).

Theorem 3.1. *Let $f(t) \geq 0$ with*

$$\sum_{k=0}^{\infty} \int_{k\tau}^{\infty} f(t)\,\mathrm{d}t < \infty. \tag{11}$$

Then eq. (8) *admits a bounded negative solution if and only if*

$$\sum_{k=0}^{\infty} \int_{k\tau}^{\infty} Q(t)\,\mathrm{d}t < \infty. \tag{H$_5$}$$

Proof. Suppose that eq. (8) admits a bounded negative solution $y(t)$ on $[T_y, \infty)$, $T_y \geq 0$. Setting $z(t) = y(t) - y(t-\tau)$ and $w(t) = z(t) + F(t)$, where

$$F(t) = \int_t^{\infty} f(s)\,\mathrm{d}s,$$

for $t \geq t_0 > T_y + r$, we obtain $w(t) > z(t)$, $w(t)$ is bounded, $F(t) \to 0$ as $t \to \infty$ and

$$w'(t) = -Q(t)\, G(y(t-\sigma)) \geq 0. \tag{12}$$

Hence $w(t) > 0$ or < 0 for $t \geq t_1 > t_0$. If $w(t) > 0$ for $t \geq t_1$, then $\lim_{t\to\infty} w(t)$ exists and hence $\lim_{t\to\infty} z(t)$ exists. Since $\liminf_{t\to\infty} y(t)$ $\left(\text{or } \limsup_{t\to\infty} y(t)\right)$ exists, then $\lim_{t\to\infty} z(t) = 0$ by Lemma 1.5.1 in [2]. Thus $\lim_{t\to\infty} w(t) = 0$, a contradiction to the fact that $w(t) > 0$ and nondecreasing. Hence $w(t) < 0$ for $t \geq t_1$. Consequently, $\lim_{t\to\infty} w(t)$ exists and $z(t) < 0$ for $t \geq t_1$. There exists $\alpha > 0$ such that $y(t) < -\alpha$ for $t \geq t_1$. Integrating (12) from s to t $(s > t \geq t_2 \geq t_1 + \sigma)$ and then taking limit as $s \to \infty$ we obtain

$$w(t) \leq G(-\alpha) \int_t^{\infty} Q(u)\,\mathrm{d}u,$$

that is,

$$y(t-\tau) > y(t) + \int_t^\infty f(s)\,\mathrm{d}s - G(-\alpha)\int_t^\infty Q(s)\,\mathrm{d}s.$$

Putting the values of t successively one may obtain

$$y(t-\tau) > y(t+n\tau) + \sum_{k=0}^{n}\int_{t+k\tau}^{\infty} f(s)\,\mathrm{d}s - G(-\alpha)\sum_{k=0}^{n}\int_{t+k\tau}^{\infty} Q(s)\,\mathrm{d}s.$$

Since $y(t)$ is bounded, then using (11) we get

$$\sum_{k=0}^{\infty}\int_{t_2+k\tau}^{\infty} Q(s)\,\mathrm{d}s < \infty.$$

From this (H_5) follows.

Next we assume that (H_5) holds. It is possible to choose $m > 0$ sufficiently large such that

$$\sum_{k=m}^{\infty}\int_{k\tau}^{\infty} Q(t)\,\mathrm{d}t < -\frac{1}{2G(-1)}$$

and

$$\sum_{k=m}^{\infty}\int_{k\tau}^{\infty} f(t)\,\mathrm{d}t < \frac{1}{2},$$

that is,

$$\sum_{k=0}^{\infty}\int_{T+k\tau}^{\infty} Q(t)\,\mathrm{d}t < \frac{-1}{2G(-1)} \text{ and } \sum_{k=0}^{\infty}\int_{T+k\tau}^{\infty} f(t)\,\mathrm{d}t < \frac{1}{2},$$

where $T = m\tau$. Define

$$L(t) = \begin{cases} 0, & 0 \le t < T \\ G(-1)\displaystyle\int_t^\infty Q(s)\,\mathrm{d}s - \int_t^\infty f(s)\,\mathrm{d}s, & t \ge T. \end{cases}$$

Hence $L(t) < 0$ for $t \ge T$. Further, define

$$u(t) = \begin{cases} 0, & 0 \le t < T \\ \displaystyle\sum_{i=0}^{\infty} L(t-i\tau), & t \ge T. \end{cases}$$

Thus $u(t) < 0$ and $u(t) - u(t-\tau) = L(t), t \ge T$. For $t \ge T$, there exists an integer $k \ge 0$ such that $T + k\tau \le t < T + (k+1)\tau$. Hence $T \le t - k\tau < T + \tau$ and $T - \tau \le t - (k+1)\tau < T$. Then

$$\begin{aligned}u(t) &= L(t)+L(t-\tau)+\cdots+L(t-k\tau)\\ &= G(-1)\int_t^\infty Q(s)\mathrm{d}s-\int_t^\infty f(s)\mathrm{d}s+\cdots+G(-1)\int_{t-k\tau}^\infty Q(s)\mathrm{d}s-\int_{t-k\tau}^\infty f(s)\mathrm{d}s\\ &\geq G(-1)\int_{T+k\tau}^\infty Q(s)\mathrm{d}s-\int_{T+k\tau}^\infty f(s)\mathrm{d}s+\cdots+G(-1)\int_T^\infty Q(s)\mathrm{d}s-\int_T^\infty f(s)\mathrm{d}s\\ &\geq G(-1)\sum_{k=0}^\infty\int_{T+k\tau}^\infty Q(t)\mathrm{d}t-\sum_{k=0}^\infty\int_{T+k\tau}^\infty f(t)\mathrm{d}t\\ &\geq -1.\end{aligned}$$

Let $X = BC([T,\infty), R)$, the space of all real-valued, bounded continuous functions on $[T,\infty)$. It is a Banach space with respect to supremum norm. Let $K = \{x \in X : x(t) \geq 0, t \geq T\}$. For $u, v \in X$, we define $u \leq v$ if and only if $v - u \in K$. Thus X is a partially ordered Banach space (see pp. 30, [2]). Define

$$M = \{x \in X : u(t) \leq x(t) \leq 0\}.$$

Clearly, $u \in M$ and $u = \inf M$. If $\phi \subset A \subseteq M$, then $A = \{x \in M : u(t) \leq v(t) \leq x(t) \leq w(t) \leq 0\}$. Setting $w_0(t) = \sup\{w(t) : x(t) \leq w(t) \leq 0, x \in A\}$, we notice that $w_0 = \sup A$ and $w_0 \in M$. Define $S : M \to X$ by

$$S\,x(t) = \begin{cases} x(t-\tau)+\displaystyle\int_t^\infty Q(s)G(x(s-\sigma))\mathrm{d}s-\int_t^\infty f(s)\mathrm{d}s, & t\geq T_1\\[2ex] \dfrac{t\,u(t)}{T_1\,u(T_1)}S\,x(T_1)+\left(1-\dfrac{t}{T_1}\right)u(t), & T\leq t\leq T_1\end{cases}, \tag{13}$$

where $T_1 = T + r$ and $r = \max\{\tau, \sigma\}$. Clearly, Sx is continuous on $[T,\infty)$ and $S\,x(t) \leq 0$ for $t \geq T$. For $t \geq T_1$,

$$\begin{aligned}S\,x(t) &\geq x(t-\tau)+G(-1)\int_t^\infty Q(s)\mathrm{d}s-\int_t^\infty f(s)\mathrm{d}s\\ &\geq u(t-\tau)+L(t)=u(t).\end{aligned}$$

For $T \leq t \leq T_1$,

$$S\,x(t) \geq \frac{t}{T_1}u(t)+\left(1-\frac{t}{T_1}\right)u(t)=u(t).$$

Thus $S : M \to M$. Moreover, $x_1 \geq x_2$ implies that $S\,x_1 \geq S\,x_2$. From the Knaster–Tarski fixed-point theorem (see pp. 30, [2]) it follows that S has a fixed point $y \in M$ which is a solution of (8) on $[T_1,\infty)$. Since $y(T_1-\tau) \leq 0$, then from (13) it follows that

$$y(T_1) \leq y(T_1-\tau)-\int_{T_1}^\infty f(s)\mathrm{d}s < 0.$$

Thus $y(t) < 0$ for $t \in [T, T_1]$. For $t \in [T_1, T_1 + \tau]$, $y(t) < 0$. Consequently, $y(t) < 0$ for $t \geq T_1$. This completes the proof of the theorem.

Theorem 3.2. *Let* $f(t) \leq 0$ *with*

$$\sum_{k=0}^{\infty} \int_{K\tau}^{\infty} f(t)\mathrm{d}t > -\infty.$$

Then eq. (8) *admits a bounded positive solution if and only if* (H_5) *holds.*

The proof is similar to that of Theorem 3.1.

Remark. Theorems 3.1 and 3.2 hold if $f(t) \equiv 0$. Hence, we have the following corollary.

COROLLARY 3.3

Every bounded solution of (7) *oscillates if and only if*

$$\sum_{k=0}^{\infty} \int_{k\tau}^{\infty} Q(t)\mathrm{d}t = \infty.$$

This follows from Theorems 3.1 *and* 3.2.

Remark. We may note that

$$\sum_{k=0}^{\infty} \int_{k\tau}^{\infty} |f(t)|\,\mathrm{d}t < \infty \text{ implies that } \int_{0}^{\infty} |f(t)|\,\mathrm{d}t < \infty$$

and

$$\sum_{k=0}^{\infty} \int_{k\tau}^{\infty} Q(t)\,\mathrm{d}t < \infty \text{ implies that } \int_{0}^{\infty} Q(t)\,\mathrm{d}t < \infty.$$

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Monotone iterative technique for impulsive delay differential equations

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Abstract. In this paper, by proving a new comparison result, we present a result on the existence of extremal solutions for nonlinear impulsive delay differential equations.

Keywords. Contraction mapping theorem; extremal solutions; impulsive delay differential equations.

1. Introduction

In this paper, we discuss the impulsive retarded functional differential equation (IRFDE)

$$\begin{cases} x' = f(t, x_t), & t \in [0, T], t \neq t_k; \\ \Delta x|_{t=t_k} = I_k(x(t_k)), & k = 1, 2, \ldots, m; \\ x_0 = \Phi, \end{cases} \tag{1.1}$$

where $\Phi \in PC([-\tau, 0], R) = \{x, x$ is a mapping from $[\tau, 0]$ into $R, x(t^-) = x(t)$ for all $t \in (-\tau, 0], x(t^+)$ exists for all $t \in [-\tau, 0)$, and $x(t^+) = x(t)$ for all but at most a finite number of points $t \in [-\tau, 0)\}$ and $M([-\tau, 0], R) = \{x, x$ is a bounded and measurable function from $[-\tau, 0]$ into $R\}$ with norm $\|x\| = \sup_{t\in[-\tau,0]} |x(t)|, \tau > 0, x_t(\theta) = x(t + \theta), \theta \in [-\tau, 0], 0 = t_0 < t_1 < t_2 < \cdots < t_m < T, J = [0, T], J' = J - \{t_i\}_{i=1}^m$. It is easy to see that $PC_0([-\tau, 0], R) \subseteq M([-\tau, 0], R)$ and $M([-\tau, 0], R)$ is a Banach space. Now we suppose that $f \in C(J \times M([-\tau, 0], R), R), I_k \in C(R, R) (k = 1, 2, \ldots, m)$ throughout this paper.

In [1] and [2], some existence and uniqueness results were obtained for eq. (1.1) by the Tonelli's method or fixed point theorems. And it is well-known that the method of upper and lower solutions and its associated monotone iteration is powerful technique for establishing existence-comparison for differential equations (see [4, 5, 6]). But to impulsive differential equations with delay as eq. (1.1), this method has not been used yet as far as we know. In this paper, we discuss eq. (1.1) by the method and we can find that the delay and impulses make the discussions more difficult.

2. Main results

Assume $M([-\tau, T], R) = \{x, x$ is a bounded and measurable function from $[-\tau, T]$ into $R\}$ with norm $\|x\| = \sup_{t\in[-\tau,T]} |x(t)|$, $PC_0([-\tau, T], R) = \{x, x$ is a mapping from $[\tau, 0]$ into $R, x(t^-) = x(t)$ for all $t \in (-\tau, 0], x(t^+)$ exists for all $t \in [-\tau, 0), x(t^+) = x(t)$

for all but at most a finite number of points $t \in [-\tau, 0)$, and $x(t)$ is continuous at $t \in [0, T] - \{t_i\}_{i=1}^m$ left continuous at $t = t_k$, and $x(t_k^+)$ exists $(k = 1, 2, \ldots, m)\}$.

DEFINITION 2.1

A function $x \in PC_0([-\tau, T], R)$ is said to be a solution of (1.1) if x satisfies the first expression of eq. (1.1) for all $t \in J$ except on a set of Lebesgue measure zero (the exceptional points will generally include but may not be limited to impulse times t_k) and satisfies the second one of eq. (1.1) for all $t \in \{t_k\}_{k=1}^m$, and x is piecewise absolutely continuous on $[0, T]$ with $x_0 = \Phi$.

DEFINITION 2.2

A function $G : M([-\tau, 0], R) \to R$ is said to be weakly continuous at $\phi_0 \in M([-\tau, 0], R)$ if for any $\{\phi_n\} \subseteq M([-\tau, 0], R)$ with $\lim_{n \to +\infty} \phi_n(s) = \phi_0(s)$, a.e. $s \in [-\tau, 0]$, then

$$\lim_{n \to +\infty} G(\phi_n) = G(\phi_0).$$

And G is said to be weakly continuous on $M([-\tau, 0], R)$ if G is weakly continuous at ϕ for any $\phi \in M([-\tau, 0], R)$.

Remark 2.1. This condition is more direct than that in [1] and is different from that in [2], which need that $f(t, \psi)$ is continuous at each $(t, \psi_0) \in (0, T] \times L^1([-r, 0], R^n)$.

Lemma 2.1. *Assume that a function $g : J \times M([-\tau, 0], R) \to R$ is continuous at every $t \in J$ for each fixed $\phi \in M([-\tau, 0], R)$ and is weakly continuous at every $\phi \in M([-\tau, 0], R)$ for each fixed $t \in J$. Then for every $x \in PC([-\tau, T], R)$, $g(t, x_t)$ is measurable on $[0, T]$.*

Proof. Choose a continuous function sequence $\{x_n\}$ such that

$$\lim_{n \to +\infty} x_n(t) = x(t), \quad \text{for all } t \in [-\tau, T].$$

By Lemma 4 in [3], x_{nt} is continuous at $t \in [0, T]$. So $g(t, x_{nt})$ is measurable on $[0, T]$. Since $\lim_{n \to +\infty} x_{nt}(s) = x_t(s)$, for all $s \in [-\tau, 0]$, then

$$\lim_{n \to +\infty} g(t, x_{nt}) = g(t, x_t), \quad \text{for all } t \in [0, T].$$

So $g(t, x_t)$ is measurable on $[0, T]$. □

Set $B \in M([-\tau, 0], R)^*$. Moreover, suppose that there exists a $\gamma \in L^1([-\tau, 0], R)$ with $\gamma(t) \geq 0$ for all most $t \in [-\tau, 0]$ such that

$$B\psi = \int_{-\tau}^{0} \psi(t)\gamma(t)\mathrm{d}t$$

for all $\psi \in M([-\tau, 0], R)$ and $\|B\| = \int_{-\tau}^{0} \gamma(t)\mathrm{d}t$.

Now we list a main lemma.

Lemma 2.2 (*Comparison result*). *Assume that* $p \in PC([-\tau, T], R) \cap C^1(J', R)$ *satisfies*

$$\begin{cases} p' \le -Mp(t) - Bp_t, & t \in J, t \ne t_k \\ \Delta p|_{t=t_k} \le -L_k p(t_k), & (k = 1, 2, \ldots, m) \end{cases}, \tag{2.1}$$

where constants $M \ge 0$, $0 \le L_k \le 1$ $(k = 1, 2, \ldots, m)$ *and* $M_0 = \int_{-\tau}^{0} \mathrm{e}^{-Mt}\gamma(t)\mathrm{d}t$. *And suppose further that*

(a) *either* $p(0) \le p_0(s) \le 0$, $s \in [-\tau, 0]$ *and*

$$M_0\Delta_1 \le \frac{\Pi_{k=1}^{m}(1 - L_k)}{1 + \sum_{j=1}^{m} \Pi_{k=1}^{j}(1 - L_k)}, \tag{2.2}$$

where $\Delta_1 = \max\{t_1, t_2 - t_1, \ldots, T - t_m\}$; *or*

(b) $p(0) \ge -\lambda$, $p_0 \in PC_0([-\tau, 0], R) \cap C^1(I', R)$ *where* $I' = [-\tau, 0] - \{t_l\}_{l=-r}^{-1}$, $\{t_l\}_{l=-r}^{-1}$ *is the set of the discontinuous points of* P_0, $p'(t) \le M_0\lambda$,

$$p(t_{-i}^{+}) - p(t_{-i}) \le -L_{-i}p(t_{-i}), \tag{2.3}$$

$\inf_{s \in [-\tau, 0]} p(s) = -\lambda < 0$ *and*

$$M_0\Delta_2 \le \frac{\Pi_{k=-r}^{m}(1 - L_k)}{1 + \sum_{j=-r}^{m} \Pi_{k=j}^{m}(1 - L_k)}, \tag{2.4}$$

where $\Delta_2 = \max\{t_{-r} + \tau, t_{-r+1} - t_r, \ldots, -t_{-1}, t_1, t_2 - t_1, \ldots, T - t_m\}$. *Then* $p(t) \le 0$ *for a.e.* $t \in J$.

Proof. Now let $v(t) = \mathrm{e}^{Mt}u(t), t \in [-\tau, 0]$. By the definition of B, the eq. (2.1) can be listed as

$$\begin{cases} v'(t) \le -\int_{t-\tau}^{t} \mathrm{e}^{M(t-s)}v(s)\gamma(s - t)\mathrm{d}s, & t \in J, t \ne t_k, \\ \Delta v|_{t=t_k} \le -L_k v(t_k), & (k = 1, 2, \ldots, m). \end{cases} \tag{2.5}$$

Now we will prove $v(t) \le \theta, t \in [-\tau, T]$.

In fact, if there exists a $0 < t^*$ with $v(t^*) > 0$, we might well suppose $t^* \ne t_1, t_2, \ldots, t_m$ (otherwise, we can choose a $\bar{t}$ nearing t^* enough with $v(\bar{t}) > 0$), let

$$\inf_{-\tau \le t \le t^*} v(t) = -b. \tag{2.6}$$

First we consider the case (a).

(A) *In case of* $b = 0$: $v(t) \ge 0, t \in [0, t^*]$. Then $v'(t) \le 0, t \in [0, t^*]$. So $v'(t^*) \le 0$. This is a contradiction.

(B) *In case of* $b > 0$: Assume $t^* \in (t_i, t_{i+1}]$. It is clear that there exists a $0 \le t_* < t^*$ with $v(t_*) = -b$, where t_* in some $J_j (j \le i)$ or $v(t_j^+) = -b$. We may assume that $v(t_*) = -b$ (in case of $v(t_j^+) = -b$, the proof is similar). By mean value theorem, we have

$$\begin{cases} v(t^*) - v(t_i^+) = v'(\zeta_i)(t^* - t_i), & t_i < \zeta_i < t^*; \\ v(t_i) - v(t_{i-1}^+) = v'(\zeta_{i-1})(t_i - t_{i-1}), & t_{i-1} < \zeta_{i-1} < t_i; \\ \ldots, & \ldots \\ v(t_{j+2}) - v(t_{j+1}^+) = v'(\zeta_{j+1})(t_{j+2} - t_{j+1}), & t_{j+1} < \zeta_{j+1} < t_{j+2}; \\ v(t_{j+1}) - v(t_*) = v'(\zeta_*)(t_{j+1} - t_*), & t_* < \zeta_* < t_{j+1}. \end{cases}$$

On the other hand, for $t \in [0, t^*]$

$$v'(t) \leq -\int_{t-\tau}^{t} e^{M(t-s)} v(s)\gamma(s-t)ds \leq bM_0. \tag{2.7}$$

Now from (2.1), we get

$$v(t_k^+) \leq (1 - L_k)v(t_k), \ (k = 1, 2\ldots, m),$$

and

$$\begin{cases} v(t^*) - (1 - L_i)v(t_i) \leq bM_0\Delta_1, \\ v(t_i) - (1 - L_{i-1})v(t_{i-1}) \leq bM_0\Delta_1, \\ \ldots, \ldots \\ v(t_{j+2}) - (1 - L_{j+1})v(t_{j+1}) \leq bM_0\Delta_1, \\ v(t_{j+1}) + b \leq bM_0\Delta_1, \end{cases} \tag{2.8}$$

which implies

$$0 < v(t^*) \leq -b\Pi_{k=j+1}^{i}(1 - L_k) + bM_0\Delta_1 \left\{ 1 + \sum_{l=j+1}^{i} \Pi_{k=l}^{i}(1 - L_k) \right\}.$$

Moreover,

$$\begin{aligned} M_0\Delta_1 &> \frac{\Pi_{k=j+1}^{i}(1 - L_k)}{1 + \sum_{l=j+1}^{i} \Pi_{k=l}^{i}(1 - L_k)} \\ &\geq \frac{\Pi_{k=j+1}^{m}(1 - L_k)}{\Pi_{k=i+1}^{m} + \sum_{l=j+1}^{i} \Pi_{k=l}^{m}(1 - L_k)} \\ &\geq \frac{\Pi_{k=1}^{m}(1 - L_k)}{1 + \sum_{l=1}^{m} \Pi_{k=l}^{m}(1 - L_k)}, \end{aligned}$$

which contradicts (2.2).

By virtue of (A) and (B), $v(t) \leq 0, t \in J$.

Next we consider the case (b).

(A′) If $-b = \inf_{t\in[0,t^*]} v(t)$, we can obtain a contraction similarly as (a).

(B′) If $-b < \inf_{t\in[0,t^*]} v(t)$, then $b = \lambda$ and there exists a $t_* \in (t_{-j-1}, t_{-j}]$ with $v(t_*) = -b$ (or $v(t_{-j-1}^+) = -b$, the proof is similar). So

$$\begin{cases} v(t^*) - v(t_i^+) = v'(\zeta_i)(t^* - t_i), & t^i < \zeta_i < t^*; \\ v(t_i) - v(t_{i-1}^+) = v'(\zeta_{i-1})(t_i - t_{i-1}), & t_{i-1} < \zeta_{i-1} < t_i; \\ \dots, & \dots \\ v(t_1) - v(t_{-1}^+) = v'(\zeta_{-1})(t_1 - t_{-1}), & t_{-1} < \zeta_{-1} < t_1, \\ v(t_{-1}) - v(t_{-2}^+) = v'(\zeta_{-2})(t_{-1} - t_{-2}), & t_{-2} < \zeta_{-2} < t_{-1}, \\ \dots, & \dots; \\ v(t_{-j+1}) - v(t_{-j}^+) = v'(\zeta_{-j})(t_{-j+1} - t_{-j}), & t_{-j} < \zeta_{-j} < t_{-j+1}, \\ v(t_{-j}^1) - v(t_*) = v'(\zeta_*)(t_{-j} - t_*), & t_* < \zeta_* < t_{-j}. \end{cases} \tag{2.9}$$

By (2.9) and (2.3), one has

$$\begin{cases} v(t^*) - (1 - L_i)v(t_i) \le bM_0\Delta_2, \\ v(t_i) - (1 - L_{i-1})v(t_{i-1}) \le bM_0\Delta_2, \\ \dots, \dots \\ v(t_1) - (1 - L_{-1})v(t_{-1}) \le bM_0\Delta_2, \\ v(t_{-1}) - (1 - L_{-2})v(t_{-2}) \le bM_0\Delta_2, \\ \dots, \dots \\ v(t_{-j+1}) - (1 - L_{-j})v(t_{-j}) \le bM_0\Delta_2, \\ v(t_{-j}) + b \le bM_0\Delta_2, \end{cases}$$

which implies

$$\begin{aligned} 0 &< v(t^*) \\ &< -b\Pi_{k=-j}^{i}(1 - L_k) + bM_0\Delta_2 \left\{ 1 + \sum_{l=-j}^{i} \Pi_{k=l}^{i}(1 - L_k) \right\}. \end{aligned}$$

Similarly we get

$$\begin{aligned} M_0\Delta_2 &> \frac{\Pi_{k=-j}^{i}(1 - L_k)}{1 + \sum_{l=-j}^{i} \Pi_{k=l}^{i}(1 - L_k)} \\ &\ge \frac{\Pi_{k=-j}^{m}(1 - L_k)}{\Pi_{k=-j}^{m} + \sum_{l=-j}^{i} \Pi_{k=l}^{m}(1 - L_k)} \\ &\ge \frac{\Pi_{k=-r}^{m}(1 - L_k)}{1 + \sum_{l=-r}^{m} \Pi_{k=l}^{m}(1 - L_k)}, \end{aligned}$$

which contradicts (2.4).

By virtue of (A′) and (B′), $v'(t) \le 0$, a.e. $t \in J$. And the proof is complete. □

Lemma 2.3. *Let* $\sigma, \eta \in M([-\tau, T], R)$. *Then* $x \in PC_0([-\tau, T], R)$ *is a solution of the equation*

$$\begin{cases} x' + Mx + Bx_t = \sigma(t), & t \in J, t \ne t_k, \\ \Delta x|_{t=t_k} = I_k(\eta_k) - L_k[x(t_k) - \eta(t_k)], & (k = 1, 2, \dots, m), \\ x_{t_0} = \Phi \end{cases} \tag{2.10}$$

if and only if $x \in PC_0([-\tau, T], R)$ *is a solution of the following integral equation*

$$\begin{aligned} x(t) = \Phi(0)\mathrm{e}^{-Mt} &+ \int_0^t \mathrm{e}^{-M(t-s)}[\sigma(s) - Bx_s]\mathrm{d}s \\ &+ \sum_{0<t_k<t} \mathrm{e}^{-M(t-t_k)}\{I_k(\eta(t_k)) - L_k[x(t_k) - \eta(t_k)]\}, \; t \in J, \end{aligned} \tag{2.11}$$

where $x_t(s) = x(t + s) = \Phi(t + s)$ *if* $t + s \le 0$.

Proof. Assume that $x \in PC_0([-\tau, T], R)$ is a solution of IRFDE (2.10). Let $z(t) = x(t)\mathrm{e}^{-Mt}$. Then $z \in PC([-\tau, T], R)$ and

$$z'(t) = [\sigma(t) - Bx_t)]\mathrm{e}^{-Mt},\ t \in [0, T],\ t \neq t_k\ (k = 1, 2, \dots, m).$$

Since $(\sigma(t) - Bx_t)\mathrm{e}^{-Mt}$ is measurable on $[0, T]$, it is easy to establish the following formula:

$$z(t) = z(0) + \int_0^t z'(s)\mathrm{d}s + \sum_{0<t_k<t} [z(t_k^+) - z(t_k)],\ t \in [0, T].$$

And from the second expression of (2.10), we have

$$z(t_k^+) - z(t_k) = \{I_k(\eta(t_k)) - L_k[x(t_k) - \eta(t_k)]\}\mathrm{e}^{Mt_k}.$$

Consequently,

$$x(t)\mathrm{e}^{Mt} = \Phi(0) + \int_0^t [\sigma(t) - Bx_s]\mathrm{d}s$$
$$+ \sum_{0<t_k<t} \{I_k(\eta(t_k)) - L_k[x(t_k) - \eta(t_k)]\}\mathrm{e}^{Mt_k}, t \in [0, T],$$

i.e., $x(t)$ satisfies (2.11).

Conversely, if $x \in PC([-\tau, T])$ is a solution of eq. (2.11), by direct differentiation, it is easy to see the first expression of (2.10) is true for all $t \in [0, T] - \{t_k\}_{k=1}^m$ except on a set of Lebesgue measure zero and the second one and the third one of (2.10) are true. The proof is complete. □

Lemma 2.4. *Equation* (2.11) *has a unique solution in* $PC_0([-\tau, T], R)$ *with* $x_0 = \Phi$.

Proof. For $x \in C([0, t_1], R)$, let $\|x\| = \max\{\mathrm{e}^{-M_1 t}|x(t)|, t \in [0, t_1]\}$ and

$$(A_1x)(t) = \Phi(0)\mathrm{e}^{-Mt} + \int_0^t \mathrm{e}^{-M(t-s)}[\sigma(s) - (Bx_s)]\mathrm{d}s,\ t \in J,$$

where $x(t+s) = \Phi(t+s)$ if $t+s \le 0$ and $M_1 = \|B\| + 1$. Obviously $A_1 : C([0, t_1], R) \to C([0, t_1], R)$ is a continuous operator. For $x, y \in C([0, t_1], R)$,

$$\begin{aligned}
&|(A_1x)(t) - (A_1y)(t)| \\
= &\int_0^t [(Bx_s) - (By_s)]\mathrm{d}s \\
= &\int_0^t \int_{-\tau}^0 |x_s(r) - y_s(r)|\gamma(r)\mathrm{d}r\mathrm{d}s \\
= &\int_{-\tau}^0 \int_0^t |x_s(r) - y_s(r)|\gamma(r)\mathrm{d}s\mathrm{d}r \\
= &\int_{-\tau}^0 \int_0^t |x_s(r) - y_s(r)|\mathrm{d}s\gamma(r)\mathrm{d}r
\end{aligned}$$

$$
\begin{aligned}
&= \int_{-\tau}^{0}\int_{r}^{t+r} |x(s) - y(s)|\mathrm{d}s\gamma(r)\mathrm{d}r \\
&= \int_{-\tau}^{0}\int_{0}^{t+r} |x(s) - y(s)|\mathrm{d}s\gamma(r)\mathrm{d}r \\
&\le \int_{-\tau}^{0}\int_{0}^{t} |x(s) - y(s)|\mathrm{d}s\gamma(r)\mathrm{d}r \\
&= \int_{0}^{t} |x(s) - y(s)|\mathrm{d}s \int_{-\tau}^{0} \gamma(r)\mathrm{d}r \\
&= \|B\| \int_{0}^{t} \mathrm{e}^{M_1 s}\mathrm{e}^{-M_1 s}\|x(s) - y(s)\|\mathrm{d}s \\
&\le \frac{\|B\|}{M_1}\mathrm{e}^{M_1 t}\|x - y\|.
\end{aligned}
$$

So

$$\mathrm{e}^{-M_1 t}|(A_1x)(t) - (A_1y)(t)| \le \frac{\|B\|}{M_1}\|x - y\|,$$

i.e.,

$$\|(A_1x - A_1y)\| \le \frac{\|B\|}{M_1}\|x - y\|. \tag{2.12}$$

By contraction mapping theorem, A_1 has a unique fixed point $x_1 \in C([0, t_1], R)$. For $x \in C([t_1, t_2], R)$, let $\|x\| = \max\{\mathrm{e}^{-M_2 t}|x(t)|, t \in [t_1, t_2]\}$ and

$$
\begin{aligned}
(A_2x)(t) = (x_1(t_1)) &+ [I_1(\eta(t_1)) - L_1(x_1(t_1) - \eta(t_1))]\mathrm{e}^{-M(t-t_1)} \\
&+ \int_{t_1}^{t} \mathrm{e}^{-M(t-s)}[\sigma(s) - Bx_s]\mathrm{d}s, t \in [t_1, t_2],
\end{aligned} \tag{2.13}
$$

where $x(t + s) = \Phi(t + s)$ if $t + s \le 0, x(t + s) = x_1(t + s)$ if $t + s \in (0, t_1]$ and $M_2 = \|B\| + 1$. Similarly, A_2 has a unique fixed point x_2 in $C([t_1, t_2], R)$. So forth and so on, for $x \in C([t_n, T], R)$, let $\|x\| = \max\{\mathrm{e}^{-M_{n+1} t}|x(t)|, t \in [t_n, T]\}$ and

$$
\begin{aligned}
(A_{n+1}x)(t) = (x_n(t_n)) &+ [I_n(\eta(t_n)) - L_n(x_n(t_n)) - \eta(t_n)]\mathrm{e}^{-M(t-t_n)} \\
&+ \int_{0}^{t} \mathrm{e}^{-M(t-s)}[\sigma(s) - Bx_s]\mathrm{d}s, t \in [t_n, T],
\end{aligned} \tag{2.14}
$$

where $x(t+s) = \Phi(t+s)$ if $t+s \le 0, x(t+s) = x_1(t+s)$ if $t+s \in (0, t_1], \dots, x(t+s) = x_{n-1}(t+s)$ if $t+s \in (t_{n-2}, t_n]$ and $M_{n+1} = \|B\| + 1$.

Similarly A_{n+1} has a unique fixed point $x_{n+1} \in C([t_n, T], R)$. Let

$$
x^*(t) = \begin{cases}
\Phi(t), & t \in [-\tau, 0]; \\
x_1(t), & t \in (0, t_1]; \\
x_2(t), & t \in (t_1, t_2]; \\
\dots, & \dots; \\
x_{n+1}(t), & t \in (t_n, T].
\end{cases}
$$

Then $x^* \in PC([-\tau, T], R)$ is a solution. If $y^* \in PC([-\tau, T], R)$ is another solution of equation, by $x^*(t) = y^*(t)$ for $t \in [-\tau, 0]$, it is easy to verify $x^*(t) = y^*(t)$ for $t \in [0, t_1]$.

And so on, $x^*(t) = y^*(t)$ for $t \in (t_1, t_2]$. Continuing as before, we get $x^*(t) = y^*(t)$ for $t \in (t_n, T]$. Therefore $x^* = y^*$. The proof is complete. □

Now we list some independent conditions for convenience.

(A$_1$) There exist $u, v \in PC_0([-\tau, T], R)$ satisfying $u(t) \le v(t)$ $(t \in J)$ and

$$\begin{cases} u'(t) \le f(t, u_t), & t \in J, t \ne t_k; \\ \Delta u|_{t=t_k} \le I_k(u(t_k)), & (k = 1, 2, \dots, m), \\ u_0 \le \Phi, \end{cases}$$

$$\begin{cases} v'(t) \ge f(t, v_t), & t \in J, t \ne t_k; \\ \Delta v|_{t=t_k} \ge I_k(v(t_k)), & (k = 1, 2, \dots, m), \\ v_0 \ge \Phi. \end{cases}$$

Moreover, $\Phi - u_0$ and $v_0 - \Phi$ satisfy either the assumption (a) or (b) of Lemma 2.1.

(A$_2$) There exist constants $M \ge 0$ such that

$$f(t, \phi) - f(t, \psi) \ge -M(\phi(0) - \psi(0)) - B(\phi - \psi),$$

whenever $t \in J$, $\phi, \psi \in \{x_t, u(t) \le x(t) \le v(t), t \in J\}$ with $\phi \ge \psi$.

(A$_3$) There exist constants $0 \le L_k \le 1$ $(k = 1, 2, \dots, m)$ such that

$$I_k(x) - I_k(y) \ge -L_k(x - y),$$

whenever $u(t_k) \le y \le x \le v(t_k)$, $(k = 1, 2, \dots, m)$.

(A$_4$) $f : J \times M([-\tau, 0], R) \to R$ is continuous at every $t \in J$ for each fixed $\phi \in M([-\tau, 0], R)$ and is weakly continuous at every $\phi \in M([-\tau, 0], R)$ for each fixed $t \in J$.

Theorem 2.1. *Let the conditions* (A$_1$)–(A$_4$) *be satisfied and* $f \in C([0, T] \times M([-\tau, 0], R), R)$ *and* $[u, v] \subseteq PC_0([-\tau, 0], R)$. *Then there exist monotone sequence* $\{u_n\}, \{v_n\} \subseteq PC_0([-\tau, T], R)$ *which converge on* J *to the minimal and maximal solutions* $x_*, x^* \in PC_0([-\tau, T], R)$ *in* $[u, v]$ *respectively. That is, if* $x \in PC_0([-\tau, T], R)$ *is any solution satisfying* $x \in [u, v]$, *then*

$$u(t) \le u_1(t) \le \dots \le x_*(t) \le x(t) \le x^*(t) \le \dots \le v_n(t) \le \dots \le v_1(t) \le v(t), \ t \in J.$$

Proof. For any $\eta \in [u, v]$, consider the linear eq. (2.10), where

$$\sigma(t) = f(t, \eta_t) + M\eta(t) + B\eta_t, \ t \in J.$$

By the condition (A$_4$) and Lemma 2.1, one has $\sigma \in M([-\tau, T], R)$. By Lemma 2.3, IRFDE (2.10) has a unique solution $x \in PC_0([-\tau, T], R)$ with $x_0 = \Phi$. Let

$$x(t) = (A\eta)(t), \ t \in J. \tag{2.15}$$

Then A is a continuous operator from $[u, v]$ into $PC_0([-\tau, T], R)$. Now we show

(a) $u \le Au$, $Av \le v$;

(b) A is nondecreasing in $[u, v]$.

To prove (a), we set $u_1 = Au$ and $p = u - u_1$. By Lemma 2.3, we have

$$\begin{cases} u_1'(t) + Mu_1(t) + Bu_{1t} = f(t, u_t) + Mu(t) + Bu_t, & t \in J, t \neq t_k, \\ \Delta u_1|_{t=t_k} = I_k(u(t_k)) - L_k[u_1(t_k) - u(t_k)], & k = 1, 2, \ldots, m, \\ u_{10} = \Phi. \end{cases} \tag{2.16}$$

So

$$\begin{cases} p'(t) = u'(t) - u_1'(t) \leq -Mp(t) - Bp_t, & t \in J, t \neq t_k, \\ \Delta p|_{t=t_k} = \Delta u|_{t=t_k} - \Delta u_1|_{t=t_k} \leq -L_k p(t_k), & (k = 1, 2, \ldots, m) \\ p_0 = u_0 - u_{10} \leq 0, \end{cases} \tag{2.17}$$

which implies by virtue of Lemma 2.2 that $p(t) \leq 0$ for $t \in J$, i.e. $u \leq u_1 = Au$. Similarly, we can show $v_1 = Av \leq v$.

To prove (b), for $\eta_1, \eta_2 \in [u, v]$ with $\eta_1 \leq \eta_2$, let $p = x_1 - x_2$, where $x_1 = A\eta_1, x_2 = A\eta_2$. From Lemma 2.2, we get

$$\begin{aligned} p' &= x_1' - x_2' \\ &= [f(t, \eta_{1t}) + M(\eta_1(t) - x_1(t)) + (B\eta_{1t} - Bx_{1t})] \\ &\quad -[f(t, \eta_{2t}) + M(\eta_2(t) - x_2(t)) + (B\eta_{2t} - Bx_{2t})] \\ &= -[f(t, \eta_{2t}) - f(t, \eta_{1t}) \\ &\quad +M(\eta_2(t) - \eta_1(t)) + (B\eta_{2t} - B\eta_{1t})] \\ &\quad -Mp - Bp_t \\ &\leq -Mp(t) - Bp_t, t \in J, t \neq t_k, \end{aligned}$$

$$\begin{aligned} \Delta p|_{t=t_k} &= \Delta x_1|_{t=t_k} - \Delta x_2|_{t=t_k} \\ &= \{I_k(\eta_1(t_k)) - L_k[x_1(t_k) - \eta_1(t_k)]\} - \{I_k(\eta_2(t_k)) - L_k[x_2(t_k) - \eta_2(t_k)]\} \\ &= -\{I_k(\eta_2(t_k)) - I_k(\eta_1(t_k)) + L_k[\eta_2(t_k) - \eta_1(t_k)]\} - L_k p(t_k) \\ &\leq -L_k p(t_k), (k = 1, 2, \ldots, m), \end{aligned}$$

and

$$p_0 = x_{10} - x_{20} = 0.$$

Hence, by Lemma 2.2, $p(t) \leq 0$ for all $t \in J$, i.e., $A\eta_1 \leq A\eta_2$, and (b) is proved.

Let $u_n = Au_{n-1}$, and $v_n = Av_{n-1}$ $(n = 1, 2, \ldots, m)$. By (a) and (b), we get

$$u(t) \leq u_1(t) \leq \ldots \leq u_n(t) \leq \ldots \leq v_n(t) \leq \ldots \leq \ldots \leq v_1(t) \leq v(t), \ t \in J, \tag{2.18}$$

and $u_n, v_n \in PC_0([-\tau, T], R)$ with $u_{n0} = v_{n0} = \Phi, n = 1, 2, \ldots$. So there exist x_* and x^* such that

$$u_n(t) \to x_*(t), t \in [-\tau, T], \ n \to +\infty, \tag{2.19}$$

$$v_n(t) \to x^*(t), t \in [-\tau, T], \ n \to +\infty. \tag{2.20}$$

Therefore

$$u_{nt}(s) \to x_{*t}(s), t \in J, s \in [-\tau, 0], \ n \to +\infty,$$

$$v_{nt}(s) \to x_t^*(s), t \in J, s \in [-\tau, 0], \ n \to +\infty.$$

So

$$f(t,u_{nt})+Mu_{n-1}(t)-(Bu_{nt}-Bu_{n-1t}) \\ \to f(t,x_{*t})+Mx_*(t),\ n\to+\infty.$$

By the Lebesgue dominated convergence theorem, we get

$$\int_0^t \mathrm{e}^{-M(t-s)}[f(s,u_{ns})+Mu_{n-1}(s)-(Bu_{ns}-Bu_{n-1s})]\mathrm{d}s \\ \to \int_0^t \mathrm{e}^{-M(t-s)}[f(s,x_{*s})+Mx_*(s)]\mathrm{d}s,\ n\to+\infty. \tag{2.21}$$

So

$$x_*(t)=\Phi(0)\mathrm{e}^{-Mt}+\int_0^t \mathrm{e}^{-M(t-s)}[f(s,x_{*s})+Mx_*(s)]\mathrm{d}s,\ t\in[0,t_1], \tag{2.22}$$

where $x_{*0}=\Phi$. And by virtue of the continuity of I_1, we get

$$I_1(u_n(t_1))\to I_1(x_*(t_1)),\ n\to+\infty. \tag{2.23}$$

Similarly, one has

$$x_*(t)=[x_*(t_1)+I_1(x_*(t_1))]\mathrm{e}^{-M(t-t_1)} \\ +\int_{t_1}^t \mathrm{e}^{-M(t-s)}[f(s,x_{*s})+Mx_*(s)]\mathrm{d}s,\ t\in(t_1,t_2], \tag{2.24}$$

where $x_{*0}=\Phi$. So forth and so on,

$$x_*(t)=[x_*(t_n)+I_n(x_*(t_n))]\mathrm{e}^{-M(t-t_n)} \\ +\int_{t_n}^t \mathrm{e}^{-M(t-s)}[f(s,x_{*s})+Mx_*(s)]\mathrm{d}s,\ t\in(t_n,T], \tag{2.25}$$

where $x_{*0}=\Phi$. Then

$$x_*(t)=\Phi\mathrm{e}^{-Mt}+\int_0^t \mathrm{e}^{-M(t-s)}[f(s,x_{*s})+Mx_*(s)]\mathrm{d}s \\ +\sum_{0<t_k<t}\mathrm{e}^{-M(t-t_k)}I_k(x_*(t_k)),\ t\in J. \tag{2.26}$$

By the similar proof, we get

$$x^*(t)=\Phi(0)\mathrm{e}^{-Mt}+\int_0^t \mathrm{e}^{-M(t-s)}[f(s,x_s^*)+Mx^*(s)]\mathrm{d}s \\ +\sum_{0<t_k<t}\mathrm{e}^{-M(t-t_k)}I_k(x^*(t_k)), \tag{2.27}$$

where $x_0^*=\Phi$.

Finally, if $x\in PC([-\tau,T],R)$ is a solution of eq. (1.1) in $[u,v]$, Now let $p=u_n-x$ and use mathematics induction. Obviously $u\leq x$. Suppose $u_{n-1}\leq x$. Then

$$\begin{aligned} p' &= u_n{}'-x' \\ &= f(t,u_{n-1t})-M(u_n(t)-u_{n-1}(t))-(Bu_{nt}-B_{u_{n-1t}})-f(t,x_t) \\ &= -Mp-Bp_t-[f(t,x_t)-f(t,u_{n-1t})] \\ &\quad +M(-x(t)+u_{n-1}(t))+(-Bx_t+Bu_{n-1t}) \\ &\leq -Mp-Bp_t,\ t\in J,\ t\neq t_k, \end{aligned}$$

$$\begin{aligned}\Delta p|_{t=t_k} &= \Delta u_n|_{t=t_k} - \Delta x|_{t=t_k}\\ &= I_k(u_{n-1}(t_k)) - L_k[u_n(t_k) - u_{n-1}(t_k)] - I_k(x(t_k))\\ &= -\{I_k(x(t_k)) - I_k(u_{n-1}(t_k) + L_k[x(t_k) - u_{n-1}(t_k)]\} - L_k p(t_k)\\ &\le -L_k p(t_k),\ (k = 1, 2, \dots, m),\end{aligned}$$

and

$$p_0 = u_{n0} - x_0 = \theta.$$

Hence, by Lemma 2.2, $p(t) \le 0$ for all $t \in J$, i.e. $u_n(t) \le x(t), t \in J$. So $u_n(t) \le x(t)$, $t \in J, n = 1, 2, \dots$. By the same proof, we can show $x(t) \le v^{(n)}(t), t \in J, n = 1, 2, \dots$. Consequently, $x_*(t) \le x(t) \le x^*(t), t \in J$. The proof is complete. □

3. An example

We consider.

$$\begin{cases} x' = \dfrac{1}{72}(t - x(t))^3 + \dfrac{1}{40}(t^2 - x(t-1))^5 \\ \qquad + \dfrac{1}{144}\left(\sin^2 t - \displaystyle\int_{-1}^{0} x(t+s)\mathrm{d}s\right)^3, t \ne \dfrac{1}{2}, t \in (0, 1]; \\ \Delta x|_{t=\frac{1}{2}} = -\dfrac{1}{6}x\left(\dfrac{1}{2}\right), \\ x_0 = \phi, \end{cases} \tag{3.1}$$

where

$$\phi(t) = \begin{cases} 1, & t \in [-1, -\frac{1}{2}), \\ \dfrac{1}{2}, & t \in (-\frac{1}{2}, 0]. \end{cases}$$

Conclusion. IRFDE (3.1) admits minimal and maximal solutions.

Proof. Let

$$u(t) = 0,\ t \in [-1, 1]$$

and

$$v(t) = \begin{cases} 1, & t \in [-1, 0]; \\ 1 + t, & t \in (0, \frac{1}{2}]; \\ t + \dfrac{5}{6}, & t \in (\frac{1}{2}, 1]. \end{cases}$$

It is easy to see that u, v are not solutions of eq. (3.1) and $u(t) \le v(t)$, $t \in [-1, 1]$. Moreover,

$$
\begin{aligned}
&\Delta u|_{t=\frac{1}{2}} = -\frac{1}{6}u\left(\frac{1}{2}\right),\\
&\Delta v|_{t=\frac{1}{2}} = -\frac{1}{6} > -\frac{1}{2} = -\frac{1}{6}u\left(\frac{1}{2}\right)\\
&u'(t) = 0,\ t \in [0,1];\\
&v'(t) = 1,\ t \in \left[0,\frac{1}{2}\right) \cup \left(\frac{1}{2},1\right],\\
&f(t,u_t) = \frac{1}{72}t^3 + \frac{1}{40}t^{10} + \frac{1}{144}\sin^6 t,\ t \in [0,1],\\
&f(t,v_t) = \frac{1}{72}(t-(1+t))^3 + \frac{1}{40}(t^2-1)^5\\
&\qquad + \frac{1}{144}\left(\sin^2 t - \int_{-1}^{0} v(t+s)\mathrm{d}s\right)^3,\ t \in [0,1].
\end{aligned}
$$

Then

$$
\begin{cases}
u'(t) \le f(t,u_t), & t \in (0,1), t \ne \frac{1}{2};\\
\Delta u|_{t=\frac{1}{2}} \le -\frac{1}{6}u\left(\frac{1}{2}\right), & \\
u_0 \le \Phi, &
\end{cases}
$$

$$
\begin{cases}
v'(t) \ge f(t,v_t), & t \in J, t \ne t_k;\\
\Delta v|_{t=t_k} > -\frac{1}{6}v\left(\frac{1}{2}\right), & \\
v_0 \ge \Phi, &
\end{cases}
$$

i.e. the condition (A_3) is true.

By mean value theorem, we get

$$
\begin{aligned}
\frac{1}{72}((t-x)^3 - (t-y)^3) &= -\frac{1}{24}(t-\eta(x,y))^2(x-y),\\
\frac{1}{40}((t^2-x)^5 - (t^2-y)^5) &= -\frac{1}{8}(t-\zeta(x,y))^4(x-y)
\end{aligned}
$$

and

$$\frac{1}{144}((\sin^2 t - x)^3 - (\sin^2 t - y)^3) = -\frac{1}{48}(\sin^2 t - \gamma(x,y))^2(x-y).$$

For any $\psi \in M([-1,0],R)$, let

$$B\psi = \frac{1}{8}\psi(-1) + \frac{1}{48}\int_{-1}^{0}\psi(s)\mathrm{d}s.$$

Then

$$f(t,\phi) - f(t,\psi) \ge -\frac{1}{24}(\phi(0) - \psi(0)) - (B\phi - B\psi)$$

for all $\phi, \psi \in \{x_t, u(t) \le x(t) \le v(t), t \in [0,1]\}$ with $\phi \le \psi$.

So the condition (A_2) is true.

For $u\left(\frac{1}{2}\right) \le y \le x \le v\left(\frac{1}{2}\right)$,

$$I(x) - I(y) = -\frac{1}{6}(x-y).$$

So the condition (A_3) is true. So $M = \frac{1}{24}, L_1 = \frac{1}{6}, \Delta_1 = \frac{1}{2}, \Delta_2 = 1$,

$$M_0 < \frac{1}{24} + \frac{1}{8} = \frac{1}{6}.$$

For $p_1(t) = u(t) - \Phi(t), t \in [-1, 0]$, we get

$$L_{-1} = \frac{1}{2}, \Delta = \max\left\{\frac{1}{2}, 1, \frac{1}{2}\right\} = 1, \inf_{t\in[-1,0]} p_1(t) = -1 < p_1(0)$$

and

$$p_1'(t) = 0 < M_0,\ t \in \left[-1, -\frac{1}{2}\right) \cap \left(-\frac{1}{2}, 0\right].$$

Moreover,

$$M_0\Delta_1 < \frac{5}{23} = \frac{(1 - L_{-1})(1 - L_1)}{1 + (1 - L_{-1}) + (1 - L_{-1})(1 - L_1)}.$$

For $p_2 = \Phi(t) - v(t)$, we get

$$p_2(0) = -\frac{1}{2} \le p_2(t),\ t \in [-1, 0]$$

and

$$M_0 < \frac{5}{11} = \frac{(1 - L_1)}{1 + (1 - L_1)}.$$

And thus it is easy to see that (A_4) is true. By Theorem 2.1, eq. (3.1) has a maximal solution and a minimal solution. The proof is complete. □

Remark. Our result can be extended to impulsive delay differential equations in Banach spaces.

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On initial conditions for a boundary stabilized hybrid Euler–Bernoulli beam

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Abstract. We consider here small flexural vibrations of an Euler–Bernoulli beam with a lumped mass at one end subject to viscous damping force while the other end is free and the system is set to motion with initial displacement $y^0(x)$ and initial velocity $y^1(x)$. By investigating the evolution of the motion by Laplace transform, it is proved (in dimensionless units of length and time) that

$$\int_0^1 y_{xt}^2 \,\mathrm{d}x \le \int_0^1 y_{xx}^2 \,\mathrm{d}x, \quad t > t_0,$$

where t_0 may be sufficiently large, provided that $\{y^0, y^1\}$ satisfy very general restrictions stated in the concluding theorem. This supplies the restrictions for uniform exponential energy decay for stabilization of the beam considered in a recent paper.

Keywords. Euler–Bernoulli beam equation; hybrid system; initial conditions; small deflection; exponential energy decay.

1. Introduction

In a recent paper, Gorain and Bose [2] investigated the possibility of stabilization of transverse vibrations of a hybrid system consisting of an Euler–Bernoulli beam held by a lumped mass movable hub attached to one of its ends. The beam is assumed to be initially set in vibration by a displacement y^0 and velocity y^1 in the transverse direction and stabilization is sought by applying viscous damping force to the moving lumped mass. The system equations for simplicity can be written in dimensionless form by suitably choosing the units of length and time. If $y(x, t)$ be the transverse displacement of a point of the beam distant x from the lumped mass at time t, the equations are [2]

$$y_{tt}(x, t) + y_{xxxx}(x, t) = 0, \quad 0 \le x \le 1, \ t \ge 0, \tag{1}$$

along the length of the beam, while at the lumped mass and free ends,

$$y_{xxx}(0, t) + \alpha y_{tt}(0, t) + \lambda y_t(0, t) = 0, \quad y_x(0, t) = 0, \quad t \ge 0, \tag{2}$$

$$y_{xx}(1, t) = 0, \quad y_{xxx}(1, t) = 0, \quad t \ge 0, \tag{3}$$

where α is the dimensionless mass of the lump and similarly λ the damping coefficient. The system is set to vibration with initial conditions

$$y(x, 0) = y^0(x), \quad y_t(x, 0) = y^1(x), \quad 0 \le x \le 1. \tag{4}$$

We note in (1)–(4) that without loss of generality we can assume

$$y^0(0) = 0. \tag{5}$$

Such hybrid systems for general $y^0(x)$ and $y^1(x)$ have been investigated in detail in search of uniform exponential decay of total energy (kinetic and potential) for proving stability of the process. However Littman and Marcus [5] and Chen and Zhou [1] have found by calculating the eigenvalues of their hybrid systems that uniform stabilization is not possible because infinitely large wave number k, during the passage of a wave along the beam are present in the general case. Rao [6] arrives at the same conclusion by applying semigroup theory to the evolving system.

In [2] it was noted that eq. (1) is arrived at by assuming that the beam remains approximately straight during vibration, precluding infinitely large wave numbers. From this observation, heuristically an additional condition was suggested, which in nondimensional form is

$$\int_0^1 y_{xt}^2 \,\mathrm{d}x \leq \int_0^1 y_{xx}^2 \,\mathrm{d}x, \quad t > t_0, \tag{6}$$

where t_0 may be as large as we please. Subject to this condition, it was proved in [2], that uniform exponential decay of total energy indeed takes place.

The condition (6) places restrictions on the initial conditions $y^0(x)$, $y^1(x)$ from which the system evolves. It is the purpose of this paper to determine them by investigating the actual evolution of the system (1)–(5) by Laplace transformation in the complex frequency domain s and invoking the final value theorem for the system behaviour for t tending to infinity.

2. System evolution

Let the Laplace transform of $y(x, t)$ be

$$Y(x, s) = \int_0^\infty y(x, t)\mathrm{e}^{-st}\mathrm{d}t, \tag{7}$$

then according to the final value theorem, if s be complex (with x fixed) and $Y(x, s)$ be analytic in $\mathrm{Re}\{s\} \geq c,\ c < 0$,

$$\lim_{t\to\infty} y(x, t) = \lim_{s\to 0} sY(x, s) \tag{8}$$

and so we would be interested in the transformed quantities as $s \to 0$. The transformation of equations (1)–(4) in the usual way yield

$$Y_{xxxx}(x, s) + s^2 Y(x, s) = sy^0(x) + y^1(x), \tag{9}$$

with boundary conditions, using (5):

$$Y_{xxx}(0, s) + \alpha s^2 Y(0, s) + \lambda s Y(0, s) = \alpha y^1(0), \quad Y_x(0, s) = 0, \tag{10}$$

$$Y_{xx}(1, s) = 0, \quad Y_{xxx}(1, s) = 0. \tag{11}$$

In order to solve (9)–(11), we introduce 'wave number' k by the relation

$$s = -ik^2 : s^2 = -k^4. \tag{12}$$

The general solution of (9) is then

$$\begin{aligned}Y(x,-ik^2) &= C_0\sin kx + C_1\cos kx + C_2\sinh kx + C_3\cosh kx\\ &\quad+\frac{1}{2k^3}\int_0^x[-ik^2y^0(\xi)+y^1(\xi)][\sin k(\xi-x)-\sinh k(\xi-x)]\,\mathrm{d}\xi. \qquad (13)\end{aligned}$$

For the differentiability of the particular solution of (9) represented by the integral in (13) we require that $y^0(x)$ and $y^1(x)$ are C^1 smooth. The boundary conditions (10), (11) yield for the coefficients C_0, C_1, C_2, C_3 the four equations

$$C_0 = -C_2, \qquad (14a)$$

$$-k^2(\alpha k^2+i\lambda)(C_1+C_3)+2k^3C_2 = \alpha y^1(0), \qquad (14b)$$

$$\begin{aligned}&-C_1\cos k + C_2(\sin k+\sinh k)+C_3\cosh k =\\ &\qquad\frac{1}{2k^3}\int_0^1[-ik^2y^0(\xi)+y^1(\xi)][\sin k(\xi-1)+\sinh k(\xi-1)]\mathrm{d}\xi, \qquad (14c)\end{aligned}$$

$$\begin{aligned}&C_1\sin k + C_2(\cos k+\cosh k)+C_3\sinh k =\\ &\qquad-\frac{1}{2k^3}\int_0^1[-ik^2y^0(\xi)+y^1(\xi)][\cos k(\xi-1)+\cosh k(\xi-1)]\mathrm{d}\xi. \qquad (14d)\end{aligned}$$

The exact solution of (14) can be explicitly written down by Cramer's rule. But here we are interested in the solution for large t, that is to say, for small s or k and so we expand the determinants formally in powers of k and do the same for the trigonometric and hyperbolic functions appearing in (13). Thus, restoring s in place of k defined in eq. (12) we obtain,

$$\begin{aligned}Y_x(x,s) = \frac{1}{4[\lambda+(\alpha+1)s+O(s^2)]}\Bigg[&-2I_1(s)(\lambda+s)x^2\\ &+2\left\{I_2(s)[\lambda+(\alpha+1)s]+I_1(s)\left[\lambda-i+\left(\alpha+\frac{1}{2}\right)s\right]\right.\\ &\left.+\alpha y^1(0)\left(1-\frac{is}{2}\right)\right\}x\left(1-\frac{isx^2}{6}\right)\\ &+2\left\{I_2(s)[\lambda+(\alpha+1)s]+I_1(s)\left[\lambda+i+\left(\alpha+\frac{1}{2}\right)s\right]\right.\\ &\left.-\alpha y^1(0)\left(1+\frac{is}{2}\right)\right\}x\left(1+\frac{isx^2}{6}\right)+O(s^2)\Bigg]\\ &-\frac{1}{2}\int_0^x[sy^0(\xi)+y^1(\xi)][(\xi-x)^2+O(s^4)]\mathrm{d}\xi, \qquad (15)\end{aligned}$$

where

$$I_1(s)=\int_0^1[sy^0(\xi)+y^1(\xi)]\mathrm{d}\xi,\quad I_2(s)=\int_0^1(\xi-1)[sy^0(\xi)+y^1(\xi)]\mathrm{d}\xi. \qquad (16)$$

In §4 we shall prove that poles of $Y_x(x,s)$ for each x lie in $\mathrm{Re}\{s\}<c$, $c<0$ when $\lambda>0$. Hence, by the final value theorem of Laplace transform, we find that since $\lambda\neq 0$,

$$\lim_{t\to\infty} y_x(x,t)=\lim_{s\to 0} sY_x(x,s)=0. \qquad (17)$$

The limiting operation in (15) is essentially justified by expansion in powers of s therein and the assumed C^1 continuity of $y^0(x)$ and $y^1(x)$. The limit (17) means that in the presence of the viscous damping, as t becomes large, the beam approaches its original straight shape.

3. Validity of condition (6)

In order to prove that condition (6) holds for the motion, consider the functions $ty_{xx}(x, t)$ and $t^2 y_{xt}(x, t)$. The Laplace transforms of the two functions are respectively

$$-\frac{\partial}{\partial s}[Y_{xx}(x, s)] \quad \text{and} \quad \frac{\partial^2}{\partial s^2}[sY_x(x, s) - y_x^0(x)] = \frac{\partial^2}{\partial s^2}[sY_x(x, s)].$$

Hence by the final value theorem,

$$\lim_{t\to\infty} \frac{t^2 \int_0^1 y_{xt}^2 \mathrm{d}x}{\int_0^1 y_{xx}^2 \mathrm{d}x} = \lim_{s\to 0} \frac{\int_0^1 \left\{\frac{\partial^2}{\partial s^2}[sY_x(x, s)]\right\}^2 \mathrm{d}x}{\int_0^1 \left\{\frac{\partial}{\partial s}[Y_{xx}(x, s)]\right\}^2 \mathrm{d}x}. \tag{18}$$

The limit of the numerator in (18), from equations (15), (16) turns out to be

$$\int_0^1 \left[2x \int_0^1 \xi y^0(\xi)\mathrm{d}\xi - x^2 \int_0^1 y^0(\xi)\mathrm{d}\xi - \frac{1}{\lambda}\left(x - x^2 + \frac{x^3}{3}\right)\right.$$
$$\left. \times \int_0^1 y^1(\xi)\mathrm{d}\xi - \frac{\alpha}{\lambda} y^1(0)\left(x + \frac{x^3}{3}\right) + \int_0^x y^0(\xi)(\xi - x)^2 \mathrm{d}\xi\right]^2 \mathrm{d}x, \tag{19}$$

while that of the denominator turns out to be

$$\frac{1}{4}\int_0^1 \left[2\int_0^1 \xi y^0(\xi)\mathrm{d}\xi - 2(x+1)\int_0^1 y^0(\xi)\mathrm{d}\xi + \frac{1}{\lambda}(2\lambda - 1 + 2x - x^2)\right.$$
$$\left. \times \int_0^1 y^1(\xi)\mathrm{d}\xi - \frac{\alpha}{\lambda} y^1(0)(1 + x^2) + \int_0^x y^0(\xi)(\xi - x)\mathrm{d}\xi\right]^2 \mathrm{d}x. \tag{20}$$

If the latter limit vanishes, it follows by differentiating twice that

$$y^0(x) = \frac{2}{\lambda}\left[\int_0^1 y^1(x)\mathrm{d}x + \alpha y^1(0)\right] = 0, \quad 0 \leq x \leq 1, \tag{21}$$

since $y^0(0) = 0$. If this is the case, (19) and (20) respectively become

$$\frac{1}{5}\left[\frac{\alpha}{\lambda} y^1(0)\right]^2 \quad \text{and} \quad \left[\frac{\alpha}{\lambda} y^1(0)\right]^2 \left[\left(\lambda + \frac{1}{2}\right)^2 + \frac{1}{12}\right]. \tag{22}$$

Hence the limit in (18) exists finitely even in the case when the initial values $y^0(x)$ and $y^1(x)$ satisfy (21) together with the provision that $y^1(0) \neq 0$. This last condition means that the velocity at the end where viscous damping is applied should not vanish when the initial displacement is zero. Let the limit in (18) be $l \geq 0$. It then follows that given $\epsilon > 0$ however small, there exists t_0 such that

$$\frac{\int_0^1 y_{xt}^2 \mathrm{d}x}{\int_0^1 y_{xx}^2 \mathrm{d}x} < \frac{l + \epsilon}{t^2} < \frac{l + \epsilon}{t_0^2}, \quad \text{for } t > t_0.$$

Hence for $t > t_0 > \sqrt{l+\epsilon}$, the condition (6) must hold. Thus we have proved the following theorem.

Theorem. *Let* $y(x,t)$ *be the solution of the system* (1)–(5) *corresponding to the initial conditions* $\{y^0(x), y^1(x)\}$ *which are* $C^1[0,1]$ *continuous. Then condition* (6) *holds, provided that if* $y^0(x) = 0$ *on* $[0,1]$ *then, either* $\int_0^1 y^1(x)\mathrm{d}x \neq -\alpha y^1(0)$ *or* $\int_0^1 y^1(x)\mathrm{d}x = -\alpha y^1(0) \neq 0$.

4. Poles of $Y_x(x, s)$

When s is considered complex, $Y(x,s)$ given by (13) together with (12) has poles at those of the coefficients C_0, C_1, C_2, C_3. These are at zeroes of the determinant of the coefficients on the right hand side of the equations (14b)–(14d), satisfying the equation (in terms of k),

$$k^2\Big[k(\sin k\cosh k + \cos k\sinh k) + (\alpha k^2 + i\lambda)(1+\cos k\cosh k)\Big] = 0. \quad (23)$$

When a differentiation of (13) is performed, $k = 0$ no longer remains a pole of $Y_x(x,s)$ as is reflected in (15). The poles of $Y_x(x,s)$ are thus the nonzero zeroes of (23). We investigate their domain by a method similar to that of Krall [4] as given in Gorain [3].

The zeroes of (23) result from (14b)–(14d) when the right hand sides are taken zero. In other words, they crop up from the boundary value problem (9)–(11) with the right hand sides set to zero:

$$Y_{xxxx}(0,s) + s^2 Y(x,s) = 0, \quad s = u + iv \neq 0, \quad (24)$$

$$Y_{xxx}(0,s) = -(\alpha s^2 + \lambda s)Y(0,s), \quad Y_x(0,s) = 0, \quad (25a)$$

$$Y_{xx}(1,s) = 0, \quad Y_{xxx}(1,s) = 0. \quad (25b)$$

If we multiply (24) by the complex conjugate Y^* and then take its conjugate, we obtain

$$Y^* Y_{xxxx} + s^2|Y|^2 = 0 \quad \text{and} \quad YY^*_{xxxx} + s^{*2}|Y|^2 = 0.$$

Subtracting one from the other and integrating from 0 to 1, we have

$$\left(s^2 - s^{*2}\right)\int_0^1 |Y|^2\mathrm{d}x = \int_0^1 \left(YY^*_{xxxx} - Y^*Y_{xxxx}\right)\mathrm{d}x.$$

Integrating by parts and applying boundary conditions (25), we obtain from the above after simplification,

$$\left(s^2 - s^{*2}\right)\int_0^1 |Y|^2\mathrm{d}x = -\left(s - s^*\right)\left[\alpha\left(s + s^*\right) + \lambda\right]|Y(0,s)|^2.$$

If now $s - s^* = 2iv \neq 0$, it follows that

$$u = -\frac{1}{2}\frac{\lambda|Y(0,s)|^2}{\int_0^1 |Y|^2\mathrm{d}x + \alpha|Y(0,s)|^2} < 0. \quad (26)$$

In (26) $u \neq 0$, since otherwise $Y(0,s) = 0$ and then (24), (25) yield $Y(x,s)$ identical to zero.

If $s - s^* = 2iv = 0$, we have $s = u$ and the boundary value problem (24), (25) becomes one of real value. Equation (24) then yields

$$YY_{xxxx} + u^2Y^2 = 0.$$

Integrating by parts from 0 to 1 and applying the boundary conditions (25) with u in place of s, we obtain since $Y(0, s) \neq 0$ as before,

$$u = -\frac{u^2\left[\int_0^1 Y^2\mathrm{d}x + \alpha\{Y(0,s)\}^2\right] + \int_0^1 Y_{xx}^2\mathrm{d}x}{\lambda\{Y(0.s)\}^2} < 0. \tag{27}$$

In (27) $u \neq 0$, since otherwise $\int_0^1 Y_{xx}^2\mathrm{d}x = 0$, which implies that $Y_{xx} = 0$, that is to say, $y_{xx} = 0$ on $0 \leq x \leq 1$, $t \geq 0$, meaning that the beam is not bent.

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Cyclic codes of length 2^m

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Abstract. In this paper explicit expressions of $m + 1$ idempotents in the ring $R = F_q[X]/\langle X^{2^m} - 1\rangle$ are given. Cyclic codes of length 2^m over the finite field F_q, of odd characteristic, are defined in terms of their generator polynomials. The exact minimum distance and the dimension of the codes are obtained.

Keywords. Cyclotomic cosets; generator polynomial; idempotent generator; $[n, k, d]$ cyclic codes.

1. Introduction

Throughout in this paper we consider F_q to be a field of odd characteristic and the ring $R = F_q[X]/\langle X^{2^m} - 1\rangle$. The ring R can be viewed as semi-simple group ring $F_q C_{2^m}$ where C_{2^m} is a cyclic group of order 2^m generated by x. It is assumed that reader is familiar with the properties of cyclic codes based on the theory of idempotents [3]. In §2 of this paper complete set of equivalence classes (modulo 2^m) is given and also the construction of explicit expressions of idempotents is given. In §3, we completely describe the cyclic codes of length 2^m in terms of their generator polynomials. In §4 we obtain q-cyclotomic cosets (modulo 2^m) when order of q modulo $2^m = 2^{m-2}$. An example has been given to illustrate the results.

2. Construction of idempotents

For any positive integer m, consider the set $S = \{1, 2, 3, \ldots, 2^m - 1\}$. Divide the set S into disjoint classes S_i (modulo 2^m) as follows:

For $1 \le i \le m$, consider the set

$$S_i = \{2^{i-1}, 2^{i-1}3, \ldots, 2^{i-1}(2n_i - 1)\}, 1 \le n_i \le 2^{m-i}$$

Clearly the elements of S_i are incongruent to each other modulo 2^m. Note that the elements of S_i are the product of 2^{i-1} with odd numbers. So these are divisible by 2^{i-1} but no higher power of 2. In the set S, the number of elements divisible by 2^{i-1} but no higher power of 2 are

$$(2^{m-i+1} - 1) - (2^{m-i} - 1) = 2^{m-i+1} - 2^{m-i} = 2^{m-i}(2 - 1) = 2^{m-i}.$$

Hence the number of elements in the set S_i is

$$\#S_i = 2^{m-i}.$$

Clearly for $i \neq j$, $S_i \cap S_j = \Phi$ and so

$$\#\left(\bigcup_{i=1}^{m} S_i\right) = \sum_{i=1}^{m} (\#S_i) = \sum_{i=1}^{m} (2^{m-i}) = 2^m - 1.$$

Hence the sets $S_i (1 \leq i \leq m)$ form the partitioning of the set S (modulo 2^m).

For $1 \leq i \leq m$, define the element $S_i(x)$ as

$$S_i(X) = \sum_{s \in S_i} x^s = \sum_{n_i=1}^{2^{m-i}} x^{2^{i-1}(2n_i-1)}.$$

Let α be a primitive 2^mth root of unity in an extension of the field F_q. To prove the main theorem we require the following facts:

Fact 2.1 For $1 \leq i \leq m$,

$$S_i(\alpha^j) = \left| \begin{array}{ccc} 0 & \text{if} & 2^{m-i} \nmid j \\ -2^{m-i} & \text{if} & j = 2^{m-i} \\ 2^{m-i} & \text{if} & 2^{m-i+1} | j \end{array} \right. .$$

Proof. By definition, for $1 \leq i \leq m$,

$$\begin{aligned} S_i(X) &= \sum_{n_i=1}^{2^{m-i}} x^{2^{i-1}(2n_i-1)} \\ &= \sum_{n_i=1}^{2^{m-i}} x^{2^{i-1}(2n_i-1)} + \sum_{n_i=1}^{2^{m-i}} x^{2^{i-1}(2n_i-2)} - \sum_{n_i=1}^{2^{m-i}} x^{2^{i-1}(2n_i-2)} \\ &= \sum_{k=0}^{2^{m-i+1}-1} (x^{2^{i-1}})^k - \sum_{n_i=1}^{2^{m-i}} x^{2^i(n_i-1)}. \end{aligned}$$

Therefore,

$$S_i(\alpha^j) = \sum_{k=0}^{2^{m-i+1}-1} (\alpha^{2^{i-1}j})^k - \sum_{n_i=1}^{2^{m-i}} \alpha^{2^i j(n_i-1)}. \qquad (1)$$

Case 1. If $2^{m-i} \nmid j$, then $2^{m-1} \nmid 2^{i-1}j$ so $2^{i-1}j \not\equiv 0 \pmod{2^m}$ hence $\alpha^{2^{i-1}j} \not\equiv 1$. Similarly $\alpha^{2^i j} \neq 1$. Therefore (1) gives that

$$S_i(\alpha^j) = \frac{(\alpha^{2^{i-1}j})^{2^{m-i+1}} - 1}{\alpha^{2^{i-1}j} - 1} - \frac{(\alpha^{2^i j})^{2^{m-i}} - 1}{\alpha^{2^i j} - 1} = 0 - 0 = 0$$

(denominator being non-zero). This proves the Case 1.

Case 2. If $j = 2^{m-i}$, then $2^{i-1}j = 2^{m-1}$ and $2^i j = 2^m$. Since α is a primitive 2^mth root of unity in an extension of F_q, so $\alpha^{2^i j} = \alpha^{2^m} = 1$ and $\alpha^{2^{i-1}j} = \alpha^{2^{m-1}} = -1$. Again (1) gives that

$$
\begin{aligned}
S_i(\alpha^j) &= \sum_{k=0}^{2^{m-i+1}-1} (-1)^k - \sum_{n_i=0}^{2^{m-i}} (+1)^{n_i-1} \\
&= 0 - 2^{m-i} = -2^{m-i}.
\end{aligned}
$$

This proves the Case 2.

Case 3. If $2^{m-i+1}/j$ then $2^m/2^{i-1}j$ implies that $\alpha^{2^{i-1}j} = 1$ and also $\alpha^{2^i j} = 1$. Again from (1) we have

$$
\begin{aligned}
S_i(\alpha^j) &= \sum_{k=0}^{2^{m-i+1}-1} (1)^k - \sum_{n_i=0}^{2^{m-i}} (1)^{n_i-1} \\
&= 2^{m-i+1} - 2^{m-i} = 2^{m-i}(2-1) = 2^{m-i}.
\end{aligned}
$$

This proves the Fact 2.1.

Fact 2.2. For $0 \le i \le m-1$,

$$
1 + \sum_{r=i+1}^{m} S_r(\alpha^j) = \left| \begin{array}{lll} 0 & \text{if} & 2^{m-i} \not| \, j \\ 2^{m-i} & \text{if} & 2^{m-i} \mid j \end{array} \right. .
$$

Proof. By definition

$$
1 + \sum_{r=i+1}^{m} S_r(\alpha^j) = \sum_{k=0}^{2^{m-i}-1} (\alpha^{2^i j})^k.
$$

If $2^{m-i} \not| \, j$ then $2^m \not| \, 2^i j$ implies that $\alpha^{2^i j} \neq 1$. Hence the required sum takes the value zero. Secondly if $2^{m-i}/j$, then $2^m/2^i j$ implies that $\alpha^{2^i j} = 1$ in the extension field and hence the required sum takes the value

$$
\sum_{k=0}^{2^{m-i}-1} (1)^k = 2^{m-i}.
$$

This proves the Fact 2.2.

Our construction of idempotents is based on the following two facts developed in §2 and 3 of chapter 8 of [3].

Fact 2.3. An expression $e(x)$ in R is an idempotent iff $e(\alpha^j) = 0$ or 1.

Fact 2.4. An idempotent $e_i(x)$ is primitive iff

$$
e_i(\alpha^j) = \left| \begin{array}{ll} 1 & \text{if } j \in Y_r \text{ for some } r, 0 \le r \le m \\ 0 & \text{otherwise,} \end{array} \right.
$$

where Y_r is some q-cyclotomic coset (modulo 2^m) with $Y_0 = \{0\}$.

Theorem 2.5. *The following polynomial expressions are* $(m+1)$ *idempotents in the ring* R,

$$e_o(x) = \frac{1}{2^m}\sum_{j=0}^{2^m-1} x^j = \frac{1}{2^m}\left\{1+\sum_{k=1}^{m} S_k(x)\right\}$$

and for $1 \le i \le m$

$$e_i(x) = \frac{1}{2^{m-i+1}}\left\{1+\sum_{k=i+1}^{m} S_k(x) - S_i(x)\right\}.$$

Proof. By Fact 2.2

$$e_0(\alpha^j) = \frac{1}{2^m}\left\{1+\sum_{k=1}^{m} S_k(\alpha^j)\right\} = \left|\begin{array}{lll} 0 & \text{if} & 2^m \nmid j \\ 1 & \text{if} & 2^m \mid j \end{array}\right.$$

$$= \left|\begin{array}{lll} 0 & \text{if} & j \in S_k \\ 1 & \text{if} & 2^m \mid j \end{array}\right. .$$

By Fact 2.4, $e_0(x)$ is a primitive idempotent with single non-zero $\alpha^0 = 1$. For $1 \le i \le m$, Facts 2.1 and 2.2 show that

$$e_i(\alpha^j) = \left|\begin{array}{lll} 0 & \text{if} & 2^{m-i} \nmid j \\ 1 & \text{if} & 2^{m-i} = j \\ 0 & \text{if} & 2^{m-i+1} \mid j \end{array}\right| .$$

Thus for $1 \le i \le m$, $e_i(\alpha^j) = 0$ or 1 and $e_i(\alpha^j) = 1$ only if $j = 2^{m-i}$ or equivalently by definition only if $j \in S_{m-i+1}$. Hence by the Fact 2.3 the expressions $e_i(x)$ are idempotents.

3. Cyclic codes of length 2^m

Let for $0 \le i \le m$, E_i denotes the cyclic code of length 2^m with idempotent generator $e_i(x)$. By (Theorem 56, [4]), (Remark 6.3, [6]) the generator polynomial $g_i(x)$ of the cyclic code E_i is given by

$$g_i(x) = \text{g.c.d.}(e_i(x), x^{2^m} - 1). \tag{2}$$

Define

$$g_0(x) = \sum_{t=0}^{2^m-1} x^t = \frac{1-x^{2^m}}{1-x}$$

and for $1 \le i \le m$,

$$g_i(x) = (1 - x^{2^{i-1}})[1 + S_{i+1} + \cdots + S_m].$$

Then to show $g_i(x)$ $(0 \le i \le m)$ is the generating polynomial of the cyclic code E_i. In view of (2) it is sufficient to prove the following two facts:

Fact 3.1. $g_i(\alpha^j) = 0$ iff $e_i(\alpha^j) = 0$.

Fact 3.2. $g_i(x)/x^{2^m}-1$.

To prove the Fact 3.1, consider for $1 \le i \le m$,

$$\begin{aligned}
e_i(x) &= \frac{1}{2^{m-i+1}}\{1+S_{i+1}+\cdots+S_m-S_i\}\\
&= \frac{1}{2^{m-i+1}}\left\{\sum_{k=0}^{2^{m-i}-1}(x^{2^i})^k-\sum_{n_i=1}^{2^{m-i}}(x^{2^{i-1}})^{(2n_i-1)}\right\}\\
&= \frac{1}{2^{m-i+1}}\left\{\sum_{k=0}^{2^{m-i}-1}(x^{2^i})^k-x^{2^{i-1}}\sum_{n_i=1}^{2^{m-i}}(x^{2^{i-1}})^{(2n_i-2)}\right\}\\
&= \frac{1}{2^{m-i+1}}\left\{\sum_{k=0}^{2^{m-i}-1}(x^{2^i})^k-x^{2^{i-1}}\sum_{k=0}^{2^{m-i}-1}(x^{2^i})^k\right\}\\
&= \frac{1}{2^{m-i+1}}(1-x^{2^{i-1}})\left\{\sum_{k=0}^{2^{m-i}-1}(x^{2^i})^k\right\}\\
&= \frac{1}{2^{m-i+1}}(1-x^{2^{i-1}})\{1+S_{i+1}+\cdots+S_m\}\\
&= \frac{1}{2^{m-i+1}}g_i(x).
\end{aligned}$$

Thus for $1 \le i \le m$, $e_i(x)$ is a constant multiple of $g_i(x)$. Also by definition $e_0(x)$ is a constant multiple of $g_0(x)$. Hence $g_i(\alpha^j)=0$ iff $e_i(\alpha^j)=0$.

To prove the Fact 3.2, consider for $0 \le i \le m$,

$$\begin{aligned}
1-x^{2^m} &= 1-(x^{2^i})^{2^{m-i}} = (1-x^{2^i})\{(x^{2^i})^{2^{m-i}-1}+(x^{2^i})^{2^{m-i}-2}+\cdots+(x^{2^i})+1\}\\
&= (1+x^{2^{i-1}})(1-x^{2^{i-1}})\{1+S_{i+1}+\cdots+S_m\}\\
&= (1+x^{2^{i-1}})g_i(x).
\end{aligned}$$

Thus $g_i(x)$ is a factor of $(1-x^{2^m})$. Hence the assertion follows.

Theorem 3.3. *E_i is a $[2^m, 2^{i-1}, 2^{m-i+1}]$ cyclic code over $GF(q)$.*

Proof. By Corollary 3 ([3], p. 218) (generalized to non binary case) for $0 \le i \le m$, $\dim E_i = \#\alpha^j$ such that $e_i(\alpha^j)=1$.

By Theorem 2.5, we have $e_i(\alpha^j)=1$ only if $j \in S_{m-i+1}$. So $\dim E_i = \#S_{m-i+1} = 2^{i-1}$.

As shown in [5, 6, 1] it is easy to prove that the repetition code E_i generated by $g_i(x)$ has the minimum distance 2^{m-i+1} and $d(E_0)=2^m=\#$ non-zero terms in $g_0(x)$.

4. q-Cyclotomic cosets (modulo 2^m) when order $(q)=2^{m-2}$

First note that such a q exists due to the following facts [2]. Obviously in this case $m \ge 3$. So throughout this section assume that $m \ge 3$.

Fact 4.1. The integer 2^m has no primitive root.

Fact 4.2. Let a be any odd integer, then it is always true that $a^{2^{m-2}} \equiv 1 \pmod{2^m}$.

Fact 4.3. If $\operatorname{ord}(a) = 2 \pmod{2^3}$ and $a^2 \not\equiv 1 \pmod{2^4}$, then $\operatorname{ord}(a) = 2^{m-2} \pmod{2^m}$ for every $m \geq 3$.

Computation of q-cyclotomic cosets (modulo 2^m) depend upon the following facts:

Fact 4.4. If $\operatorname{ord}(q) = 2^{m-2}$ (modulo 2^m) for every $m \geq 3$, (Fact 4.3), then $q^t \not\equiv -1 \pmod{2^m}$ for $1 \leq t \leq 2^{m-2}$.

Proof. For $t \geq 2^{m-2}$, we have $q^t \equiv 1 \pmod{2^m}$.

If possible let $q^t \equiv -1 \pmod{2^m}$ for some non-negative integer $t < 2^{m-2}$, then $q^{2t} \equiv 1 \pmod{2^m}$. But $\operatorname{ord}(q) = 2^{m-2}$ implies that $2^{m-2} | 2t$ or $2^{m-3} | t \Rightarrow t = 2^{m-3}a$, but $t < 2^{m-2}$. So we must have $a = 1$. So we have

$$\Rightarrow q^{2^{m-3}} \equiv -1 \pmod{2^m}$$

$$\Rightarrow q^{2^{m-3}} \equiv -1 \pmod{2^{m-1}}. \tag{3}$$

But we are assuming that $\operatorname{ord}(q) = 2^{m-2}$ for all $m \geq 3$. So we have

$$q^{2^{m-3}} \equiv 1 \pmod{2^{m-1}}. \tag{4}$$

From (3) and (4)

$$-1 \equiv 1 \pmod{2^{m-1}} \quad \text{for all } m \geq 3$$

which is not possible. Hence the result follows.

Fact 4.5. Thus in this case q cyclotomic cosets modulo 2^m are given by:

For $1 \leq i \leq m$,

$$\begin{aligned} X_i &= \{2^{i-1}, 2^{i-1}q, 2^{i-1}q^2, \ldots, 2^{i-1}q^{2^{m-(i+1)}-1}\}, \\ X_i^* &= \{-2^{i-1}, -2^{i-1}q, -2^{i-1}q^2, \ldots, -2^{i-1}q^{2^{m-(i+1)}-1}\}. \end{aligned}$$

Remark 4.6. By definition of S_i it is clear that for $1 \leq i \leq m$,

$$S_i = X_i \cup X_i^*.$$

Note that integers of the type $q = 8\lambda + 3 (\lambda \geq 0)$ satisfy the above facts. In particular we may consider $q = 3$, then order $(3) = 2^{m-2}$ (modulo 2^m) for all $m \geq 3$. In this case observe the following.

Fact 4.7. For $1 \leq i \leq m - 2$,

$$3^{2^{m-(i+1)}} \equiv 1 \pmod{2^{m-i+1}}$$

or

$$2^{i-1}3^{2^{m-(i+1)}} \equiv 2^{i-1} \pmod{2^m}.$$

Fact 4.8. Since 3 is primitive root of unity modulo 4

$$3^2 \equiv 1(\text{mod } 2^2) \Rightarrow 2^{m-2}3^2 \equiv 2^{m-2}(\text{modulo } 2^m).$$

Fact 4.9. Since $3 \equiv -1$ (mod 2^2),

$$2^{m-2}.3 \equiv -2^{m-2}(\text{mod } 2^m)$$

and

$$2^{m-2}.3^2 \equiv -2^{m-2}.3(\text{modulo } 2^m).$$

Fact 4.10.

$$\begin{aligned} & 1 \equiv -1(\text{mod } 2), \\ & \Rightarrow 2^{m-1} \equiv -2^{m-1}(\text{mod } 2^m). \end{aligned}$$

Using the facts of §4, the 3-cyclotomic cosets modulo 2^m are given as follows:
For $1 \leq i \leq m-2$,

$$\begin{aligned} X_i &= \{2^{i-1}, 2^{i-1}.3, 2^{i-1}3^2, \ldots, 2^{i-1}3^{2^{m-(i+1)}-1}\}, \\ X_i^* &= \{-2^{i-1}, -2^{i-1}3, -2^{i-1}3^2, \ldots, -2^{i-1}3^{2^{m-(i+1)}-1}\} \end{aligned}$$

and

$$\begin{aligned} X_{m-1} &= X_{m-1}^* = \{2^{m-2}, 2^{m-2}.3\} = \{-2^{m-2}, -2^{m-2}.3\}, \\ X_m &= X_m^* = \{2^{m-1}\}. \end{aligned}$$

Example. Consider $q = 5$ and C_{2^5} be a cyclic group of order 2^5 generated by x. Then the q-cyclotomic cosets (modulo 2^5) are given by

$$\begin{aligned} X_1 &= \{1, 5, 25, 29, 17, 21, 9, 13\}, \\ X_1^* &= \{-1, -5, -25, -29, -17, -21, -9, -13\} \\ &= \{31, 27, 7, 3, 15, 11, 23, 19\}, \\ X_2 &= \{2, 10, 18, 26\}, \\ X_2^* &= \{30, 22, 14, 6\} \\ X_3 &= \{4, 20\}, \\ X_3^* &= \{28, 12\}, \\ X_4 &= \{8\}, \\ X_4^* &= \{24\}, \\ X_5 &= \{6\} = X_5^*. \end{aligned}$$

By Remark 4.6,

$$\begin{aligned} S_1 &= \{1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 31\}, \\ S_2 &= \{2, 6, 10, 14, 18, 22, 26, 30\}, \\ S_3 &= \{4, 12, 20, 28\}, \\ S_4 &= \{8, 24\}, \\ S_5 &= \{16\}. \end{aligned}$$

The six distinct idempotents in this case can be read as follows:

$$\begin{aligned}
e_0(x) &= \frac{1}{2^5}\{1+S_1+S_2+S_3+S_4+S_5\}(x),\\
e_1(x) &= \frac{1}{2^5}\{1+S_2+S_3+S_4+S_5-S_1\}(x),\\
e_2(x) &= \frac{1}{2^4}\{1+S_3+S_4+S_5-S_2\}(x),\\
e_3(x) &= \frac{1}{2^3}\{1+S_4+S_5-S_3\}(x),\\
e_4(x) &= \frac{1}{2^2}\{1+S_5-S_4\}(x),\\
e_5(x) &= \frac{1}{2}\{1-S_5\}(x).
\end{aligned}$$

The important parameters of the codes E_0, E_1, E_2, E_3, E_4, E_5 of length 2^5 over the field GF(5) are listed in the table below.

Code	Non-zero	Dimension K	Minimum distance, d	Generator polynomial, $g_i(x)$
E_0	$\alpha^0 = 1$	1	2^5	$1+x+x^2+\cdots+x^{31}$
E_1	α^{16}	1	2^5	$(1-x)\{1+S_2+S_3+S_4+S_5\}$
E_2	α^8, α^{24}	2	2^4	$(1-x^2)\{1+S_3+S_4+S_5\}$
E_3	$\alpha^4, \alpha^{12}, \alpha^{20}, \alpha^{28}$	4	2^3	$(1-x^4)\{1+x^8+x^{24}+x^{16}\}$
E_4	$\alpha^2, \alpha^6, \alpha^{10}, \alpha^{14}, \alpha^{18}, \alpha^{22}, \alpha^{26}, \alpha^{30}$	8	2^2	$(1-x^8)\{1+x^{16}\}$
E_5	$\alpha^j, j \in S_1$	16	2	$(1-x^{16})$

Example. Consider $q = 3$ and C_2^3 be a cyclic group of order 2^3 generated by x. Then the q-cyclotomic cosets (modulo 2^3) are given by

$$\begin{aligned}
X_1 &= \{1,3\},\\
X_1^* &= \{5,7\},\\
X_2 &= \{2,6\},\\
X_3 &= \{4\},\\
X_0 &= \{0\}.
\end{aligned}$$

The five primitive idempotents in the group algebra GF(3) C_2^3 are given with their non-zeroes:

Primitive idempotents	Non-zeroes
$e_0(x) = \frac{1}{2^3}\{1+X_1+X_1^*+X_2+X_3\}(x)$	α^0
$e_1(x) = \frac{1}{2^3}\{1+X_3+X_2-(X_1+X_1^*)\}(x)$	$\alpha^j, j \in X_3$
$e_2(x) = \frac{1}{2^2}\{1+X_3-X_2\}(x)$	$\alpha^j, j \in X_2$
$e_3(x) = \frac{1}{2^2}\{(1-X_3)-(X_1-X_1^*)\}(x)$	$\alpha^j, j \in X_1$
$e_4(x) = \frac{1}{2^2}\{(1-X_3)+(X_1-X_1^*\}(x)$	$\alpha^j, j \in X_1^*$

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Unitary tridiagonalization in $M(4, \mathbb{C})$

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Abstract. A question of interest in linear algebra is whether all $n \times n$ complex matrices can be unitarily tridiagonalized. The answer for all $n \neq 4$ (affirmative or negative) has been known for a while, whereas the case $n = 4$ seems to have remained open. In this paper we settle the $n = 4$ case in the affirmative. Some machinery from complex algebraic geometry needs to be used.

Keywords. Unitary tridiagonalization; 4×4 matrices; line bundle; degree; algebraic curve.

1. Main Theorem

Let $V = \mathbb{C}^n$, and $\langle \ , \ \rangle$ be the usual euclidean hermitian inner product on V. $U(V) = U(n)$ denotes the group of unitary automorphisms of V with respect to $\langle \ , \ \rangle$. $\{e_i\}_{i=1}^n$ will denote the standard orthonormal basis of V. $A \in M(n, \mathbb{C})$ will always denote an $n \times n$ complex matrix.

A matrix $A = [a_{ij}]$ is said to be *tridiagonal* if $a_{ij} = 0$ for all $1 \leq i, j \leq n$ such that $|i - j| \geq 2$. Then we have:

Theorem 1.1. *For $n \leq 4$, and $A \in M(n, \mathbb{C})$, there exists a unitary $U \in U(n)$ such that UAU^* is tridiagonal.*

Remark 1.2. The case $n = 3$, and counterexamples for $n \geq 6$, are due to Longstaff, [3]. In the paper [1], Fong and Wu construct counterexamples for $n = 5$, and provide a proof in certain special cases for $n = 4$. The article §4 of [1] poses the $n = 4$ case in general as an open question. Our main theorem above answers this question in the affirmative. In passing, we also provide another elementary proof for the $n = 3$ case.

2. Some Lemmas

We need some preliminary lemmas, which we collect in this section. In the sequel, we will also use the letter A to denote the unique linear transformation determined by the matrix $A = [a_{ij}]$ (satisfying $Ae_j = \sum_{i=1}^n a_{ij} e_i$).

Lemma 2.1. *Let $A \in M(n, \mathbb{C})$. For all n, the following are equivalent:*

(i) *There exists a unitary $U \in U(n)$ such that UAU^* is tridiagonal.*

(ii) *There exists a flag (= ascending sequence of* $\mathbb{C}$*-subspaces) of* $V = \mathbb{C}^n$:

$$0 = W_0 \subset W_1 \subset W_2 \subset \ldots \subset W_n = V$$

such that $\dim W_i = i,\ AW_i \subset W_{i+1}$ *and* $A^*W_i \subset W_{i+1}$ *for all* $0 \leq i \leq n-1$.

(iii) *There exists a flag in* V:

$$0 = W_0 \subset W_1 \subset W_2 \subset \ldots \subset W_n = V$$

such that $\dim W_i = i,\ AW_i \subset W_{i+1}$ *and* $A(W_{i+1}^{\perp}) \subset W_i^{\perp}$ *for all* $0 \leq i \leq n-1$.

Proof. (i) ⇒ (ii). Set $W_i = \mathbb{C}\text{-span}(f_1, f_2, \ldots, f_i)$, where $f_i = U^*e_i$ and e_i is the standard basis of $V = \mathbb{C}^n$. Since the matrix $[b_{ij}] := UAU^*$ is tridiagonal, we have

$$Af_i = b_{i-1,i}f_{i-1} + b_{ii}f_i + b_{i+1,i}f_{i+1}, \quad \text{for} \quad 1 \leq i \leq n$$

(where b_{ij} is understood to be $= 0$ for $i, j \leq 0$ or $\geq n+1$). Thus $AW_i \subset W_{i+1}$. Since $\{f_i\}_{i=1}^n$ is an orthonormal basis for $V = \mathbb{C}^n$, we also have

$$A^*f_i = \bar{b}_{i,i-1}f_{i-1} + \bar{b}_{ii}f_i + \bar{b}_{i,i+1}f_{i+1} \quad 1 \leq i \leq n$$

which shows $A^*(W_i) \subset W_{i+1}$ for all i as well, and (ii) follows.

(ii) ⇒ (iii). $A^*W_i \subset W_{i+1}$ implies $(A^*W_i)^{\perp} \supset W_{i+1}^{\perp}$ for $1 \leq i \leq n-1$. But since $(A^*W_i)^{\perp} = A^{-1}(W_i^{\perp})$, we have $A(W_{i+1}^{\perp}) \subset W_i^{\perp}$ for $1 \leq i \leq n-1$ and (iii) follows.

(iii) ⇒ (i). Inductively choose an orthonormal basis f_i of $V = \mathbb{C}^n$ so that W_i is the span of $\{f_1, \ldots, f_i\}$. Since $A(W_i) \subset W_{i+1}$, we have

$$Af_i = a_{1i}f_1 + a_{2i}f_2 + \cdots + a_{i+1,i}f_{i+1}. \tag{1}$$

Since $f_i \in (W_{i-1})^{\perp}$, and by hypothesis $A(W_{i-1}^{\perp}) \subset W_{i-2}^{\perp}$, and $W_{i-2}^{\perp} = \mathbb{C}\text{-span}(f_{i-1}, f_i, \ldots, f_n)$, we also have

$$Af_i = a_{i-1,i}f_{i-1} + a_{ii}f_i + \cdots + a_{ni}f_n \tag{2}$$

and by comparing the two equations (1), (2) above, it follows that

$$Af_i = a_{i-1,i}f_{i-1} + a_{ii}f_i + a_{i+1,i}f_{i+1}$$

for all i, and defining the unitary U by $U^*e_i = f_i$ makes UAU^* tridiagonal, so that (i) follows. □

Lemma 2.2. *Let* $n \leq 4$. *If there exists a 2-dimensional* $\mathbb{C}$*-subspace* W *of* $V = \mathbb{C}^n$ *such that* $AW \subset W$ *and* $A^*W \subset W$, *then* A *is unitarily tridiagonalizable.*

Proof. If $n \leq 2$, there is nothing to prove. For $n = 3$ or 4, the hypothesis implies that A maps $W^{\perp}$ onto itself. Then, in an orthonormal basis $\{f_i\}_{i=1}^n$ of V which satisfies $W = \mathbb{C}\text{-span}(f_1, f_2)$ and $W^{\perp} = \mathbb{C}\text{-span}(f_3, \ldots, f_n)$ the matrix of A is in $(1, 2)$ (resp. $(2, 2)$) block-diagonal form for $n = 3$ (resp. $n = 4$), which is clearly tridiagonal. □

Lemma 2.3. *Every matrix* $A \in M(3, \mathbb{C})$ *is unitarily tridiagonalizable.*

Proof. For $A \in M(3, \mathbb{C})$, consider the homogeneous cubic polynomial in $v = (v_1, v_2, v_3)$ given by

$$F(v_1, v_2, v_3) := \det(v, Av, A^*v).$$

Note $v \wedge Av \wedge A^*v = F(v_1, v_2, v_3)e_1 \wedge e_2 \wedge e_3$. By a standard result in dimension theory (see [4], p. 74, Theorem 5) each irreducible component of $V(F) \subset \mathbb{P}^2_{\mathbb{C}}$ is of dimension ≥ 1, and $V(F)$ is non-empty. Choose some $[v_1 : v_2 : v_3] \in V(F)$, and let $v = (v_1, v_2, v_3)$ which is non-zero. Then we have the two cases:

Case 1. v is a common eigenvector for A and A^*. Then the 2-dimensional subspace $W = (\mathbb{C}v)^{\perp}$ is an invariant subspace for both A and A^*, and applying the Lemma 2.2 to W yields the result.

Case 2. v is not a common eigenvector for A and A^*. Say it is not an eigenvector for A (otherwise interchange the roles of A and A^*). Set $W_1 = \mathbb{C}v$, $W_2 = \mathbb{C}\text{-span}(v, Av)$, $W_3 = V = \mathbb{C}^3$. Then $\dim W_i = i$, for $i = 1, 2, 3$, and the fact that $v \wedge Av \wedge A^*v = 0$ shows that $A^*W_1 \subset W_2$. Thus, by (ii) of Lemma 2.1, we are done. ☐

Note. From now on, $V = \mathbb{C}^4$ and $A \in M(4, \mathbb{C})$.

Lemma 2.4. *If A and A^* have a common eigenvector, then A is unitarily tridiagonalizable.*

Proof. If $v \neq 0$ is a common eigenvector for A and A^*, the 3-dimensional subspace $W = (\mathbb{C}v)^{\perp}$ is invariant under both A and A^*, and unitary tridiagonalization of $A_{|W}$ exists from the $n = 3$ case of Lemma 2.3 by a $U_1 \in U(W) = U(3)$. The unitary $U = 1 \oplus U_1$ is the desired unitary in $U(4)$ tridiagonalizing A. ☐

Lemma 2.5. *If the main theorem holds for all $A \in S$, where S is any dense (in the classical topology) subset of $M(4, \mathbb{C})$, then it holds for all $A \in M(4, \mathbb{C})$.*

Proof. This is a consequence of the compactness of the unitary group $U(4)$. Indeed, let T denote the closed subset of tridiagonal (with respect to the standard basis) matrices.

Let $A \in M(4, \mathbb{C})$ be any general element. By the density of S, there exist $A_n \in S$ such that $A_n \to A$. By hypothesis, there are unitaries $U_n \in U(4)$ such that $U_n A_n U_n^* = T_n$, where T_n are tridiagonal. By the compactness of $U(4)$, and by passing to a subsequence if necessary, we may assume that $U_n \to U \in U(4)$. Then $U_n A_n U_n^* \to UAU^*$. That is $T_n \to UAU^*$. Since T is closed, and $T_n \in T$, we have UAU^* is in T, viz., is tridiagonal. ☐

We shall now construct a suitable dense open subset $S \subset M(4, \mathbb{C})$, and prove tridiagonalizability for a general $A \in S$ in the remainder of this paper. More precisely:

Lemma 2.6. *There is a dense open subset $S \subset M(4, \mathbb{C})$ such that*:

(i) *A is nonsingular for all $A \in S$.*
(ii) *A has distinct eigenvalues for all $A \in S$.*

(iii) *For each* $A \in S$, *the element* $(t_0 I + t_1 A + t_2 A^*) \in M(4, \mathbb{C})$ *has rank* ≥ 3 *for all* $(t_0, t_1, t_2) \neq (0, 0, 0)$ *in* $\mathbb{C}^3$.

Proof. The subset of singular matrices in $M(4, \mathbb{C})$ is the complex algebraic subvariety of complex codimension one defined by $Z_1 = \{A : \det A = 0\}$. Let S_1, (which is just $GL(4, \mathbb{C})$) be its complement. Clearly S_1 is open and dense in the classical topology (in fact, also in the Zariski topology).

A matrix A has distinct eigenvalues iff its characteristic polynomial ϕ_A has distinct roots. This happens iff the discriminant polynomial of ϕ_A, which is a 4th degree homogeneous polynomial $\Delta(A)$ in the entries of A, is not zero. The zero set $Z_2 = V(\Delta)$ is again a codimension-1 subvariety in $M(4, \mathbb{C})$, so its complement $S_2 = (V(\Delta))^c$ is open and dense in both the classical and Zariski topologies.

To enforce (iii), we claim that the set defined by

$$Z_3 := \{A \in M(4, \mathbb{C}) : \operatorname{rank}(t_0 I + t_1 A + t_2 A^*) \leq 2 \text{ for some } (t_0, t_1, t_2) \neq (0, 0, 0) \text{ in } \mathbb{C}^3\}$$

is a proper *real* algebraic subset of $M(4, \mathbb{C})$. The proof hinges on the fact that three general cubic curves in $\mathbb{P}^2_{\mathbb{C}}$ having a point in common imposes an algebraic condition on their coefficients.

Indeed, saying that $\operatorname{rank}(t_0 I + t_1 A + t_2 A^*) \leq 2$ for some $(t_0, t_1, t_2) \neq (0, 0, 0)$ is equivalent to saying that the third exterior power $\bigwedge^3 (t_0 I + t_1 A + t_2 A^*)$ is the zero map, for some $(t_0, t_1, t_2) \neq 0$. This is equivalent to demanding that there exist a $(t_0, t_1, t_2) \neq 0$ such that the determinants of all the 3×3-minors of $(t_0 I + t_1 A + t_2 A^*)$ are zero.

Note that the (determinants of) the (3×3)-minors of $(t_0 I + t_1 A + t_2 A^*)$, denoted as $M_{ij}(A, t)$ (where the ith row and jth column are deleted) are complex valued, complex algebraic and $\mathbb{C}$-homogeneous of degree 3 in $t = (t_0, t_1, t_2)$, with coefficients real algebraic of degree 3 in the variables $(A_{ij}, \bar{A}_{ij})$ (or, equivalently, in $\operatorname{Re} A_{ij}$, $\operatorname{Im} A_{ij}$), where $A = [A_{ij}]$.

We know that the space of all homogeneous polynomials of degree 3 with complex coefficients in (t_0, t_1, t_2) (up to scaling) is parametrized by the projective space $\mathbb{P}^9_{\mathbb{C}}$ (the Veronese variety, see [4], p. 52). We first consider the complex algebraic variety:

$$X = \{(P, Q, R, [t]) \in \mathbb{P}^9_{\mathbb{C}} \times \mathbb{P}^9_{\mathbb{C}} \times \mathbb{P}^9_{\mathbb{C}} \times \mathbb{P}^2_{\mathbb{C}} : P(t) = Q(t) = R(t) = 0\},$$

where $[t] := [t_0 : t_1 : t_2]$, and (P, Q, R) denotes a triple of homogeneous polynomials. This is just the subset of those $(P, Q, R, [t])$ in the product $\mathbb{P}^9_{\mathbb{C}} \times \mathbb{P}^9_{\mathbb{C}} \times \mathbb{P}^9_{\mathbb{C}} \times \mathbb{P}^2_{\mathbb{C}}$ such that the point $[t]$ lies on all three of the plane cubic curves $V(P)$, $V(Q)$, $V(R)$. Since X is defined by multihomogenous degree $(1, 1, 1, 3)$ equations, it is a complex algebraic subvariety of the quadruple product. Its image under the first projection $Y := \pi_1(X) \subset \mathbb{P}^9_{\mathbb{C}} \times \mathbb{P}^9_{\mathbb{C}} \times \mathbb{P}^9_{\mathbb{C}}$ is therefore an algebraic subvariety inside this triple product (see [4], p. 58, Theorem 3). Y is a proper subvariety because, for example, the cubic polynomials $P = t_0^3$, $Q = t_1^3$, $R = t_2^3$ have no common non-zero root.

Denote pairs (i, j) with $1 \leq i, j \leq 4$ by capital letters like I, J, K etc. From the minorial determinants $M_I(A, t)$, we can define various *real algebraic* maps:

$$\begin{aligned} \Theta_{IJK} : M(4, \mathbb{C}) &\to \mathbb{P}^9_{\mathbb{C}} \times \mathbb{P}^9_{\mathbb{C}} \times \mathbb{P}^9_{\mathbb{C}} \\ A &\mapsto (M_I(A, t), M_J(A, t), M_K(A, t)) \end{aligned}$$

for I, J, K distinct. Clearly, $\bigwedge^3(t_0 I + t_1 A + t_2 A^*) = 0$ for some $t = (t_0, t_1, t_2) \neq (0, 0, 0)$ iff $\Theta_{IJK}(A)$ lies in the complex algebraic subvariety Y of $\mathbb{P}^9_{\mathbb{C}} \times \mathbb{P}^9_{\mathbb{C}} \times \mathbb{P}^9_{\mathbb{C}}$, for all I, J, K distinct. Hence the subset $Z_3 \subset M(4, \mathbb{C})$ defined above is the intersection:

$$Z_3 = \bigcap_{I,J,K} \Theta^{-1}_{IJK}(Y),$$

where I, J, K runs over all distinct triples of pairs (i, j), $1 \leq i, j \leq 4$.

We claim that Z_3 is a proper real algebraic subset of $M(4, \mathbb{C})$. Clearly, since each $M_I(A, t)$ is real algebraic in the variables $\operatorname{Re} A_{ij}$, $\operatorname{Im} A_{ij}$ the map Θ_{IJK} is real algebraic. Since Y is complex and hence real algebraic, its inverse image $\Theta^{-1}_{IJK}(Y)$, defined by the real algebraic equations obtained upon substitution of the components $M_I(A, t)$, $M_J(A, t)$, $M_K(A, t)$ in the equations that define Y, is also real algebraic. Hence the set Z_3 is a real algebraic subset of $M(4, \mathbb{C})$.

To see that Z_3 is a *proper* subset of $M(4, \mathbb{C})$, we simply consider the matrix (defined with respect to the standard orthonormal basis $\{e_i\}_{i=1}^4$ of $\mathbb{C}^4$):

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

For $t = (t_0, t_1, t_2) \neq 0$, we see that

$$t_0 I + t_1 A + t_2 A^* = \begin{bmatrix} t_0 & t_1 & 0 & 0 \\ t_2 & t_0 & t_1 & 0 \\ 0 & t_2 & t_0 & t_1 \\ 0 & 0 & t_2 & t_0 \end{bmatrix}.$$

For the above matrix the minorial determinant $M_{41}(A, t) = t_1^3$, whereas $M_{14}(A, t) = t_2^3$. The only common zeros to these two minorial determinants are points $[t_0 : 0 : 0]$. Setting $t_1 = t_2 = 0$ in the matrix above gives $M_{ii}(A, t) = t_0^3$ for $1 \leq i \leq 4$. Thus t_0 must also be 0 for all the minorial determinants to vanish. Hence the matrix A above lies outside the real algebraic set Z_3.

It is well-known that a proper real algebraic subset in euclidean space cannot have a non-empty interior. Thus the complement Z_3^c is dense and open in the classical and real-Zariski topologies. Take $S_3 = Z_3^c$.

Finally, set

$$S := S_1 \cap S_2 \cap S_3 = \left(\bigcup_{i=1}^{3} Z_i \right)^c$$

which is also open and dense in the classical topology in $M(4, \mathbb{C})$. Hence the lemma. □

Remark 2.7. One should note here that for *each* matrix $A \in M(4, \mathbb{C})$, there will be at least a curve of points $[t] = [t_0 : t_1 : t_2] \in \mathbb{P}^2_{\mathbb{C}}$ (defined by the vanishing of $\det(t_0 I + t_1 A + t_2 A^*)$), on which $(t_0 I + t_1 A + t_2 A^*)$ is singular. Similarly for each A there is at least a curve of points on which the trace $\operatorname{tr}\left(\bigwedge^3(t_0 I + t_1 A + t_2 A^*)\right)$ vanishes, and so a non-empty (and generally a finite) set on which *both* these polynomials vanish, by dimension theory ([4], Theorem 5, p. 74). Thus for *each* $A \in M(4, \mathbb{C})$, there is at least a non-empty finite set of points $[t]$ such that $(t_0 I + t_1 A + t_2 A^*)$ has 0 as a repeated eigenvalue. For example, for the matrix A constructed at the end of the previous lemma, we see that the matrix $(t_0 I + t_1 A + t_2 A^*)$ is strictly upper-triangular and thus has 0 as an eigenvalue of multiplicity 4 for all $(0, t_1, 0) \neq 0$, but nevertheless has rank 3 for all $(t_0, t_1, t_2) \neq (0, 0, 0)$.

Indeed, as (iii) of the lemma above shows, for A in the open dense subset S, the kernel $\ker(t_0 I + t_1 A + t_2 A^*)$ is at most 1-dimensional *for all* $[t] = [t_0 : t_1 : t_2] \in \mathbb{P}^2_{\mathbb{C}}$.

3. The varieties C, Γ, and D

Notation 3.1. In the light of Lemmas 2.5 and 2.6 above, we shall henceforth assume $A \in S$. As is easily verified, this implies $A^* \in S$ as well. We will also henceforth assume, in view of Lemma 2.4 above, that *A and A^* have no common eigenvectors*. (For example, this rules out A being normal, in which case we know that the main result for A is true by the spectral theorem.) Also, in view of Lemma 2.2, we shall assume that A and A^* do not have a common 2-dimensional invariant subspace.

In $\mathbb{P}^3_{\mathbb{C}}$, the complex projective space of $V = \mathbb{C}^4$, we denote the equivalence class of $v \in V \setminus 0$ by $[v]$. For a $[v] \in \mathbb{P}^3_{\mathbb{C}}$, we define $W([v])$ (or simply $W(v)$ when no confusion is likely) by

$$W([v]) := \mathbb{C}\text{-span}(v, Av, A^*v).$$

Since we are assuming that A and A^* have no common eigenvectors, we have $\dim W([v]) \geq 2$ for all $[v] \in \mathbb{P}^3_{\mathbb{C}}$.

Denote the four distinct points in $\mathbb{P}^3_{\mathbb{C}}$ representing the four linearly independent eigenvectors of A (resp. A^*) by E (resp. E^*). By our assumption above, $E \cap E^* = \phi$.

Lemma 3.2. *Let* $A \in M(4, \mathbb{C})$ *be as in* 3.1 *above. Then the closed subset*:

$$C = \{[v] \in \mathbb{P}^3_{\mathbb{C}} : v \wedge Av \wedge A^*v = 0\}$$

is a closed projective variety. This variety C is precisely the subset of $[v] \in \mathbb{P}^3_{\mathbb{C}}$ *for which the dimension* $\dim W([v]) = \dim(\mathbb{C}\text{-span}\{v, Av, A^*v\})$ *is exactly* 2.

Proof. That C is a closed projective variety is clear from the fact that it is defined as the set of common zeros of all the four (3×3)-minorial determinants of the (3×4)-matrix

$$\Lambda := \begin{bmatrix} v \\ Av \\ A^*v \end{bmatrix}$$

(which are all degree-3 homogeneous polynomials in the components of v with respect to some basis). Also C is nonempty since it contains $E \cup E^*$.

Also, since A and A^* are nonsingular by the assumptions in 3.1, the wedge product $v \wedge Av \wedge A^*v$ of the three non-zero vectors v, Av, A^*v vanishes precisely when the space $W([v]) = \mathbb{C}\text{-span}(v, Av, A^*v)$ is of dimension ≤ 2. Since by 3.1, A, A^* have no common eigenvectors, the dimension $\dim W([v]) \geq 2$ for all $[v] \in \mathbb{P}^3_{\mathbb{C}}$, so C is precisely the locus of $[v] \in \mathbb{P}^3_{\mathbb{C}}$ for which the space $W([v])$ is 2-dimensional. □

Now we shall show that for A as in 3.1, the variety C defined above is of pure dimension one. For this, we need to define some more associated algebraic varieties and regular maps.

DEFINITION 3.3

Let us define the bilinear map:

$$\begin{array}{rcl} B : \mathbb{C}^4 \times \mathbb{C}^3 & \to & \mathbb{C}^4 \\ (v, t_0, t_1, t_2) & \mapsto & B(v, t) := (t_0 I + t_1 A + t_2 A^*)v. \end{array}$$

We then have the linear maps $B(v, -) : \mathbb{C}^3 \to \mathbb{C}^4$ for $v \in \mathbb{C}^4$ and $B(-, t) : \mathbb{C}^4 \to \mathbb{C}^3$ for $t \in \mathbb{C}^3$.

Note that the image $\operatorname{Im} B(v, -)$ is the span of $\{v, Av, A^*v\}$, which was defined to be $W(v)$. For a fixed t, denote the kernel

$$K(t) := \ker(B(-, t) : \mathbb{C}^4 \to \mathbb{C}^4).$$

Denoting $[t_0 : t_1 : t_2]$ by $[t]$ and $[v_1 : v_2 : v_3 : v_4]$ by $[v]$ for brevity, we define

$$\Gamma := \{([v], [t]) \in \mathbb{P}^3_{\mathbb{C}} \times \mathbb{P}^2_{\mathbb{C}} : B(v, t) = 0\}.$$

Finally, define the variety D by

$$D \subset \mathbb{P}^2_{\mathbb{C}} := \{[t] \in \mathbb{P}^2_{\mathbb{C}} : \ \det B(-, t) = \det(t_0 I + t_1 A + t_2 A^*) = 0\}.$$

Let

$$\pi_1 : \mathbb{P}^3_{\mathbb{C}} \times \mathbb{P}^2_{\mathbb{C}} \to \mathbb{P}^3_{\mathbb{C}}, \quad \pi_2 : \mathbb{P}^3_{\mathbb{C}} \times \mathbb{P}^2_{\mathbb{C}} \to \mathbb{P}^2_{\mathbb{C}}$$

denote the two projections.

Lemma 3.4. *We have the following facts*:

(i) $\pi_1(\Gamma) = C$, *and* $\pi_2(\Gamma) = D$.

(ii) $\pi_1 : \Gamma \to C$ *is* 1-1, *and the map* g *defined by*

$$g := \pi_2 \circ \pi_1^{-1} : C \to D$$

is a regular map so that Γ *is the graph of* g *and isomorphic as a variety to* C.

(iii) $D \subset \mathbb{P}^2_{\mathbb{C}}$ *is a plane curve, of pure dimension one. The map* $\pi_2 : \Gamma \to D$ *is* 1-1, *and the map* $\pi_1 \circ \pi_2^{-1} : D \to C$ *is the regular inverse of the regular map* g *defined above in* (ii). *Again* Γ *is also the graph of this regular inverse* g^{-1}, *and* D *and* Γ *are isomorphic as varieties. In particular,* C *and* D *are isomorphic as varieties, and thus* C *is a curve in* $\mathbb{P}^3_{\mathbb{C}}$ *of pure dimension one.*

(iv) *Inside* $\mathbb{P}^3_{\mathbb{C}} \times \mathbb{P}^2_{\mathbb{C}}$, *each irreducible component of the intersection of the four divisors* $D_i := (B_i(v, t) = 0)$ *for* $i = 1, 2, 3, 4$ (*where* $B_i(v, t)$ *is the* i*-th component of* $B(v, t)$ *with respect to a fixed basis of* $\mathbb{C}^4$) *occurs with multiplicity* 1. (*Note that* Γ *is set-theoretically the intersection of these four divisors, by definition*).

Proof. It is clear that $\pi_1(\Gamma) = C$, because $B(v,t) = t_0 v + t_1 Av + t_2 A^* v = 0$ for some $[t_0 : t_1 : t_2] \in \mathbb{P}^2_{\mathbb{C}}$ iff $\dim W(v) \leq 2$, and since A and A^* have no common eigenvectors, this means $\dim W(v) = 2$. That is, $[v] \in C$.

Clearly $[t] \in \pi_2(\Gamma)$ iff there exists a $[v] \in \mathbb{P}^3_{\mathbb{C}}$ such that $B(v,t) = 0$. That is, iff $\dim \ker B(-,t) \geq 1$, that is, iff

$$G(t_0, t_1, t_2) := \det B(-,t) = 0.$$

Thus $D = \pi_2(\Gamma)$ and is defined by a single degree 4 homogeneous polynomial G inside $\mathbb{P}^2_{\mathbb{C}}$. It is a curve of pure dimension 1 in $\mathbb{P}^2_{\mathbb{C}}$ by standard dimension theory (see [4], p. 74, Theorem 5) because, for example $[1 : 0 : 0] \notin D$ so $D \neq \mathbb{P}^2_{\mathbb{C}}$. So $\pi_2(\Gamma) = D$, and this proves (i).

To see (ii), for a given $[v] \in C$, we claim there is exactly one $[t]$ such that $([v],[t]) \in \Gamma$. Note that $([v],[t]) \in \Gamma$ iff the linear map:

$$\begin{aligned} B(v,-) : \mathbb{C}^3 &\rightarrow \mathbb{C}^4 \\ t &\mapsto (t_0 I + t_1 A + t_2 A^*) v \end{aligned}$$

has a non-trivial kernel containing the line $\mathbb{C}t$. That is, $\dim \operatorname{Im} B(v,-) \leq 2$. But the image $\operatorname{Im} B(v,-) = W(v)$, which is of dimension 2 for all $v \in C$ by our assumptions. Thus its kernel must be exactly one dimensional, defined by $\ker B(v,-) = \mathbb{C}t$. Thus $([v],[t])$ is the unique point in Γ lying in $\pi_1^{-1}[v]$, viz. for each $[v] \in C$, the vertical line $[v] \times \mathbb{P}^2_{\mathbb{C}}$ intersects Γ in a single point, call it $([v], g[v])$. So $\pi_1 : \Gamma \rightarrow C$ is 1-1, and Γ is the graph of a map $g : C \rightarrow D$. Since $g([v]) = \pi_2 \pi_1^{-1}([v])$ for $[v] \in C$, and Γ is algebraic, g is a regular map. This proves (ii).

To see (iii), note that for $[t] \in D$, by definition, the dimension $\dim \ker B(-,t) \geq 1$. By the fact that $A \in S$, and (iii) of Lemma 2.6, we know that $\dim \ker B(-,t) \leq 1$ for all $[t] \in \mathbb{P}^2_{\mathbb{C}}$. Thus, denoting $K(t) := \ker B(-,t)$ for $[t] \in D$, we have

$$\dim K(t) = 1 \quad \text{for all} \quad t \in D. \tag{3}$$

Hence we see that the unique projective line $[v]$ corresponding to $\mathbb{C}v = K(t)$ yields the unique element of C, such that $([v],[t]) \in \Gamma$. Thus $\pi_2 : \Gamma \rightarrow D$ is 1-1, and the regular map $\pi_1 \circ \pi_2^{-1} : D \rightarrow C$ is the regular inverse to the map g of (ii) above. Γ is thus also the graph of g^{-1} and, in particular, is isomorphic to D. Since g is an isomorphism of curves, and D is of pure dimension 1, it follows that C is of pure dimension one. This proves (iii).

To see (iv), we need some more notation.

Note that $D \subset \mathbb{P}^2_{\mathbb{C}} \setminus \{[1;0;0]\}$, (because there exists no $[v] \in \mathbb{P}^3_{\mathbb{C}}$ such that $I.v = 0$!). Thus there is a regular map:

$$\begin{aligned} \theta : D &\rightarrow \mathbb{P}^1_{\mathbb{C}} \\ [t_0 : t_1 : t_2] &\mapsto [t_1 : t_2]. \end{aligned} \tag{4}$$

Let $\Delta(t_1, t_2)$ be the discriminant polynomial of the characteristic polynomial $\phi_{t_1 A + t_2 A^*}$ of $t_1 A + t_2 A^*$. Clearly $\Delta(t_1, t_2)$ is a homogeneous polynomial of degree 4 in (t_1, t_2), and it is not the zero polynomial because, for example, $\Delta(1,0) \neq 0$, for $\Delta(1,0)$ is the discriminant of ϕ_A, which has distinct roots (=the distinct eigenvalues of A) by the assumptions 3.1 on A. Let $\Sigma \subset \mathbb{P}^1_{\mathbb{C}}$ be the zero locus of Δ, which is a finite set of points. Note that the fibre $\theta^{-1}([1 : \mu])$ consists of all $[t : 1 : \mu] \in D$ such that $-t$ is an eigenvalue of $A + \mu A^*$, which are at most four in number. Similarly the fibres $\theta^{-1}([\lambda : 1])$ are also finite. Thus the subset of D defined by

$$F := \theta^{-1}(\Sigma)$$

is a finite subset of D. F is precisely the set of points $[t] = [t_0 : t_1 : t_2]$ such that $B(-, t) = (t_0 I + t_1 A + t_2 A^*)$ has 0 as a repeated eigenvalue.

Since $\pi_2 : \Gamma \to D$ is 1-1, the inverse image:

$$F_1 = \pi_2^{-1}(F) \subset \Gamma$$

is a finite subset of Γ.

We will now prove that for each irreducible component Γ_α of Γ, and each point $x = ([a], [b])$ in $\Gamma_\alpha \setminus F_1$, the four equations $\{B_i(v, t) = 0\}_{i=1}^4$ are the generators of the ideal of the variety Γ_α in an affine neighbourhood of x, where $B_i(v, t)$ are the components of $B(v, t)$ with respect to a fixed basis of $\mathbb{C}^4$. Since F_1 is a finite set, this will prove (iv), because the multiplicity of Γ_α in the intersection cycle of the four divisors $D_i = (B_i(v, t) = 0)$ in $\mathbb{P}^3_{\mathbb{C}} \times \mathbb{P}^2_{\mathbb{C}}$ is determined by generic points on Γ_α, for example all points of $\Gamma_\alpha \setminus F_1$. We will prove this by showing that for $x = ([a], [b]) \in \Gamma_\alpha \setminus F_1$, the four divisors $(B_i(v, t) = 0)$ intersect transversely at x.

So let Γ_α be some irreducible component of Γ, with $x = ([a], [b]) \in \Gamma_\alpha \setminus F_1$.

Fix an $a \in \mathbb{C}^4$ representing $[a] \in C_\alpha := \pi_1(\Gamma_\alpha)$, and also fix $b \in \mathbb{C}^3$ representing $[b] = g([a]) \in g(C_\alpha)$. Also fix a 3-dimensional linear complement $V_1 := T_{[a]}(\mathbb{P}^3_{\mathbb{C}}) \subset \mathbb{C}^4$ to a and similarly, fix a 2-dimensional linear complement $V_2 = T_{[b]}(\mathbb{P}^2_{\mathbb{C}}) \subset \mathbb{C}^3$ to b. (The notation comes from the fact that $T_{[v]}(\mathbb{P}^n_{\mathbb{C}}) \simeq \mathbb{C}^{n+1}/\mathbb{C}v$, which we are identifying non-canonically with these respective complements V_i.) These complements also provide local coordinates in the respective projective spaces as follows. Set coordinate charts ϕ around $[a] \in \mathbb{P}^3_{\mathbb{C}}$ by $[v] = \phi(u) := [a+u]$, and ψ around $[b] \in \mathbb{P}^2_{\mathbb{C}}$ by $[t] = \psi(s) := [b+s]$, where $u \in V_1 \simeq \mathbb{C}^3$, and $s \in V_2 \simeq \mathbb{C}^2$. The images $\phi(V_1)$ and $\psi(V_2)$ are affine neighbourhoods of $[a]$ and $[b]$ respectively. These charts are like 'stereographic projection' onto the tangent space and depend on the initial choice of a (resp. b) representing $[a]$ (resp. $[b]$), and are *not* the standard coordinate systems on projective space, but more convenient for our purposes.

Then the local affine representation of $B(v, t)$ on the affine open $V_1 \times V_2 = \mathbb{C}^3 \times \mathbb{C}^2$, which we denote by β, is given by

$$\beta(u, s) := B(a + u, b + s).$$

Note that $\ker B(a, -) = \mathbb{C}b$, where $[b] = g([a])$, so that $B(a, -)$ passes to the quotient as an isomorphism:

$$B(a, -) : V_2 \xrightarrow{\sim} W(a), \tag{5}$$

where $W(a)$ is 2-dimensional.

Similarly, since $B(-, b)$ has one dimensional kernel $\mathbb{C}a = K(b) \subset \mathbb{C}^4$, by (3) above, we also have the other isomorphism:

$$B(-, b) : V_1 \xrightarrow{\sim} \operatorname{Im} B(-, b), \tag{6}$$

where $\operatorname{Im} B(-, b)$ is 3-dimensional, therefore.

Now one can easily calculate the derivative $D\beta(0,0)$ of β at $(u,s)=(0,0)$. Let $(X,Y)\in V_1\times V_2$. Then, by bilinearity of B, we have

$$\begin{aligned}\beta(X,Y)-\beta(0,0) &= B(a+X,b+Y)-B(a,b)\\ &= B(X,b)+B(a,Y)+B(X,Y).\end{aligned}$$

Now since $B(X,Y)$ is quadratic, it follows that

$$\begin{aligned}D\beta(0,0):V_1\times V_2 &\to \mathbb{C}^4\\ (X,Y) &\mapsto B(X,b)+B(a,Y).\end{aligned}\tag{7}$$

By eqs (5) and (6) above, we see that the image of $D\beta(0,0)$ is precisely $\operatorname{Im}B(-,b)+W(a)$.

Claim. For $([a],[b])\in\Gamma_\alpha\setminus F_1$, the space $\operatorname{Im}B(-,b)+W(a)$ is all of $\mathbb{C}^4$.

Proof of Claim. Denote $T:=B(-,b)$ for brevity. Clearly $a\in W(a)$ by definition of $W(a)$. Also, $a\in\ker T=K(b)$. We claim that a is not in the image of T. For, if $a\in\operatorname{Im}T$, we would have $a=Tw$ for some $w\notin K(b)=\ker T$ and $w\neq 0$. In fact w is not a multiple of a since $Tw=a\neq 0$ whereas $a\in\ker T$. Thus we would have $T^2w=0$, and completing $f_1=a=Tw$, $f_2=w$ to a basis $\{f_i\}_{i=1}^4$ of $\mathbb{C}^4$, the matrix of T with respect to this basis would be of the form:

$$\begin{bmatrix}0&1&*&*\\0&0&*&*\\0&0&*&*\\0&0&*&*\end{bmatrix}.$$

Thus $T=B(-,b)$ would have 0 as a repeated eigenvalue. But we have stipulated that $([a],[b])\notin F_1$, so that $[b]\notin F$, and hence $B(-,b)$ does not have 0 as a repeated eigenvalue. Hence the non-zero vector $a\in W(a)$ is not in $\operatorname{Im}T$. Since $\operatorname{Im}T$ is 3-dimensional, we have $\mathbb{C}^4=\operatorname{Im}T+W(a)$, and this proves the claim. □

In conclusion, all the points of $\Gamma_\alpha\setminus F_1$ are in fact smooth points of Γ_α, and the local equations for Γ_α in a small neighbourhood of such a point are precisely the four equations $\beta_i(u,s)=0$, $1\leq i\leq 4$. This proves (iv), and the lemma. □

4. Some algebraic bundles

We construct an algebraic line bundle with a (regular) global section over C. By showing that this line bundle has positive degree, we will conclude that the section has zeroes in C. Any zero of this section will yield a flag of the kind required by Lemma 2.1. One of the technical complications is that none of the bundles we define below are allowed to use the hermitian metric on V, orthogonal complements, orthonormal bases etc., because we wish to remain in the $\mathbb{C}$-algebraic category. As a general reference for this section and the next, the reader may consult [2].

DEFINITION 4.1

For $0 \neq v \in V = \mathbb{C}^4$, we will denote the point $[v] \in \mathbb{P}^3_{\mathbb{C}}$ by v, whenever no confusion is likely, to simplify notation. We have already denoted the vector subspace $\mathbb{C}\text{-span}(v, Av, A^*v) \subset \mathbb{C}^4$ as $W(v)$. Further define $W_3(v) := W(v) + AW(v)$, and $\widetilde{W}_3(v) := W(v) + A^*W(v)$. Clearly both $W_3(v)$ and $\widetilde{W}_3(v)$ contain $W(v)$.

Since A and A^* have no common eigenvectors, we have dim $W(v) \geq 2$ for all $v \in \mathbb{P}^3_{\mathbb{C}}$, and dim $W(v) = 2$ for all $v \in C$, because of the defining equation $v \wedge Av \wedge A^*v = 0$ of C. Also, since dim $W(v) = 2 =$ dim $AW(v)$ for $v \in C$, and since $0 \neq Av \in W(v) \cap AW(v)$, we have dim $W_3(v) \leq 3$ for all $v \in C$. Similarly dim $\widetilde{W}_3(v) \leq 3$ for all $v \in C$.

If there exists a $v \in C$ such that dim $W_3(v) = 2$, then we are done. For, in this case $W_3(v)$ must equal $W(v)$ since it contains $W(v)$. Then the dimension dim $\widetilde{W}_3(v) = 2$ or $= 3$. If it is 2, $W(v)$ will be a 2-dimensional invariant space for both A and A^*, and the main theorem will follow by Lemma 2.2. If dim $\widetilde{W}_3(v) = 3$, then the flag:

$$0 = W_0 \subset W_1 = \mathbb{C}v \subset W_2 = W(v) \subset W_3 = \widetilde{W}_3(v) \subset W_4 = V$$

satisfies the requirements of (ii) in Lemma 2.1, and we are done. Similarly, if there exists a $v \in C$ with dim $\widetilde{W}_3(v) = 2$, we are again done. Hence we may assume that:

$$\dim W_3(v) = \dim \widetilde{W}_3(v) = 3 \quad \text{for all} \quad v \in C. \tag{8}$$

In the light of the above, we have the following:

Remark 4.2. We are reduced to the situation where the following condition holds: For each $v \in C$, dim $W(v) = 2$, dim $W_3(v) =$ dim $\widetilde{W}_3(v) = 3$.

Now our main task is to prove that there exists a $v \in C$ such that the two 3-dimensional subspaces $W_3(v)$ and $\widetilde{W}_3(v)$ are the *same*. In that event, the flag

$$0 = W_0 \subset W_1 = \mathbb{C}v \subset W_2 = W(v) \subset W_3 = W(v) + AW(v) = W(v) + A^*W(v) \subset W_4 = V$$

will meet the requirements of (ii) of the Lemma 2.1. The remainder of this discussion is aimed at proving this.

DEFINITION 4.3

Denote the trivial rank 4 algebraic bundle on $\mathbb{P}^3_{\mathbb{C}}$ by $\mathcal{O}^4_{\mathbb{P}^3_{\mathbb{C}}}$, with fibre $V = \mathbb{C}^4$ at each point (following standard algebraic geometry notation). Similarly, $\mathcal{O}^4_C$ is the trivial bundle on C. In $\mathcal{O}^4_{\mathbb{P}^3_{\mathbb{C}}}$, there is the tautological line-subbundle $\mathcal{O}_{\mathbb{P}^3_{\mathbb{C}}}(-1)$, whose fibre at v is $\mathbb{C}v$. Its restriction to the curve C is denoted as $\mathcal{W}_1 := \mathcal{O}_C(-1)$.

There are also the line subbundles $A\mathcal{O}_{\mathbb{P}^3_{\mathbb{C}}}(-1)$ (respectively $A^*\mathcal{O}_{\mathbb{P}^3_{\mathbb{C}}}(-1)$) of $\mathcal{O}^4_{\mathbb{P}^3_{\mathbb{C}}}$, whose fibre at v is Av (respectively A^*v). Both are isomorphic to $\mathcal{O}_{\mathbb{P}^3_{\mathbb{C}}}(-1)$ (via the global linear automorphisms A (resp. A^*) of V). Similarly, their restrictions $A\mathcal{O}_C(-1)$, $A^*\mathcal{O}_C(-1)$, both isomorphic to $\mathcal{O}_C(-1)$. *Note that throughout what follows, bundle isomorphism over any variety X will mean algebraic isomorphism, i.e. isomorphism of the corresponding sheaves of algebraic sections as $\mathcal{O}_X$-modules.*

Denote the rank 2 algebraic bundle with fibre $W(v) \subset V$ at $v \in C$ as $\mathcal{W}_2$. It is an algebraic sub-bundle of $\mathcal{O}_C^4$, for its sheaf of sections is the restriction of the subsheaf

$$\mathcal{O}_{\mathbb{P}^3_{\mathbb{C}}}(-1) + A\mathcal{O}_{\mathbb{P}^3_{\mathbb{C}}}(-1) + A^*\mathcal{O}_{\mathbb{P}^3_{\mathbb{C}}}(-1) \subset \mathcal{O}^4_{\mathbb{P}^3_{\mathbb{C}}}$$

to the curve C, which is precisely the subvariety of $\mathbb{P}^3_{\mathbb{C}}$ on which the sheaf above is locally free of rank 2 (=rank 2 algebraic bundle).

Denote the rank 3 algebraic sub-bundle of $\mathcal{O}_C^4$ with fibre $W_3(v) = W(v) + AW(v)$ (respectively $\widetilde{W}_3(v) = W(v) + A^*W(v)$) by $\mathcal{W}_3$ (respectively $\widetilde{\mathcal{W}}_3$). Both $\mathcal{W}_3$ and $\widetilde{\mathcal{W}}_3$ are of rank 3 on C because of Remark 4.2 above, and both contain $\mathcal{W}_2$ as a sub-bundle. We denote the line bundles $\bigwedge^2 \mathcal{W}_2$ by $\mathcal{L}_2$, and $\bigwedge^3 \mathcal{W}_3$ (resp. $\bigwedge^3 \widetilde{\mathcal{W}}_3$) by $\mathcal{L}_3$ (resp. $\widetilde{\mathcal{L}}_3$). Then $\mathcal{L}_2$ is a line sub-bundle of $\bigwedge^2 \mathcal{O}_C^4$, and $\mathcal{L}_3$, $\widetilde{\mathcal{L}}_3$ are line sub-bundles of $\bigwedge^3 \mathcal{O}_C^4$.

Finally, for X any variety, with a bundle $\mathcal{E}$ on X which is a sub-bundle of a trivial bundle $\mathcal{O}_X^m$, the *annihilator* of $\mathcal{E}$ is defined as

$$\mathrm{Ann}\mathcal{E} = \{\phi \in \hom_X(\mathcal{O}_X^m, \mathcal{O}_X) : \phi(\mathcal{E}) = 0\}.$$

Clearly, by taking $\hom_X(-\,, \mathcal{O}_X)$ of the exact sequence

$$0 \to \mathcal{E} \to \mathcal{O}_X^m \to \mathcal{O}_X^m/\mathcal{E} \to 0,$$

the bundle

$$\mathrm{Ann}\mathcal{E} \simeq \hom_X(\mathcal{O}_X^m/\mathcal{E}, \mathcal{O}_X) = (\mathcal{O}_X^m/\mathcal{E})^*,$$

where $*$ always denotes the (complex) dual bundle.

Lemma 4.4. *Denote the bundle* $\mathcal{W}_3/\mathcal{W}_2$ *(resp.* $\widetilde{\mathcal{W}}_3/\mathcal{W}_2$*) by* Λ *(resp.* $\widetilde{\Lambda}$*). Then we have the following identities of bundles on* C:

(i)

$$\begin{aligned}
0 &\to \mathcal{W}_2 \to \mathcal{W}_3 \to \Lambda \to 0\\
0 &\to \mathcal{W}_2 \to \widetilde{\mathcal{W}}_3 \to \widetilde{\Lambda} \to 0\\
0 &\to \mathcal{L}_3 \xrightarrow{i} \mathrm{Ann}\mathcal{W}_2 \xrightarrow{\pi} \Lambda^* \to 0\\
0 &\to \widetilde{\mathcal{L}}_3 \xrightarrow{\tilde{i}} \mathrm{Ann}\mathcal{W}_2 \xrightarrow{\tilde{\pi}} \widetilde{\Lambda}^* \to 0,
\end{aligned}$$

(ii)

$$\mathcal{L}_3 \simeq \mathcal{L}_2 \otimes \Lambda \quad \text{and} \quad \widetilde{\mathcal{L}}_3 \simeq \mathcal{L}_2 \otimes \widetilde{\Lambda},$$

(iii)

$$\bigwedge^2 \mathrm{Ann}\mathcal{W}_2 \simeq \bigwedge^2 \mathcal{W}_2,$$

(iv)

$$\Lambda \simeq \widetilde{\Lambda},$$

(v)

$$\mathcal{L}_2 \simeq \Lambda \otimes \mathcal{O}_C(-1) \simeq \widetilde{\Lambda} \otimes \mathcal{O}_C(-1),$$

(vi)

$$\mathrm{hom}_C(\mathcal{L}_3, \widetilde{\Lambda}^*) \simeq \mathcal{L}_2^* \otimes \widetilde{\Lambda}^{*2} \simeq \mathcal{L}_2^{*3} \otimes \mathcal{O}_C(-2).$$

Proof. From the definition of Λ, we have the exact sequence:

$$0 \to \mathcal{W}_2 \to \mathcal{W}_3 \to \Lambda \to 0$$

from which it follows that:

$$0 \to \Lambda \to \mathcal{O}_C^4/\mathcal{W}_2 \to \mathcal{O}_C^4/\mathcal{W}_3 \to 0$$

is exact. Taking $\mathrm{hom}_C(-,\ \mathcal{O}_C)$ of this exact sequence yields the exact sequence:

$$0 \to \mathrm{Ann}\mathcal{W}_3 \to \mathrm{Ann}\mathcal{W}_2 \to \Lambda^* \to 0.$$

Now, via the canonical isomorphism $\bigwedge^3 V \to V^*$ which arises from the non-degenerate pairing

$$\bigwedge^3 V \otimes V \to \bigwedge^4 V \simeq \mathbb{C},$$

it is clear that $\mathrm{Ann}\mathcal{W}_3 \simeq \bigwedge^3 \mathcal{W}_3 = \mathcal{L}_3$.

Thus the first and third exact sequences of (i) follow. The proofs of the second and fourth are similar. From the first exact sequence in (i), it follows that $\bigwedge^3 \mathcal{W}_3 \simeq \bigwedge^2 \mathcal{W}_2 \otimes \Lambda$. This implies the first identity of (ii). Similarly the second exact sequence of (i) implies the other identity of (ii).

Since for every line bundle γ, $\gamma \otimes \gamma^*$ is trivial, we get from the first identity of (ii) that $\mathcal{L}_2 \simeq \mathcal{L}_3 \otimes \Lambda^*$. From third exact sequence in (i) it follows that $\bigwedge^2 \mathrm{Ann}\mathcal{W}_2 \simeq \mathcal{L}_3 \otimes \Lambda^*$, and this implies (iii).

To see (iv), note that

$$\Lambda \simeq \frac{\mathcal{W}_2 + A\mathcal{W}_2}{\mathcal{W}_2} \simeq \frac{A\mathcal{W}_2}{A\mathcal{W}_2 \cap \mathcal{W}_2}.$$

The automorphism A^{-1} of V makes the last bundle on the right isomorphic to the line bundle $\mathcal{W}_2/(\mathcal{W}_2 \cap A^{-1}\mathcal{W}_2)$ (note all these operations are happening inside the rank 4 trivial bundle $\mathcal{O}_C^4$). Similarly, $\widetilde{\Lambda}$ is isomorphic (via the global isomorphism A^{*-1} of V) to the line bundle $\mathcal{W}_2/(\mathcal{W}_2 \cap A^{*-1}\mathcal{W}_2)$. But for each $v \in C$, $W(v) \cap A^{-1}W(v) = \mathbb{C}v = W(v) \cap A^{*-1}W(v)$, from which it follows that the line sub-bundles $\mathcal{W}_2 \cap A^{-1}\mathcal{W}_2$ and $\mathcal{W}_2 \cap A^{*-1}\mathcal{W}_2$ of $\mathcal{W}_2$ are the same ($= \mathcal{W}_1 \simeq \mathcal{O}_C(-1)$). Thus $\Lambda \simeq \widetilde{\Lambda}$, proving (iv).

To see (v), we need another exact sequence. For each $v \in C$, we noted in the proof of (iv) above that $\mathbb{C}v = W(v) \cap A^{-1}W(v)$. Thus the sequence of bundles:

$$0 \to \mathcal{O}_C(-1) \to \mathcal{W}_2 \to \frac{\mathcal{W}_2}{\mathcal{W}_2 \cap A^{-1}\mathcal{W}_2} \to 0$$

is exact. But, as we noted in the proof of (iv) above, the bundle on the right is isomorphic to Λ, so that

$$0 \to \mathcal{O}_C(-1) \to \mathcal{W}_2 \to \Lambda \to 0$$

is exact. Hence $\mathcal{L}_2 = \bigwedge^2 \mathcal{W}_2 \simeq \Lambda \otimes \mathcal{O}_C(-1)$. The other identity follows from (iv), thus proving (v).

To see (vi) note that we have by (ii) $\mathcal{L}_3^* \simeq \mathcal{L}_2^* \otimes \Lambda^*$. Thus

$$\hom_C(\mathcal{L}_3, \widetilde{\Lambda}^*) \simeq \mathcal{L}_3^* \otimes \widetilde{\Lambda}^* \simeq \mathcal{L}_2^* \otimes \Lambda^* \otimes \widetilde{\Lambda}^*.$$

However, since by (iv), $\Lambda \simeq \widetilde{\Lambda}$, we have $\hom_C(\mathcal{L}_3, \widetilde{\Lambda}^*) \simeq \mathcal{L}_2^* \otimes \Lambda^{*2}$. Now, substituting $\Lambda^* = \mathcal{L}_2^* \otimes \mathcal{O}_C(-1)$ from (v), we have the rest of (vi). Hence the lemma. □

We need one more bundle identity:

Lemma 4.5. *There is a bundle isomorphism*:

$$\mathcal{L}_2 \simeq \mathcal{O}_C(-2) \otimes g^*\mathcal{O}_D(1).$$

Proof. When $[t] = [t_0 : t_1 : t_2] = g([v])$, we saw in (5) that the linear map $B(v, -) : \mathbb{C}^3 \to \mathbb{C}^4$ acquires a 1-dimensional kernel, which is precisely the line $\mathbb{C}t$, which is the fibre of $\mathcal{O}_D(-1)$ at $[t]$. The image of $B(v, -)$ was the 2-dimensional span $W(v)$ of v, Av, A^*v, as noted there. Thus for $v \in C$, $B(-, -)$ induces a canonical isomorphism of vector spaces:

$$\mathcal{O}_C(-1)_v \otimes \left(\mathbb{C}^3/\mathcal{O}_D(-1)\right)_{g(v)} \to W(v) = \mathcal{W}_{2,v}$$

which, being defined by the global map $B(-, -)$, gives an isomorphism of bundles:

$$\mathcal{O}_C(-1) \otimes g^*\left(\mathcal{O}_D^3/\mathcal{O}_D(-1)\right) \simeq \mathcal{W}_2.$$

From the short exact sequence:

$$0 \to \mathcal{O}_D(-1) \to \mathcal{O}_D^3 \to \mathcal{O}_D^3/\mathcal{O}_D(-1) \to 0,$$

it follows that $\bigwedge^2(\mathcal{O}_D^3/\mathcal{O}_D(-1)) \simeq \mathcal{O}_D(1)$. Thus:

$$\begin{aligned}\mathcal{L}_2 &= \bigwedge^2 \mathcal{W}_2 \simeq \mathcal{O}_C(-2) \otimes g^*\left(\bigwedge^2(\mathcal{O}_D^3/\mathcal{O}_D(-1))\right)\\ &\simeq \mathcal{O}_C(-2) \otimes g^*\mathcal{O}_D(1).\end{aligned}$$

This proves the lemma. □

5. Degree computations

In this section, we compute the degrees of the various line bundles introduced in the previous section.

DEFNITION 5.1

Note that an *irreducible* complex projective curve C, as a topological space, is a canonically oriented pseudomanifold of real dimension 2, and has a canonical generator $\mu_C \in H_2(C, \mathbb{Z}) = \mathbb{Z}$. Indeed, it is the image $\pi_*\mu_{\widetilde{C}}$, where $\pi : \widetilde{C} \to C$ is the normalization map, and $\mu_{\widetilde{C}} \in H_2(\widetilde{C}, \mathbb{Z}) = \mathbb{Z}$ is the canonical orientation class for the smooth connected compact complex manifold $\widetilde{C}$, where $\pi_* : H_2(\widetilde{C}, \mathbb{Z}) \to H_2(C, \mathbb{Z})$ is an isomorphism for elementary topological reasons.

If $C = \cup_{\alpha=1}^{r} C_\alpha$ is a projective curve of pure dimension 1, with the curves C_α as irreducible components, then since the intersections $C_\alpha \cap C_\beta$ are finite sets of points (or empty), $H_2(C, \mathbb{Z}) = \oplus_\alpha H_2(C_\alpha, \mathbb{Z})$. Letting μ_α denote the canonical orientation classes of C_α as above, there is a *unique class* $\mu_C = \sum_\alpha \mu_\alpha \in H_2(C, \mathbb{Z})$. Thinking of C as an oriented 2-pseudomanifold, μ_C is just the sum of all the oriented 2-simplices of C.

If $\mathcal{F}$ is a complex line bundle on C, it has a first Chern class $c_1(\mathcal{F}) \in H^2(X, \mathbb{Z})$, and the *degree* of $\mathcal{F}$ is defined by

$$\deg \mathcal{F} = \langle c_1(\mathcal{F}), \mu_C \rangle \in \mathbb{Z}.$$

It is known that a complex line bundle on a pseudomanifold is topologically trivial iff its first Chern class is zero. In particular, if an algebraic line bundle on a projective variety has non-zero degree, then it is topologically (and hence algebraically) non-trivial.

Finally, if $i : C \hookrightarrow \mathbb{P}^n_{\mathbb{C}}$ is an (algebraic) embedding of a curve in some projective space, we define the degree of the bundle $\mathcal{O}_C(1) = i^*\mathcal{O}_{\mathbb{P}^n_{\mathbb{C}}}(1)$ as the *degree of the curve* C (in $\mathbb{P}^n_{\mathbb{C}}$). We note that $[C] := i_*(\mu_C) \in H_2(\mathbb{P}^n_{\mathbb{C}}, \mathbb{Z})$ is called the *fundamental class* of C in $\mathbb{P}^n_{\mathbb{C}}$, and by definition $\deg C = \langle c_1(\mathcal{O}_C(1)), \mu_C \rangle = \left\langle c_1(\mathcal{O}_{\mathbb{P}^n_{\mathbb{C}}}(1)), [C] \right\rangle$. Geometrically, one intersects C with a generic hyperplane, which intersects C away from its singular locus in a finite set of points, and then counts these points of intersection with their multiplicity.

More generally, a complex projective variety $X \subset \mathbb{P}^n_{\mathbb{C}}$ of complex dimension m has a unique orientation class $\mu_X \in H_{2m}(X, \mathbb{Z})$. Its image in $H_{2m}(\mathbb{P}^n_{\mathbb{C}}, \mathbb{Z})$ is denoted $[X]$, and the degree $\deg X$ of X is defined as $\left\langle (c_1(\mathcal{O}_{\mathbb{P}^n_{\mathbb{C}}}(1)))^m, [X] \right\rangle$. It is known that if $X = V(F)$ for a homogeneous polynomial F of degree d, then $\deg X = d$.

We need the following remark later on.

Remark 5.2. If $f : C \to D$ is a regular isomorphism of complex projective curves C and D, both of pure dimension 1, and if $\mathcal{F}$ is a complex line bundle on D, then $\deg f^*\mathcal{F} = \deg \mathcal{F}$. This is because $f_*(\mu_C) = \mu_D$, so that

$$\deg \mathcal{F} = \langle c_1(\mathcal{F}), \mu_D \rangle = \langle c_1(\mathcal{F}), f_*\mu_C \rangle = \left\langle f^*c_1(\mathcal{F}), \mu_C \right\rangle = \left\langle c_1(f^*\mathcal{F}), \mu_C \right\rangle = \deg f^*\mathcal{F}.$$

Now we can compute the degrees of all the line bundles introduced.

Lemma 5.3. *The degrees of the various line bundles above are as follows*:

(i) $\deg \mathcal{O}_C(1) = \deg C = 6$

(ii) $\deg \mathcal{O}_D(1) = \deg D = 4$

(iii) $\deg \mathcal{L}_2^* = 8$

(iv) $\deg \mathrm{hom}_C(\mathcal{L}_3, \widetilde{\Lambda}^*) = \deg \left(\mathcal{L}_2^{*3} \otimes \mathcal{O}_C(-2)\right) = 12.$

Proof. We denote the image of orientation class μ_Γ of the curve Γ (see Definition 3.3 for the definition of Γ) in $H_2(\mathbb{P}^3_{\mathbb{C}} \times \mathbb{P}^2_{\mathbb{C}}, \mathbb{Z})$ by $[\Gamma]$. By the part (iv) of Lemma 3.4, we have that the homology class $[\Gamma]$ is the same as the homology class of the intersection cycle defined by the four divisors $D_i := (B_i(v,t) = 0)$ inside $H_2(\mathbb{P}^3_{\mathbb{C}} \times \mathbb{P}^2_{\mathbb{C}}, \mathbb{Z})$. By the generalized Bezout theorem in $\mathbb{P}^3_{\mathbb{C}} \times \mathbb{P}^2_{\mathbb{C}}$, the homology class of the last-mentioned intersection cycle is the homology class Poincaré-dual to the cup product

$$d := d_1 \cup d_2 \cup d_3 \cup d_4,$$

where d_i is the first Chern class of the the line bundle L_i corresponding to D_i, for $i = 1, 2, 3, 4$ (see [4], p. 237, Ex. 2).

Since each $B_i(v,t)$ is separately linear in v, t, the line bundle defined by the divisor D_i is the bundle $\pi_1^*\mathcal{O}_{\mathbb{P}^3_{\mathbb{C}}}(1) \otimes \pi_2^*\mathcal{O}_{\mathbb{P}^2_{\mathbb{C}}}(1)$, where π_1, π_2 are the projections to $\mathbb{P}^3_{\mathbb{C}}$ and $\mathbb{P}^2_{\mathbb{C}}$ respectively. If we denote the hyperplane classes which are the generators of the cohomologies $H^2(\mathbb{P}^3_{\mathbb{C}}, \mathbb{Z})$ and $H^2(\mathbb{P}^2_{\mathbb{C}}, \mathbb{Z})$ by x and y respectively, we have

$$d_i = c_1(L_i) = \pi_1^*(x) + \pi_2^*(y).$$

Then we have, from the cohomology ring structures of $\mathbb{P}^3_{\mathbb{C}}$ and $\mathbb{P}^2_{\mathbb{C}}$ that $x \cup x \cup x \cup x = y \cup y \cup y = 0$. Hence the cohomology class in $H^8(\mathbb{P}^3_{\mathbb{C}} \times \mathbb{P}^2_{\mathbb{C}}, \mathbb{Z})$ given by the cup-product of d_i is

$$d := d_1 \cup d_2 \cup d_3 \cup d_4 = (\pi_1^*(x) + \pi_2^*(y))^4 = 4\pi_1^*(x^3)\pi_2^*(y) + 6\pi_1^*(x^2)\pi_2^*(y^2),$$

where $x^3 = x \cup x \cup x \ldots$ etc. By part (ii) of Lemma 3.4, the map $\pi_1 : \Gamma \to C$ is an isomorphism, so applying the Remark 5.2 to it, we have

$$\begin{aligned}
\deg \mathcal{O}_C(1) &= \deg \pi_1^*\mathcal{O}_C(1) \\
&= \left\langle c_1(\pi_1^*(\mathcal{O}_{\mathbb{P}^3_{\mathbb{C}}}(1)), [\Gamma] \right\rangle \\
&= \left\langle c_1(\pi_1^*(\mathcal{O}_{\mathbb{P}^3_{\mathbb{C}}}(1)) \cup d, [\mathbb{P}^3_{\mathbb{C}} \times \mathbb{P}^2_{\mathbb{C}}] \right\rangle \\
&= \left\langle \pi_1^*(x) \cup \left(4\pi_1^*(x^3)\pi_2^*(y) + 6\pi_1^*(x^2)\pi_2^*(y^2)\right), [\mathbb{P}^3_{\mathbb{C}} \times \mathbb{P}^2_{\mathbb{C}}] \right\rangle \\
&= \left\langle 6\pi_1^*(x^3) \cup \pi_2^*(y^2), [\mathbb{P}^3_{\mathbb{C}} \times \mathbb{P}^2_{\mathbb{C}}] \right\rangle \\
&= 6, \qquad (9)
\end{aligned}$$

where we have used the Poincaré duality cap-product relation $[\Gamma] = [\mathbb{P}^3_{\mathbb{C}} \times \mathbb{P}^2_{\mathbb{C}}] \cap d$ mentioned above, and that $\pi_1^*(x^3) \cup \pi_2^*(y^2)$ is the generator of $H^{10}(\mathbb{P}^3_{\mathbb{C}} \times \mathbb{P}^2_{\mathbb{C}}, \mathbb{Z})$, so evaluates to 1 on the orientation class $[\mathbb{P}^3_{\mathbb{C}} \times \mathbb{P}^2_{\mathbb{C}}]$, and $x^4 = 0$. This proves (i).

The proof of (ii) is similar, we just replace C by D, and π_1 by π_2, and $\pi_1^*(x)$ by $\pi_2^*(y)$ in the equalities of (9) above, and get 4 (as one should expect, since D is defined by a degree 4 homogeneous polynomial in $\mathbb{P}^2_{\mathbb{C}}$). This proves (ii).

For (iii), we use the identity of Lemma 4.5 that $\mathcal{L}_2 = \mathcal{O}_C(-2) \otimes g^*\mathcal{O}_D(1)$, and the Remark 5.2 applied to the isomorphism of curves $g : C \to D$ (part (iii) of Lemma 3.4) to conclude that $\deg \mathcal{L}_2 = \deg D - 2\deg C = 4 - 12 = -8$, by (i) and (ii) above, so that $\deg \mathcal{L}_2^* = 8$.

For (iv), we have by (vi) of Lemma 4.4 that $\hom_C(\mathcal{L}_3, \widetilde{\Lambda}^*) \simeq \mathcal{L}_2^{*3} \otimes \mathcal{O}_C(-2)$, so that its degree is $3\deg \mathcal{L}_2^* - 2\deg C = 24 - 12 = 12$ by (i) and (iii) above.

This proves the lemma. □

From (iv) of the lemma above, we have the following.

COROLLARY 5.4

The line bundle $\hom_C(\mathcal{L}_3, \widetilde{\Lambda}^*)$ *is a non-trivial line bundle.*

6. Proof of the main theorem

Proof of Theorem 1.1. By the third and fourth exact sequences in (i) of Lemma 4.4, we have a bundle morphism s of line bundles on C defined as the composite:

$$\mathrm{Ann}\mathcal{W}_3 = \mathcal{L}_3 \xrightarrow{i} \mathrm{Ann}\mathcal{W}_2 \xrightarrow{\widetilde{\pi}} \widetilde{\Lambda}^* = \mathrm{Ann}\mathcal{W}_2/\mathrm{Ann}\widetilde{\mathcal{W}}_3$$

which vanishes at $v \in C$ if and only if the fibre $\mathrm{Ann}\mathcal{W}_{3,v}$ is equal to the fibre $\mathrm{Ann}\widetilde{\mathcal{W}}_{3,v}$ inside $\mathrm{Ann}\mathcal{W}_{2,v}$. At such a point $v \in C$, we will have $\mathrm{Ann}\mathcal{W}_{3,v} = \mathrm{Ann}\widetilde{\mathcal{W}}_{3,v}$, so that $W_3(v) = \mathcal{W}_{3,v} = W(v) + AW(v) = \widetilde{\mathcal{W}}_{3,v} = W(v) + A^*W(v) = \widetilde{W}_3(v)$.

Now, this morphism s is a global section of the bundle $\hom_C(\mathcal{L}_3, \widetilde{\Lambda}^*)$, which is not a trivial bundle by Corollary 5.4 of the last section. Thus there does exist a $v \in C$, satisfying $s(v) = 0$, and consequently the flag

$$\begin{aligned} 0 \subset W_1 := \mathcal{W}_{1,v} &= \mathbb{C}v \subset W_2 := \mathcal{W}_{2,v} = W(v) = \mathbb{C}\text{-span}\{v, Av, A^*v\} \\ \subset W_3 &:= W_3(v) = W(v) + AW(v) = W(v) + A^*W(v) \\ &= \widetilde{W}_3(v) \subset W_4 = V = \mathbb{C}^4 \end{aligned}$$

satisfies the requirements of (ii) of Lemma 2.1, (as noted after Remark 4.2) and the main theorem 1.1 follows. □

Remark 6.1. Note that since $\dim C = 1$, the set of points $v \in C$ such that $s(v) = 0$, where s is the section above, will be a finite set. Then the set of flags that satisfy (ii) of Lemma 2.1 which tridiagonalize A of the kind considered above (viz. A satisfying the assumptions of 3.1), will only be finitely many (at most 12 in number!).

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On Ricci curvature of C-totally real submanifolds in Sasakian space forms

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Abstract. Let M^n be a Riemannian n-manifold. Denote by $S(p)$ and $\overline{\mathrm{Ric}}(p)$ the Ricci tensor and the maximum Ricci curvature on M^n, respectively. In this paper we prove that every C-totally real submanifold of a Sasakian space form $\bar{M}^{2m+1}(c)$ satisfies $S \le (\frac{(n-1)(c+3)}{4} + \frac{n^2}{4}H^2)g$, where H^2 and g are the square mean curvature function and metric tensor on M^n, respectively. The equality holds identically if and only if either M^n is totally geodesic submanifold or $n = 2$ and M^n is totally umbilical submanifold. Also we show that if a C-totally real submanifold M^n of $\bar{M}^{2n+1}(c)$ satisfies $\overline{\mathrm{Ric}} = \frac{(n-1)(c+3)}{4} + \frac{n^2}{4}H^2$ identically, then it is minimal.

Keywords. Ricci curvature; C-totally real submanifold; Sasakian space form.

1. Introduction

Let M^n be a Riemannian n-manifold isometrically immersed in a Riemannian m-manifold $\bar{M}^m(c)$ of constant sectional curvature c. Denote by g, R and h the metric tensor, Riemann curvature tensor and the second fundamental form of M^n, respectively. Then the mean curvature vector H of M^n is given by $H = \frac{1}{n}\mathrm{trace}\, h$. The Ricci tensor S and the scalar curvature ρ at a point $p \in M^n$ are given by $S(X, Y) = \sum_{i=1}^n \langle R(e_i, X)Y, e_i\rangle$ and $\rho = \sum_{i=1}^n S(e_i, e_i)$, respectively, where $\{e_1, \ldots, e_n\}$ is an orthonormal basis of the tangent space T_pM^n. A submanifold M^n is called totally umbilical if h, H and g satisfy $h(X, Y) = g(X, Y)H$ for X, Y tangent to M^n.

The equation of Gauss for the submanifold M^n is given by

$$\begin{aligned} g(R(X,Y)Z,W) &= c(g(X,W)g(Y,Z) - g(X,Z)g(Y,W)) \\ &\quad + g(h(X,W),h(Y,Z)) - g(h(X,Z),h(Y,W)), \end{aligned} \tag{1}$$

where $X, Y, Z, W \in TM^n$. From (1) we have

$$\rho = n(n-1)c + n^2H^2 - |h|^2, \tag{2}$$

where $|h|^2$ is the squared norm of the second fundamental form. From (2) we have

$$\rho \le n(n-1)c + n^2H^2,$$

with equality holding identically if and only if M^n is totally geodesic.

Let $\overline{\mathrm{Ric}}(p)$ denote the maximum Ricci curvature function on M^n defined by

$$\overline{\mathrm{Ric}}(p) = \max\{S(u, u) | u \in T_p^1 M^n, \quad p \in M^n\},$$

where $T_p^1 M^n = \{v \in T_p M^n | \langle v, v \rangle = 1\}$.

In [3], Chen proves that there exists a basic inequality on Ricci tensor S for any submanifold M^n in $\bar{M}^m(c)$, i.e.

$$S \leq \left((n-1)c + \frac{n^2}{4} H^2\right) g, \tag{3}$$

with the equality holding if and only if either M^n is a totally geodesic submanifold or $n = 2$ and M^n is a totally umbilical submanifold. And in [4], Chen proves that every isotropic submanifold M^n in a complex space form $\bar{M}^m(4c)$ satisfies $\overline{\mathrm{Ric}} \leq (n-1)c + \frac{n^2}{4}H^2$, and every Lagrangian submanifold of a complex space form satisfying the equality case identically is a minimal submanifold. In the present paper, we would like to extend the above results to the C-totally real submanifolds of a Sasakian space form, namely, we prove that every C-totally real submanifold of a Sasakian space form $\bar{M}^{2m+1}(c)$ satisfies $S \leq (\frac{(n-1)(c+3)}{4} + \frac{n^2}{4}H^2)g$, and the equality holds identically if and only if either M^n is totally geodesic submanifold or $n = 2$ and M^n is totally umbilical submanifold. Also we show that if a C-totally real submanifold M^n of a Sasakian space form $\bar{M}^{2n+1}(c)$ satisfies $\overline{\mathrm{Ric}} = \frac{(n-1)(c+3)}{4} + \frac{n^2}{4}H^2$ identically, then it is minimal.

2. Preliminary

Let $\bar{M}^{2m+1}$ be an odd dimensional Riemannian manifold with metric g. Let ϕ be a (1,1)-tensor field, ξ a vector field, and η a 1-form on $\bar{M}^{2m+1}$, such that

$$\phi^2 X = -X + \eta(X)\xi, \quad \phi\xi = 0, \quad \eta(\phi X) = 0, \quad \eta(\xi) = 1,$$
$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad \eta(X) = g(X, \xi).$$

If, in addition, $d\eta(X, Y) = g(\phi X, Y)$, for all vector fields X, Y on $\bar{M}^{2m+1}$, then $\bar{M}^{2m+1}$ is said to have a contact metric structure (ϕ, ξ, η, g), and $\bar{M}^{2m+1}$ is called a contact metric manifold. If moreover the structure is normal, that is if $[\phi X, \phi Y] + \phi^2[X, Y] - \phi[X, \phi Y] - \phi[\phi X, Y] = -2d\eta(X, Y)\xi$, then the contact metric structure is called a Sasakian structure (normal contact metric structure) and $\bar{M}^{2m+1}$ is called a Sasakian manifold. For more details and background, see the standard references [1] and [8].

A plane section σ in $T_p\bar{M}^{2m+1}$ of a Sasakian manifold $\bar{M}^{2m+1}$ is called a ϕ-section if it is spanned by X and ϕX, where X is a unit tangent vector field orthogonal to ξ. The sectional curvature $\bar{K}(\sigma)$ with respect to a ϕ-section σ is called a ϕ-sectional curvature. If a Sasakian manifold $\bar{M}^{2m+1}$ has constant ϕ-sectional curvature c, $\bar{M}^{2m+1}$ is called a Sasakian space form and is denoted by $\bar{M}^{2m+1}(c)$.

The curvature tensor $\bar{R}$ of a Sasakian space form $\bar{M}^{2m+1}(c)$ is given by [8]

$$\begin{aligned}\bar{R}(X, Y)Z &= \frac{c+3}{4}(g(Y, Z)X - g(X, Z)Y) \\ &\quad + \frac{c-1}{4}(\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi \\ &\quad + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y - 2g(\phi X, Y)\phi Z),\end{aligned}$$

for any tangent vector fields X, Y, Z to $\bar{M}^{2m+1}(c)$.

An n-dimensional submanifold M^n of a Sasakian space form $\bar{M}^{2m+1}(c)$ is called a C-totally real submanifold of $\bar{M}^{2m+1}(c)$ if ξ is a normal vector field on M^n. A direct consequence of this definition is that $\phi(TM^n) \subset T^{\perp}M^n$, which means that M^n is an anti-invariant submanifold of $\bar{M}^{2m+1}(c)$. So we have $n \leq m$.

The Gauss equation implies that

$$\begin{aligned} R(X,Y,Z,W) &= \frac{1}{4}(c+3)(g(Y,Z)g(X,W) - g(X,Z)g(Y,W)) \\ &\quad + g(h(X,W),h(Y,Z)) - g(h(X,Z),h(Y,W)), \end{aligned} \tag{4}$$

for all vector fields X, Y, Z, W tangent to M^n, where h denotes the second fundamental form and R the curvature tensor of M^n.

Let A denote the shape operator on M^n in $\bar{M}^{2m+1}(c)$. Then A is related to the second fundamental form h by

$$g(h(X,Y),\alpha) = g(A_{\alpha}X,Y), \tag{5}$$

where α is a normal vector field on M^n.

For C-totally real submanifold in $\bar{M}^{2m+1}(c)$, we also have (for example, see [7])

$$A_{\phi Y}X = -\phi h(X,Y) = A_{\phi X}Y, \quad A_{\xi} = 0. \tag{6}$$

$$g(h(X,Y),\phi Z) = g(h(X,Z),\phi Y). \tag{7}$$

3. Ricci tensor of C-totally real submanifolds

We will need the following algebraic lemma due to Chen [2].

Lemma 3.1. *Let $a_1, \ldots, a_n, c$ be $n+1$ ($n \geq 2$) real numbers such that*

$$\left(\sum_{i=1}^{n} a_i\right)^2 = (n-1)\left(\sum_{i=1}^{n} a_i^2 + c\right). \tag{8}$$

Then $2a_1a_2 \geq c$, with equality holding if and only if $a_1 + a_2 = a_3 = \cdots = a_n$.

For a C-totally real submanifold M^n of $\bar{M}^{2m+1}(c)$, we have

Theorem 3.1. *If M^n is a C-totally real submanifold of $\bar{M}^{2m+1}(c)$, then the Ricci tensor of M^n satisfies*

$$S \leq \left(\frac{(n-1)(c+3)}{4} + \frac{n^2}{4}H^2\right)g, \tag{9}$$

and the equality holds identically if and only if either M^n is totally geodesic or $n = 2$ and M^n is totally umbilical.

Proof. From Gauss' equation (4), we have

$$\rho = \frac{n(n-1)(c+3)}{4} + n^2H^2 - |h|^2. \tag{10}$$

Put $\delta = \rho - \frac{n(n-1)(c+3)}{4} - \frac{n^2}{2}H^2$. Then from (10) we obtain

$$n^2H^2 = 2(\delta + |h|^2). \tag{11}$$

Let L be a linear $(n-1)$-subspace of T_pM^n, $p \in M^n$, and $\{e_1, \dots, e_{2m}, e_{2m+1} = \xi\}$ an orthonormal basis such that (1) $e_1, \dots, e_n$ are tangent to M^n, (2) $e_1, \dots, e_{n-1} \in L$ and (3) if $H(p) \neq 0$, e_{n+1} is in the direction of the mean curvature vector at p.

Put $a_i = h_{ii}^{n+1}$, $i = 1, \dots, n$. Then from (11) we get

$$\left(\sum_{i=1}^{n} a_i\right)^2 = 2\left\{\delta + \sum_{i=1}^{n} a_i^2 + \sum_{i\neq j}(h_{ij}^{n+1})^2 + \sum_{r=n+2}^{2m+1}\sum_{i,j=1}^{n}(h_{ij}^r)^2\right\}. \tag{12}$$

Equation (12) is equivalent to

$$\left(\sum_{i=1}^{3} \bar{a}_i\right)^2 = 2\left\{\delta + \sum_{i=1}^{3} \bar{a}_i^2 + \sum_{i\neq j}(h_{ij}^{n+1})^2 + \sum_{r=n+2}^{2m+1}\sum_{i,j=1}^{n}(h_{ij}^r)^2 - \sum_{2\leq i\neq j\leq n-1} a_ia_j\right\}, \tag{13}$$

where $\bar{a}_1 = a_1$, $\bar{a}_2 = a_2 + \cdots + a_{n-1}$, $\bar{a}_3 = a_n$.

By Lemma 3.1 we know that if $(\sum_{i=1}^3 \bar{a}_i)^2 = 2(c + \sum_{i=1}^3 \bar{a}_i^2)$, then $2\bar{a}_1\bar{a}_2 \geq c$ with equality holding if and only if $\bar{a}_1 + \bar{a}_2 = \bar{a}_3$. Hence from (13) we can get

$$\sum_{1\leq i\neq j\leq n-1} a_ia_j \geq \delta + 2\sum_{i<j}(h_{ij}^{n+1})^2 + \sum_{r=n+2}^{2m+1}\sum_{i,j=1}^{n}(h_{ij}^r)^2, \tag{14}$$

which gives

$$\frac{n(n-1)(c+3)}{4} + \frac{n^2}{2}H^2 \geq \rho - \sum_{1\leq i\neq j\leq n-1} a_ia_j + 2\sum_{i<j}(h_{ij}^{n+1})^2 + \sum_{r=n+2}^{2m+1}\sum_{i,j=1}^{n}(h_{ij}^r)^2. \tag{15}$$

Using Gauss' equation we have

$$\begin{aligned}
&\rho - \sum_{1\leq i\neq j\leq n-1} a_ia_j + 2\sum_{i<j}(h_{ij}^{n+1})^2 + \sum_{r=n+2}^{2m+1}\sum_{i,j=1}^{n}(h_{ij}^r)^2 \\
&= 2S(e_n, e_n) + \frac{(n-1)(n-2)(c+3)}{4} + 2\sum_{i<n}(h_{in}^{n+1})^2 \\
&+ \sum_{r=n+2}^{2m+1}\left[(h_{nn}^r)^2 + 2\sum_{i=1}^{n-1}(h_{in}^r)^2 + \left(\sum_{j=1}^{n-1}h_{jj}^r\right)^2\right].
\end{aligned} \tag{16}$$

From (15) and (16) we have

$$\begin{aligned}
\frac{(n-1)(c+3)}{4} + \frac{n^2}{4}H^2 \quad &\geq \quad S(e_n, e_n) + 2\sum_{i<n}(h_{in}^{n+1})^2 \\
&+ \sum_{r=n+2}^{2m+1}\left[\sum_{i=1}^{n}(h_{in}^r)^2 + \left(\sum_{j=1}^{n-1}h_{jj}^r\right)^2\right].
\end{aligned} \tag{17}$$

So we have

$$\frac{(n-1)(c+3)}{4} + \frac{n^2}{4}H^2 \geq S(e_n, e_n) \tag{18}$$

with equality holding if and only if

$$h^s_{jn} = 0, \quad h^r_{in} = 0, \quad \sum_{j=1}^{n-1} h^s_{jj} = h^s_{nn} \tag{19}$$

for $1 \leq j \leq n-1$, $1 \leq i \leq n$ and $n+2 \leq r \leq 2m+1$ and, since Lemma 3.1 states that $2\bar{a}_1\bar{a}_2 = c$ if and only if $\bar{a}_1 + \bar{a}_2 = \bar{a}_3$, we also have $h^{n+1}_{nn} = \sum_{j=1}^{n-1} h^{n+1}_{jj}$. Since e_n can be any unit tangent vector of M^n, then (18) implies inequality (9).

If the equality sign case of (9) holds identically, then we have

$$\begin{aligned} &h^{n+1}_{ij} = 0 \quad (1 \leq i \neq j \leq n), \\ &h^r_{ij} = 0 \quad (1 \leq i, j \leq n; n+2 \leq r \leq 2m+1), \\ &h^{n+1}_{ii} = \sum_{k \neq i} h^{n+1}_{kk}, \quad \sum_{k \neq i} h^r_{kk} = 0, \qquad (n+2 \leq r \leq 2m+1). \end{aligned} \tag{20}$$

If $\lambda_i = h^{n+1}_{ii}(1 \leq i \leq n)$, we find $\sum_{k \neq i} \lambda_k = \lambda_i(1 \leq i \leq n)$ and, since the matrix $A^{(n)} = (a^{(n)}_{ij})$ with $a^{(n)}_{ij} = 1 - 2\delta_{ij}$ is regular for $n \neq 2$ and has kernel $R(1, 1)$ for $n = 2$, we conclude that M^n is either totally geodesic or $n = 2$ and M^n is totally umbilical.

The converse is easy to prove. This completes the proof of Theorem 3.1.

4. Minimality of C-totally real submanifolds

Theorem 4.1. *If M^n is an n-dimensional C-totally real submanifold in a Sasakian space form $\bar{M}^{2n+1}(c)$, then*

$$\overline{\mathrm{Ric}} \leq \frac{(n-1)(c+3)}{4} + \frac{n^2}{4}H^2. \tag{21}$$

If M^n satisfies the equality case of (21) *identically, then M^n is minimal.*

Clearly Theorem 4.1 follows immediately from the following Lemma.

Lemma 4.1. *If M^n is an n-dimensional totally real submanifold in a Sasakian space form $\bar{M}^{2m+1}(c)$, then we have* (21). *If a C-totally real submanifold M^n in $\bar{M}^{2m+1}(c)$ satisfies the equality case of* (21) *at a point p, then the mean curvature vector H at p is perpendicular to $\phi(T_pM^n)$.*

Proof. Inequality (21) is an immediate consequence of inequality (9).

Now let us assume that M^n is a C-totally real submanifold of $\bar{M}^{2m+1}(c)$ which satisfies the equality sign of (21) at a point $p \in M^n$. Without loss of the generality we may choose an orthonormal basis $\{\bar{e}_1, \ldots, \bar{e}_n\}$ of T_pM^n such that $\overline{\mathrm{Ric}}(p) = S(\bar{e}_n, \bar{e}_n)$. From the proof of Theorem 3.1, we get

$$h^s_{in} = 0, \quad \sum_{i=1}^{n-1} h^s_{ii} = h^s_{nn}, \quad i = 1, \ldots, n-1; s = n+1, \ldots, 2m+1, \tag{22}$$

where h^s_{ij} denote the coefficients of the second fundamental form with respect to th orthonormal basis $\{\bar{e}_1, \dots, \bar{e}_n\}$ and $\{\bar{e}_{n+1}, \dots, \bar{e}_{2m+1} = \xi\}$.

If for all tangent vectors u, v and w at p, $g(h(u, v), \phi w) = 0$, there is nothing to prove So we assume that this is not the case. We define a function f_p by

$$f_p : \ T^1_p M^n \to R : \quad v \mapsto f_p(v) = g(h(v, v), \phi v). \tag{23}$$

Since $T^1_p M^n$ is a compact set, there exists a vector $v \in T^1_p M^n$ such that f_p attains a absolute maximum at v. Then $f_p(v) > 0$ and $g(h(v, v), \phi w) = 0$ for all w perpendicula to v. So from (5), we know that v is an eigenvector of $A_{\phi v}$. Choose a frame $\{e_1, e_2, \dots, e_n$ of $T_p M^n$ such that $e_1 = v$ and e_i be an eigenvector of $A_{\phi e_1}$ with eigenvalue λ_i. The functio f_i, $i \geq 2$, defined by $f_i(t) = f_p(\cos t\, e_1 + \sin t\, e_2)$ has relative maximum at $t = 0$, s $f_i''(0) \leq 0$. This will lead to the inequality $\lambda_1 \geq 2\lambda_i$. Since $\lambda_1 > 0$, we have

$$\lambda_i \neq \lambda_1, \quad \lambda_1 \geq 2\lambda_i, \quad i \geq 2. \tag{24}$$

Thus, the eigenspace of $A_{\phi e_1}$ with eigenvalue λ_1 is 1-dimensional.

From (22) we know that $\bar{e}_n$ is a common eigenvector for all shape operators at p. O the other hand, we have $e_1 \neq \pm\bar{e}_n$ since otherwise, from (22) and $A_{\phi e_i}\bar{e}_n = \pm A_{\phi e_i} e_1 =$ $\pm A_{\phi e_1} e_i = \pm\lambda_i e_i \perp \bar{e}_n$ $(i = 2, \dots, n)$, we obtain $\lambda_i = 0$, $i = 2, \dots, n$; and hence $\lambda_1 =$ by (22), which is a contradiction. Consequently, without loss of generality we may assum $e_1 = \bar{e}_1, \dots, e_n = \bar{e}_n$.

By (6), $A_{\phi e_n} e_1 = A_{\phi e_1} e_n = \lambda_n e_n$. Comparing this with (22) we obtain $\lambda_n = 0$. Thus by applying (22) once more, we get $\lambda_1 + \cdots + \lambda_{n-1} = \lambda_n = 0$. Therefore, trace $A_{\phi e_1} = 0$

For each $i = 2, \dots, n$, we have

$$h^{n+i}_{nn} = g(A_{\phi e_i} e_n, e_n) = g(A_{\phi e_n} e_i, e_n) = h^{2n}_{in}.$$

Hence, by applying (22) again, we get $h^{n+i}_{nn} = 0$. Combining this with (22) yield trace $A_{\phi e_i} = 0$. So we have trace $A_{\phi X} = 0$ for any $X \in T_p M^n$. Therefore, we con clude that the mean curvature vector at p is perpendicular to $\phi(T_p M^n)$.

Remark 4.1. From the proof of Lemma 4.1 we know that if M^n is a C-totally real submanifold of $\bar{M}^{2n+1}(c)$ satisfying

$$\overline{\mathrm{Ric}} = \frac{(n-1)(c+3)}{4} + \frac{n^2}{4} H^2, \tag{25}$$

then M^n is minimal and $A_{\phi v} = 0$ for any unit tangent vector satisfying $S(v, v) = \overline{\mathrm{Ric}}$ Thus, by (6) we have $A_{\phi X} v = 0$. Hence, we obtain $h(v, X) = 0$ for any X tangent t M^n and any v satisfying $S(v, v) = \overline{\mathrm{Ric}}$. Conversely, if M^n is a minimal C-totally rea submanifold of $\bar{M}^{2n+1}(c)$ such that for each $p \in M^n$ there exists a unit vector $v \in T_p M$ such that $h(v, X) = 0$ for all $X \in T_p M^n$, then it satisfies (25) indentically.

For each $p \in M^n$, the kernel of the second fundamental form is defined by

$$\mathcal{D}(p) = \{Y \in T_p M^n \mid h(X, Y) = 0, \forall X \in T_p M^n\}. \tag{26}$$

From the above discussion, we conclude that M^n is a minimal C-totally real submanifol of $\bar{M}^{2m+1}(c)$ satisfying (25) at p if and only if $\dim \mathcal{D}(p)$ is at least 1-dimensional.

Following the same argument as in [4], we can prove

Theorem 4.2. *Let M^n be a minimal C-totally real submanifold of $\bar{M}^{2n+1}(c)$. Then*

(1) *M^n satisfies* (25) *at a point p if and only if* $\dim\mathcal{D}(p) \geq 1$.

(2) *If the dimension of $\mathcal{D}(p)$ is positive constant d, then $\mathcal{D}$ is a completely integral distribution and M^n is d-ruled, i.e., for each point $p \in M^n$, M^n contains a d-dimensional totally geodesic submanifold N of $\bar{M}^{2n+1}(c)$ passing through p.*

(3) *A ruled minimal C-totally real submanifold M^n of $\bar{M}^{2n+1}(c)$ satisfies* (24) *identically if and only if, for each ruling N in M^n, the normal bundle $T^{\perp}M^n$ restricted to N is a parallel normal subbundle of the normal bundle $T^{\perp}N$ along N.*

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A variational proof for the existence of a conformal metric with preassigned negative Gaussian curvature for compact Riemann surfaces of genus > 1

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Abstract. Given a smooth function $K < 0$ we prove a result by Berger, Kazhdan and others that in every conformal class there exists a metric which attains this function as its Gaussian curvature for a compact Riemann surface of genus $g > 1$. We do so by minimizing an appropriate functional using elementary analysis. In particular for K a negative constant, this provides an elementary proof of the uniformization theorem for compact Riemann surfaces of genus $g > 1$.

Keywords. Uniformization theorem; Riemann surfaces; prescribed Gaussian curvature.

1. Introduction

In this paper we present a variational proof of a result by Berger [2], Kazhdan and Warner [6] and others, namely given an arbitrary smooth function $K < 0$ we show that in every conformal class there exists a metric which attains this function as its Gaussian curvature for a compact Riemann surface of genus $g > 1$. In particular, this result includes the uniformization theorem of Poincaré [8] when K is a negative constant. In his proof Berger considers the critical points of a functional subject to the Gauss–Bonnet condition. He shows that the functional is bounded from below and uses the Friedrich's inequality to complete the proof. The functional we choose is positive definite so that it is automatically bounded from below. Our proof is elementary, using Hodge theory, i.e., the existence of the Green's operator for the Laplacian. Our proof could be useful for analysing the appropriate condition on K for a corresponding result for genus $g = 1$ and $g = 0$ [6, 10, 3], the two other cases considered by Berger, Kazdan and Warner. Another variational proof of the uniformization theorem for genus $g > 1$ can be found in a gauge-theoretic context in [5] which uses Uhlenbeck's weak compactness theorem for connections with L^p bounds on curvature [9].

Let M be a compact Riemann surface of genus $g > 1$ and let $\mathrm{d}s^2 = h\mathrm{d}z \otimes \mathrm{d}\bar{z}$ be a metric on M normalized such that the total area of M is 1. Let $K < 0$. We minimize the functional

$$S(\sigma) = \int_M (K(\sigma) - K)^2 \mathrm{e}^{2\sigma} \mathrm{d}\mu$$

over $C^\infty(M, \mathbb{R})$, where $K(\sigma)$ stands for the Gaussian curvature of the metric $\mathrm{e}^\sigma \mathrm{d}s^2$, and $\mathrm{d}\mu = \frac{\sqrt{-1}}{2} h\mathrm{d}z \wedge \mathrm{d}\bar{z}$ is the area form for the metric $\mathrm{d}s^2$. Using Sobolev embedding theorem we show that $S(\sigma)$ takes its absolute minimum on $C^{\infty}(M)$ which corresponds to a metric on M of negative curvature K.

2. The main theorem

2.1

All notations are as in §1.

The functional $S(\sigma) = \int_M (K(\sigma) - K)^2 e^{2\sigma} d\mu$ is non-negative on $C^{\infty}(M, \mathbb{R})$, so that its infimum

$$S_0 = \inf\{S(\sigma),\ \sigma \in C^{\infty}(M, \mathbb{R})\}$$

exists and is non-negative. Let $\{\sigma_n\}_{n=1}^{\infty} \subset C^{\infty}(M, \mathbb{R})$ be a corresponding minimizing sequence,

$$\lim_{n\to\infty} S(\sigma_n) = S_0.$$

Our main result is the following

Theorem 2.1 *Let M be a compact Riemann surface of genus $g > 1$. The infimum S_0 is attained at $\sigma \in C^{\infty}(M, \mathbb{R})$, i.e. the minimizing sequence $\{\sigma_n\}$ contains a subsequence that converges in $C^{\infty}(M, \mathbb{R})$ to $\sigma \in C^{\infty}(M, \mathbb{R})$ and $S(\sigma) = 0$. The corresponding metric $e^{\sigma} h dz \otimes d\bar{z}$ is the unique metric on M of negative curvature K.*

2.2 *Uniform bounds*

Since $\{\sigma_n\}$ is a minimizing sequence, we have the obvious inequality

$$S(\sigma_n) = \int_M (K_n - K)^2 e^{2\sigma_n} d\mu = \int_M \left(K_0 - \frac{1}{2}\Delta_h \sigma_n - K e^{\sigma_n}\right)^2 d\mu \le m \quad (2.1)$$

for some $m > 0$, where we denoted by K_n the Gaussian curvature $K(\sigma_n)$ of the metric $e^{\sigma_n} h$ and by K_0 that of the metric h, and used that

$$K_n = e^{-\sigma_n}\left(K_0 - \frac{1}{2}\Delta_h \sigma_n\right).$$

Note. Here $\Delta_h = 4h^{-1}(\partial^2/\partial z \partial \bar{z})$ stands for the Laplacian defined by the metric h on M.

Lemma 2.2. *There exist constants C_1 and C_2 such that, uniformly in n,*

(a) $\int_M (\Delta_h \sigma_n)^2 d\mu < C_1$,

(b) $\int_M e^{2\sigma_n} d\mu < C_2$.

Proof. By Minkowski inequality, and using (2.1), we get

$$\left[\int_M \left(-\frac{1}{2}\Delta_h \sigma_n - K e^{\sigma_n}\right)^2 d\mu\right]^{1/2} \le \left[\int_M \left(K_0 - \frac{1}{2}\Delta_h \sigma_n - K e^{\sigma_n}\right)^2 d\mu\right]^{1/2} + \left[\int_M (K_0)^2 d\mu\right]^{1/2} \le m^{1/2} + c = C,$$

so that

$$\frac{1}{4}\int_M (\Delta_h\sigma_n)^2 d\mu + \int_M K^2 e^{2\sigma_n} d\mu + \int_M \Delta_h\sigma_n e^{\sigma_n} K\, d\mu \leq C^2. \tag{2.2}$$

Let $A_+^n = \{\Delta_h\sigma_n > 0\}$ and $A_-^n = \{\Delta_h\sigma_n < 0\}$.

$$\begin{aligned}\textstyle\int_M \Delta_h\sigma_n K e^{\sigma_n} d\mu &= \int_{A_+^n} \Delta_h\sigma_n K e^{\sigma_n} d\mu + \int_{A_-^n} \Delta_h\sigma_n K e^{\sigma_n} d\mu \\ &\geq \min K \int_{A_+^n} (\Delta_h\sigma_n) e^{\sigma_n} d\mu + \max K \int_{A_-^n} (\Delta_h\sigma_n) e^{\sigma_n} d\mu \\ &= \tau_n \int_M (\Delta_h\sigma_n) e^{\sigma_n} d\mu\end{aligned}$$

where

$$\begin{aligned}\tau_n &= \frac{(\min K)\int_{A_+^n}(\Delta_h\sigma_n)e^{\sigma_n} d\mu + (\max K)\int_{A_-^n}(\Delta_h\sigma_n)e^{\sigma_n} d\mu}{\int_M(\Delta_h\sigma_n)e^{\sigma_n} d\mu} \\ &= -\tau_n \int_M |\partial_z\sigma_n|^2 e^{\sigma_n} d\mu, \quad \text{by Stoke's theorem.}\end{aligned}$$

Note that $\tau_n \leq \max K < 0$. Thus by (2.2), we get

$$\frac{1}{4}\int_M (\Delta_h\sigma_n)^2 d\mu + \int_M K^2 e^{2\sigma_n} d\mu - \tau_n \int_M |\partial_z\sigma_n|^2 e^{\sigma_n} d\mu \leq C^2. \tag{2.3}$$

Since each term is positive, the result follows. □

2.3. *Pointwise convergence of zero mean-value part*

Next, for $\sigma \in C^\infty(M)$ denote by $m(\sigma)$ its mean value,

$$m(\sigma) = \int_M \sigma\, d\mu,$$

and by $\tilde{\sigma} = \sigma - m(\sigma)$ denote its zero-mean value part. For the minimizing sequence $\{\sigma_n\}$ we denote the corresponding mean values by m_n. (*Note*: We had normalized the volume $\int_M d\mu = 1$.)

Lemma 2.3. *The mean-value-zero part* $\{\tilde{\sigma}_n\}_{n=1}^\infty$ *of the minimizing sequence* $\{\sigma_n\}_{n=1}^\infty$ *is uniformly bounded in the Sobolev space* $W^{2,2}(M)$.

Proof. By Hodge theory, there exists an operator G such that $G\Delta_h = I - P$, where I is the identity operator in $L^2(M)$ and P is the orthogonal projection onto kernel of Δ_h. We also know $\Delta_h : W^{2,2} \to L^2$ boundedly and $G : L^2 \to W^{2,2}$ is a bounded operator.

Now, by Lemma 2.2 $\{\Delta_h\sigma_n\}$ are bounded uniformly in L^2. Thus, $\{G\Delta_h\sigma_n\}$ are bounded uniformly in $W^{2,2}$. But $G\Delta_h\sigma_n = (I - P)\sigma_n = \tilde{\sigma}_n$.

Now we can formulate the main result of this subsection.

PROPOSITION 2.4

The sequence $\{\tilde{\sigma}_{l_n}\}_{n=1}^\infty$ *contains a subsequence* $\{\tilde{\sigma}_{l_n}\}_{n=1}^\infty$ *with the following properties.*

(a) *The sequences* $\{\tilde{\sigma}_{l_n}\}_{n=1}^{\infty}$ *and* $\{\mathrm{e}^{\tilde{\sigma}_{l_n}+m_{l_n}}\}$ *converge in* $C^0(M)$ *topology to continuous functions* $\tilde{\sigma}$ *and* u *respectively. Moreover,* $\tilde{\sigma} \in W^{2,2}(M)$.

(b) *The subsequence* $\{\Delta_h\tilde{\sigma}_{l_n}\}$ *converges weakly in* L^2 *to* $f \doteq \Delta_h^{\mathrm{distr}}\tilde{\sigma}$ *— a distribution Laplacian of* $\tilde{\sigma}$.

(c) *Passing to this subsequence* $\{\tilde{\sigma}_{l_n}\}$, *the following limits exist*

$$\lim_{n\to\infty} \|\Delta_h\tilde{\sigma_{l_n}}\|_2 = \|\Delta_h^{\mathrm{distr}}\tilde{\sigma}\|_2,$$

$$\lim_{n\to\infty} S(\sigma_{l_n}) = S_0 = \int_M (K_0 - \frac{1}{2}\Delta_h^{\mathrm{distr}}\tilde{\sigma} - K u)^2 \mathrm{d}\mu,$$

where

$$\lim_{n\to\infty} \mathrm{e}^{\tilde{\sigma}_{l_n}+m_n} = u.$$

In fact, the convergence in (b) *is strong in* L^2.

Proof. Part (a) follows from the Sobolev embedding theorem and Rellich lemma since, for dim $M = 2$, the space $W^{2,2}(M)$ is compactly embedded into $C^0(M)$ (see, e.g. [1, 7]). Therefore the sequence $\{\tilde{\sigma}_n\}$, which, according to Lemma 2.3, is uniformly bounded in $W^{2,2}(M)$, contains a convergent subsequence in $C^0(M)$. Passing to this subsequence $\{\tilde{\sigma}_{l_n}\}$ we can assume that there exists mean-value zero function $\tilde{\sigma} \in C^0(M)$ such that

$$\lim_{n\to\infty} \tilde{\sigma}_{l_n} = \tilde{\sigma}.$$

Since $\tilde{\sigma}_n$'s are uniformly bounded in a Hilbert space $W^{2,2}(M)$, they weakly converge to $s \in W^{2,2}(M)$ (after passing to a subsequence if necessary). The uniform limit coincides with s so that $\tilde{\sigma} = s \in W^{2,2}(M)$.

Moreover, since

$$m_n = \int_M \sigma_n \mathrm{d}\mu \le \int_M \mathrm{e}^{2\sigma_n}\mathrm{d}\mu,$$

then by part (b) of Lemma 2.2 the sequence $\{m_n\}$ is bounded above. Passing to a subsequence, if necessary, we obtain that there exists $u \in C^0(M)$ such that in the $C^0(M)$ topology $\lim_{n\to\infty} \mathrm{e}^{\tilde{\sigma}_{l_n}+m_n} = u$. The function u may be identically zero if $m_n \to -\infty$.

In order to prove (b), set $\psi_n = \Delta_h\tilde{\sigma}_{l_n}$ and observe that, according to part (a) of Lemma 2.2, the sequence $\{\psi_n\}$ is bounded in L^2. Therefore, passing to a subsequence, if necessary, there exists $f \in L^2(M)$ such that

$$\lim_{n\to\infty} \int_M \psi_n g = \int_M fg$$

for all $g \in L^2(M)$. In particular, considering $g \in C^\infty(M)$, this implies $f = \Delta_h^{\mathrm{distr}}\tilde{\sigma}$.

In order to prove (c) we use the following lemma.

Lemma 2.5. *If a sequence* $\{\psi_n\}$ *converges to* $f \in L^2$ *in the weak topology, then*

$$\lim_{n\to\infty} \|\psi_n\| \ge \|f\|.$$

Further $\lim_{n\to\infty} \|\psi_n\| = \|f\|$ *iff there is strong convergence.*

Proof. The lemma follows from considering the following inequality:

$$\lim_{n\to\infty}\int(\psi_n - f)^2\mathrm{d}\mu \geq 0.$$

$\square$

To continue with the proof of the proposition, suppose $\lim_{n\to\infty}\|\psi_n\| > \|f\|$. Using the definition of the functional, we have

$$\begin{aligned} S(\sigma_n) &= \int_M\left(K_0 - \frac{1}{2}\Delta_h\tilde{\sigma}_n - K\,\mathrm{e}^{\tilde{\sigma}_n+m_n}\right)^2\mathrm{d}\mu \\ &= \frac{1}{4}\|\psi_n\|^2 + \|K_0 - K\,\mathrm{e}^{\tilde{\sigma}_n+m_n}\|^2 - \int_M\psi_n(K_0 - K\,\mathrm{e}^{\tilde{\sigma}_n+m_n})\mathrm{d}\mu. \end{aligned}$$

From parts (a) and (b) it follows that the sequence $S(\sigma_n)$ converges to S_0 and

$$\begin{aligned} S_0 &= \lim_{n\to\infty} S(\sigma_n) \\ &= \lim_{n\to\infty}\frac{1}{4}\|\psi_n\|^2 + \|K_0 - K\,u\|^2 - \int_M f(K_0 - K\,u)\mathrm{d}\mu \\ &> \frac{1}{4}\|f\|^2 + \|K_0 - K\,u\|^2 - \int_M f(K_0 - K\,u)\mathrm{d}\mu \\ &= \left\|-\frac{1}{2}f + K_0 - K\,u\right\|^2. \end{aligned}$$

We will show that this inequality contradicts that $\{\sigma_n\}$ was a minimizing sequence, i.e. we can construct a sequence $\{\tau + m_{l_n}\} \in C^\infty(M)$ such that $S(\tau + m_{l_n})$ gets as close to $\|-\frac{1}{2}f + K_0 - K\,u\|^2$ as we like.

Namely, for any $\epsilon > 0$ we can construct, by the density of C^∞ in $W^{2,2}$, a function $\tau \in C^\infty(M)$ approximating $\tilde{\sigma} \in W^{2,2}$ such that $\|\Delta_h\tau - f\| < \epsilon$ and $\|4(v-u)\| < \epsilon/2$ where $v = \lim_{n\to\infty}\mathrm{e}^{\tau+m_{l_n}}$. Since

$$S_\tau = \lim_{n\to\infty} S(\tau + m_{l_n}) = \left\|-\frac{1}{2}\Delta_h\tau + K_0 - K\,v\right\|^2,$$

we have

$$\left|\sqrt{S_\tau} - \left\|-\frac{1}{2}f + K_0 - K\,u\right\|\right| \leq \left\|\frac{1}{2}(f - \Delta_h\tau) - K\,(v-u)\right\| \leq \epsilon.$$

Now setting $\delta = \sqrt{S_0} - \|-\frac{1}{2}f + K_0 - K\,u\| > 0$ and choosing $\epsilon < \delta/2$, and using $\sqrt{S_\tau} \leq \|-\frac{1}{2}f + K_0 - K\,u\| + \epsilon$ we get $\sqrt{S_\tau} < \sqrt{S_0} - \frac{\delta}{2}$—a contradiction, since S_0 was the infimum of the functional.

Thus, $\lim_{n\to\infty}\|\Delta_h\tilde{\sigma}_n\| = \|f\|$, so that, in fact, by Lemma 2.5, the convergence is in the strong L^2 topology. This proves part (c).

2.4 *Convergence and the non-degeneracy*

PROPOSITION 2.6

The minimizing sequence $\{\sigma_n\}_{n=1}^\infty$ contains a subsequence that converges in $C^0(M)$ to a function $\sigma \in C^0(M)$, so that the resulting metric $\mathrm{e}^\sigma h$ is non-degenerate.

Proof. Since $\sigma_n = \tilde{\sigma} + m_n$, by Proposition 2.4 and Lemma 2.3, it is enough to show that the sequence $\{m_n\}$ is bounded below. Supposing the contrary and passing, if necessary, to a subsequence, we can assume that

$$\lim_{n\to\infty} m_n = -\infty,$$

so that, in notations of Proposition 2.4, $u = 0$. By Proposition 2.4(c) we get

$$S_0 = \lim_{n\to\infty} S(\sigma_n) = \int_M \left(K_0 - \frac{1}{2}\Delta_h^{\mathrm{distr}}\tilde{\sigma}\right)^2 \mathrm{d}\mu.$$

We shall show that this contradicts the fact that S_0 is the infimum of the functional S and that $\{\sigma_n\}$ is a minimizing sequence. First we have the following lemma.

Lemma 2.7. *Let* $b = K_0 - \frac{1}{2}\Delta_h^{\mathrm{distr}}\tilde{\sigma} \in L^2(M)$, *where* $\tilde{\sigma}_n \to \tilde{\sigma}$ *and* $m_n \to -\infty$ *as* $n \to \infty$. *Then*

$$\int_M b\Delta_h\beta \mathrm{d}\mu = 0$$

for all $\beta \in C^\infty(M)$ *and* $b \equiv -L$, *where* L *is a positive constant.*

Proof. Consider $G_n(t) = S(\sigma_n + t\beta) - S_0$—a smooth function of t for a fixed β. Then by Proposition 2.4(c) we have

$$\begin{aligned} G(t) &= \lim_{n\to\infty} G_n(t) \\ &= \int_M \left(K_0 - \frac{1}{2}\Delta_h^{\mathrm{distr}}(\tilde{\sigma} + t\beta)\right)^2 \mathrm{d}\mu - \int_M \left(K_0 - \frac{1}{2}\Delta_h^{\mathrm{distr}}\tilde{\sigma}\right)^2 \mathrm{d}\mu, \end{aligned}$$

and $G(t)$ is a smooth function of t for fixed β. Since S_0 is the infimum of S, we have that $G(t) \geq 0$ for all t and $G(0) = 0$. Therefore it follows that

$$\left.\frac{\mathrm{d}G}{\mathrm{d}t}\right|_{t=0} = 0$$

for all $\beta \in C^\infty(M)$. Straightforward computation yields

$$\left.\frac{\mathrm{d}G}{\mathrm{d}t}\right|_{t=0} = -\int_M b\Delta_h\beta\mathrm{d}\mu.$$

Therefore, b satisfies the Laplace equation $\Delta_h b = 0$ in a distributional sense and from elliptic regularity it follows that b is smooth. Thus b is harmonic and therefore is a constant. Finally, by the Gauss–Bonnet theorem, we have $\int_M b\mathrm{d}\mu = 4\pi(1-g)$ and recalling that $g > 1$, we conclude that $b = 4\pi(1-g) = -L < 0$. □

To complete the proof of the proposition, we get a contradiction as follows. By Lemma 2.7, we have that $S_0 = \int_M (-L)^2\mathrm{d}\mu = L^2$ is the infimum of the functional. Since $L > 0$, and $\{m_n\} \to -\infty$, we consider $\tau = \tilde{\sigma} + m_n$ and choose n large enough so that $-K\mathrm{e}^\tau < L/2$. We have

$$S(u) = \int_M \left(K_0 - \frac{1}{2}\Delta_h\tilde{\sigma} - K\mathrm{e}^\tau\right)^2 \mathrm{d}\mu = \int_M (-L - K\mathrm{e}^\tau)^2\mathrm{d}\mu.$$

Then, since $-L + \alpha < -L - K\mathrm{e}^\tau < -L/2$, where $\alpha > 0$ is the infimum of $-K\mathrm{e}^\tau$, we have $(-L - K\mathrm{e}^\tau)^2 < (L-\alpha)^2$ so that $S(\tau) < L^2$—a contradiction. □

3. Smoothness and uniqueness

Here we complete the proof of the main Theorem 3.1.

PROPOSITION 3.1

The minimizing function $\sigma \in C^0(M)$ is smooth and corresponds to the unique Kähler metric of negative curvature K.

Proof. Let $b = (K_0 - \frac{1}{2}\Delta_h^{\mathrm{distr}}\sigma - Ke^{\sigma}) \in L^2(M)$; according to Proposition 2.4(c) and Proposition 2.6, $S_0 = \int_M b^2 \mathrm{d}\mu$. Set $G(t) = S(\sigma + t\beta) - S_0$, where $\beta \in C^{\infty}(M)$. Repeating arguments in the proof of Lemma 2.7, we conclude that $G(t)$ for fixed β is smooth, $G(0) = 0$ and $G(t) \geq 0$ for all t. Therefore,

$$\left.\frac{\mathrm{d}G}{\mathrm{d}t}\right|_{t=0} = 0.$$

A simple calculation yields

$$\left.\frac{\mathrm{d}G}{\mathrm{d}t}\right|_{t=0} = \int_M (-b\Delta_h\beta - 2Ke^{\sigma}b\beta)\mathrm{d}\mu.$$

Thus $b \in L^2(M)$ satisfies, in a distributional sense, the following equation

$$-\Delta_h b - 2Ke^{\sigma}b = 0. \tag{3.1}$$

First, we will show that $b = 0$ is the only weak L^2 solution to eq. (3.1). Indeed, by elliptic regularity b is smooth, so that multiplying (3.1) by b and integrating over M using the Stokes formula, we get

$$\int_M \mathrm{d}b \wedge * \, \mathrm{d}b + \int_M b^2 e^{\sigma}\mathrm{d}\mu = 0,$$

which implies that $b = 0$. Thus we have shown that $S_0 = 0$.

Second, equation $b = 0$ for the minimizing function $\sigma \in C^0(M)$ reads

$$\frac{1}{2}\Delta_h^{\mathrm{distr}}\sigma = K_0 - Ke^{\sigma} \in C^0(M). \tag{3.2}$$

Therefore, $\Delta_h^{\mathrm{distr}}\sigma$ belongs to $L^p(M)$ so that $\sigma \in W^{2,p}$ for all p. By the Sobolev embedding theorem it follows that $\sigma \in C^{1,\alpha}(M)$ for some $0 < \alpha < 1$. Therefore, the right hand side of eq. (3.2) actually belongs to the space $C^{1,\alpha}(M)$, and therefore $\sigma \in C^{3,\alpha}(M)$ and so on. This kind of bootstrapping argument shows that σ is smooth [7].

The equation $b \equiv 0$ satisfied by σ now translates to $K(\sigma) \equiv K$, where $K(\sigma)$ is the Gaussian curvature of the metric $e^{\sigma}h\mathrm{d}z \otimes \mathrm{d}\bar{z}$, $\sigma \in C^{\infty}(M)$.

The minimizing function σ is unique: here is the standard argument, which goes back to Poincaré. Let η be another minimizing function, which is smooth and also satisfies eq. (3.2)

$$\frac{1}{2}\Delta_h\eta = K_0 - Ke^{\eta},$$

so that

$$\Delta_h(\sigma - \eta) = -2K(\mathrm{e}^{\sigma} - \mathrm{e}^{\eta}).$$

Multiplying this equation by $\sigma - \eta$ and integrating over M with the help of Stokes formula, we get

$$-\int_M \mathrm{d}\xi \wedge * \,\mathrm{d}\xi = \int_M -2K(\sigma - \eta)(\mathrm{e}^{\sigma} - \mathrm{e}^{\eta})\mathrm{d}\mu,$$

where we set $\xi = \sigma - \eta$. Since $-2K(\sigma - \eta)(\mathrm{e}^{\sigma} - \mathrm{e}^{\eta}) \geq 0$, we conclude that $\mathrm{d}\xi = 0$ and, in fact, $\xi = 0$.

The proof of Theorem 3.1 is complete.

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Homogeneous operators and projective representations of the Möbius group: A survey

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Abstract. This paper surveys the existing literature on homogeneous operators and their relationships with projective representations of $PSL(2, \mathbb{R})$ and other Lie groups. It also includes a list of open problems in this area.

Keywords. Projective representations; homogeneous operators; reproducing kernels; Sz-Nagy–Foias characteristic functions.

1. Preliminaries

This paper is a survey of the known results on homogeneous operators. A small proportion of these results are as yet available only in preprint form. A miniscule proportion may even be new. The paper ends with a list of thirteen open problems suggesting possible directions for future work in this area. This list is not purported to be exhaustive, of course!

All Hilbert spaces in this paper are separable Hilbert spaces over the field of complex numbers. All operators are bounded linear operators between Hilbert spaces. If $\mathcal{H}$, $\mathcal{K}$ are two Hilbert spaces, $\mathcal{B}(\mathcal{H}, \mathcal{K})$ will denote the Banach space of all operators from $\mathcal{H}$ to $\mathcal{K}$, equipped with the usual operator norm. If $\mathcal{H} = \mathcal{K}$, this will be abridged to $\mathcal{B}(\mathcal{H})$. The group of all unitary operators in $\mathcal{B}(\mathcal{H})$ will be denoted by $\mathcal{U}(\mathcal{H})$. When equipped with any of the usual operator topology $\mathcal{U}(\mathcal{H})$ becomes a topological group. All these topologies induce the same Borel structure on $\mathcal{U}(\mathcal{H})$. We shall view $\mathcal{U}(\mathcal{H})$ as a Borel group with this structure.

$\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ will denote the integers, the real numbers and the complex numbers, respectively. $\mathbb{D}$ and $\mathbb{T}$ will denote the open unit disc and the unit circle in $\mathbb{C}$, respectively, and $\bar{\mathbb{D}}$ will denote the closure of $\mathbb{D}$ in $\mathbb{C}$. Möb will denote the Möbius group of all biholomorphic automorphisms of $\mathbb{D}$. Recall that Möb $= \{\varphi_{\alpha,\beta} : \alpha \in \mathbb{T}, \beta \in \mathbb{D}\}$, where

$$\varphi_{\alpha,\beta}(z) = \alpha \frac{z - \beta}{1 - \bar{\beta} z}, \quad z \in \mathbb{D}. \tag{1.1}$$

For $\beta \in \mathbb{D}$, $\varphi_\beta := \varphi_{-1,\beta}$ is the unique involution (element of order 2) in Möb which interchanges 0 and β. Möb is topologized via the obvious identification with $\mathbb{T} \times \mathbb{D}$. With this topology, Möb becomes a topological group. Abstractly, it is isomorphic to $PSL(2, \mathbb{R})$ and to $PSU(1, 1)$.

The following definition from [6] has its origin in the papers [21] and [22] by the second named author.

DEFINITION 1.1

An operator T is called *homogeneous* if $\varphi(T)$ is unitarily equivalent to T for all φ in Möb which are analytic on the spectrum of T.

It was shown in Lemma 2.2 of [6] that

Theorem 1.1. *The spectrum of any homogeneous operator T is either $\mathbb{T}$ or $\bar{\mathbb{D}}$. Hence $\varphi(T)$ actually makes sense (and is unitarily equivalent to T) for all elements φ of* Möb.

Let $*$ denote the involution (i.e. automorphism of order two) of Möb defined by

$$\varphi^*(z) = \overline{\varphi(\bar{z})}, \quad z \in \mathbb{D}, \ \varphi \in \text{Möb}. \tag{1.2}$$

Thus $\varphi^*_{\alpha,\beta} = \varphi_{\bar{\alpha},\bar{\beta}}$ for $(\alpha, \beta) \in \mathbb{T} \times \mathbb{D}$. It is known that essentially (i.e. up to multiplication by arbitrary inner automorphisms), $*$ is the only outer automorphism of Möb. It also satisfies $\varphi^*(z) = \varphi(z^{-1})^{-1}$ for $z \in \mathbb{T}$. It follows that for any operator T whose spectrum is contained in $\bar{\mathbb{D}}$, we have

$$\varphi(T^*) = \varphi^*(T)^*, \quad \varphi(T^{-1}) = \varphi^*(T)^{-1}, \tag{1.3}$$

the latter in case T is invertible, of course. It follows immediately from (1.3) that the adjoint T^* – as well as the inverse T^{-1} in case T is invertible – of a homogeneous operator T is again homogeneous.

Clearly a direct sum (more generally, direct integral) of homogeneous operators is again homogeneous.

2. Characteristic functions

Recall that an operator T is called a *contraction* if $\|T\| \leq 1$, and it is called *completely non-unitary* (cnu) if T has no non-trivial invariant subspace $\mathcal{M}$ such that the restriction of T to $\mathcal{M}$ is unitary. T is called a *pure contraction* if $\|Tx\| < \|x\|$ for all non-zero vectors x. To any cnu contraction T on a Hilbert space, Sz-Nagy and Foias associate in [25] a pure contraction valued analytic function θ_T on $\mathbb{D}$, called the *characteristic function* of T.

Reading through [25] one may get the impression that the characteristic function is only contraction valued and its value at 0 is a pure contraction. However, if θ is a contraction valued analytic function on $\mathbb{D}$ and the value of θ at some point is pure, its value at all points must be pure contractions. This is immediate on applying the strong maximum modulus principle to the function $z \to \theta(z)x$, where x is an arbitrary but fixed non-zero vector.

Two pure contraction valued analytic functions $\theta_i : \mathbb{D} \to \mathcal{B}(\mathcal{K}_i, \mathcal{L}_i)$, $i = 1, 2$ are said to *coincide* if there exist two unitary operators $\tau_1 : \mathcal{K}_1 \to \mathcal{K}_2$, $\tau_2 : \mathcal{L}_1 \to \mathcal{L}_2$ such that $\theta_2(z)\tau_1 = \tau_2\theta_1(z)$ for all $z \in \mathbb{D}$. The theory of Sz-Nagy and Foias shows that (i) two cnu contractions are unitarily equivalent if and only if their characteristic functions coincide, (ii) any pure contraction valued analytic function is the characteristic function of some cnu contraction. In general, the model for the operator associated with a given function θ is difficult to describe. However, if θ is an inner function (i.e., θ is isometry-valued on the boundary of $\mathbb{D}$), the description of the Sz-Nagy and Foias model simplifies as follows:

Theorem 2.1. *Let $\theta : \mathbb{D} \to \mathcal{B}(\mathcal{K}, \mathcal{L})$ be a pure contraction valued inner analytic function. Let $\mathcal{M}$ denote the invariant subspace of $H^2(\mathbb{D}) \otimes \mathcal{L}$ corresponding to θ in the sense of*

Beurling's theorem. That is, $\mathcal{M} = \{z \mapsto \theta(z)f(z) : f \in H^2(\mathbb{D}) \otimes \mathcal{K}\}$. *Then* θ *coincides with the characteristic function of the compression of multiplication by* z *to the subspace* $\mathcal{M}^\perp$.

From the general theory of Sz-Nagy and Foias outlined above, it follows that if T is a cnu contraction with characteristic function θ then, letting $T[\mu]$ denote the cnu contraction with characteristic function $\mu\theta$ for $0 < \mu \le 1$, we find that $\{T[\mu] : 0 < \mu \le 1\}$ is a continuum of mutually unitarily inequivalent cnu contractions (provided θ is not the identically zero function, of course). In general, it is difficult to describe these operators explicitly in terms of T alone. But, in [7], we succeeded in obtaining such a description in case θ is an inner function (equivalently, when T is in the class $C_{.0}$, i.e., $T^{*n}x \to 0$ as $n \to \infty$ for every vector x) – so that T has the description in terms of θ given in Theorem 2.1. Namely, for a suitable Hilbert space $\mathcal{L}$, T may be identified with the compression of M to $\mathcal{M}^\perp$, where $M : H^2_{\mathcal{L}} := H^2(\mathbb{D}) \otimes \mathcal{L} \to H^2_{\mathcal{L}}$ is multiplication by the co-ordinate function and $\mathcal{M}$ is the invariant subspace for M corresponding to the inner function θ. Let $M = \begin{pmatrix} M_{11} & 0 \\ M_{21} & M_{22} \end{pmatrix}$ be the block matrix representation of M corresponding to the decomposition $H^2_{\mathcal{L}} = \mathcal{M}^\perp \oplus \mathcal{M}$. (Thus, in particular, $T = M_{11}$ and M_{22} is the restriction of M to $\mathcal{M}$.) Finally, let $\mathcal{K}$ denote the co-kernel of M_{22}, $N : H^2_{\mathcal{K}} \to H^2_{\mathcal{K}}$ be multiplication by the co-ordinate function and let $E : H^2_{\mathcal{K}} \to \mathcal{M}$ be defined by $Ef = f(0) \in \mathcal{K}$. In terms of these notations, we have

Theorem 2.2. *Let* T *be a cnu contraction in the class* $C_{.0}$ *with characteristic function* θ. *Let* μ *be a scalar in the range* $0 < \mu < 1$ *and put* $\delta = \sqrt{1 - \mu^2}$. *Then, with respect to the decomposition* $\mathcal{M}^\perp \oplus \mathcal{M} \oplus H^2_{\mathcal{K}}$ *of its domain, the operator* $T[\mu] : H^2_{\mathcal{L}} \oplus H^2_{\mathcal{L}} \oplus H^2_{\mathcal{K}} \to H^2_{\mathcal{K}}$ *has the block matrix representation*

$$T[\mu] = \begin{pmatrix} M_{11} & 0 & 0 \\ \delta M_{21} & M_{22} & \mu E \\ 0 & 0 & N^* \end{pmatrix}.$$

In Theorem 2.9 of [6], it was noted that

Theorem 2.3. *A pure contraction valued analytic function* θ *on* $\mathbb{D}$ *is the characteristic function of a homogeneous cnu contraction if and only if* $\theta \circ \varphi$ *coincides with* θ *for every* φ *in* Möb.

From this theorem, it is immediate that whenever T is a homogeneous cnu contraction, so are the operators $T[\mu]$ given by Theorem 2.2. Some interesting examples of this phenomenon were worked out in [7]. See §6 for these examples.

As an interesting particular case of Theorem 2.3, one finds that any cnu contraction with a constant characteristic function is necessarily homogeneous. These operators are discussed in [11] and [6]. Generalizing a result in [6], Kerchy shows in [19] that

Theorem 2.4. *Let* θ *be the characteristic function of a homogeneous cnu contraction. If* $\theta(0)$ *is a compact operator then* θ *must be a constant function.*

(Actually Kerchy proves the same theorem with the weaker hypothesis that all the points in the spectrum of $\theta(0)$ are isolated from below.)

Sketch of Proof. Let $\theta : \mathbb{D} \to \mathcal{B}(\mathcal{K}, \mathcal{L})$ be the characteristic function of a homogeneous operator. Assume $C := \theta(0)$ is compact. Replacing θ by a coincident analytic function if neceesary, we may assume without loss of generality that $\mathcal{K} = \mathcal{L}$ and $C \geq 0$. By Theorem 2.3 there exists unitaries U_z, V_z such that $\theta(z) = U_z C V_z$, $z \in \mathbb{D}$. Let $\lambda_1 > \lambda_2 > \cdots$ be the non-zero eigenvalues of the compact positive operator C. At this point Kerchy shows that (as a consequence of the maximum modulus principle for Hilbert space valued analytic functions) the eigenspace $\mathcal{K}_1$ corresponding to the eigenvalue λ_1 is a common reducing subspace for U_z, V_z, $z \in \mathbb{D}$ (as well as for C of course) and hence for $\theta(z)$, $z \in \mathbb{D}$. So we can write $\theta(z) = \theta_1(z) \oplus \theta_2(z)$ where θ_1 is an analytic function into $\mathcal{B}(\mathcal{K}_1)$. Since θ_1 is a unitary valued analytic function, it must be a constant. Repeating the same argument with θ_2, one concludes by induction on n that the eigenspace $\mathcal{K}_n$ corresponding to the eigenvalue λ_n is reducing for $\theta(z)$, $z \in \mathbb{D}$, and the projection of θ to each $\mathcal{K}_n$ is a constant function. Since the same is obviously true of the zero eigenvalue, we are done.

3. Representations and multipliers

Let G be a locally compact second countable topological group. Then a measurable function $\pi : G \to \mathcal{U}(\mathcal{H})$ is called a *projective representation* of G on the Hilbert space $\mathcal{H}$ if there is a function (necessarily Borel) $m : G \times G \to \mathbb{T}$ such that

$$\pi(1) = I, \ \pi(g_1 g_2) = m(g_1, g_2)\pi(g_1)\pi(g_2) \tag{3.1}$$

for all g_1, g_2 in G. (More precisely, such a function π is called a projective unitary representation of G; however, we shall often drop the adjective unitary since all representations considered in this paper are unitary.) The projective representation π is called an ordinary representation (and we drop the adjective 'projective') if m is the constant function 1. The function m associated with the projective representation π via (3.1) is called the multiplierof π. The ordinary representation π of G which sends every element of G to the identity operator on a one dimensional Hilbert space is called the identity (or trivial) representation of G. It is surprising that although projective representations have been with us for a long time (particularly in the Physics literature), no suitable notion of equivalence of projective representations seems to be available. In [7], we offered the following:

DEFINITION 3.1

Two projective representations π_1, π_2 of G on the Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$ (respectively) will be called equivalent if there exists a unitary operator $U : \mathcal{H}_1 \to \mathcal{H}_2$ and a function (necessarily Borel) $f : G \to \mathbb{T}$ such that $\pi_2(\varphi)U = f(\varphi)U\pi_1(\varphi)$ for all $\varphi \in G$.

We shall identify two projective representations if they are equivalent. This has the some what unfortunate consequence that any two one dimensional projective representations are identified. But this is of no importance if the group G has no ordinary one dimensional representation other than identity representation (as is the case for all semi-simple Lie groups G.) In fact, the above notion of equivalence (and the resulting identifications) saves us from the following disastrous consequence of the above (commonly accepted) notion of projective representations: Any Borel function from G into $\mathbb{T}$ is a (one dimensional) projective representation of the group!!

3.1 *Multipliers and cohomology*

Notice that the requirement (3.1) on a projective representation implies that its associated multiplier m satisfies

$$m(\varphi, 1) = 1 = m(1, \varphi), \ \ m(\varphi_1, \varphi_2)m(\varphi_1\varphi_2, \varphi_3) = m(\varphi_1, \varphi_2\varphi_3)m(\varphi_2, \varphi_3) \quad (3.2)$$

for all elements $\varphi, \varphi_1, \varphi_2, \varphi_3$ of G. Any Borel function $m : G \times G \to \mathbb{T}$ satisfying (3.2) is called a *multiplier* of G. The set of all multipliers on G form an abelian group $M(G)$, called the multiplier group of G. If $m \in M(G)$, then taking $\mathcal{H} = L^2(G)$ (with respect to Haar measure on G), define $\pi : G \to \mathcal{U}(\mathcal{H})$ by

$$\Big(\pi(\varphi)f\Big)(\psi) = m(\psi, \varphi)f(\psi\varphi) \quad (3.3)$$

for φ, ψ in G, f in $L^2(G)$. Then one readily verifies that π is a projective representation of G with associated multiplier m. Thus each element of $M(G)$ actually occurs as the multiplier associated with a projective representation. A multiplier $m \in M(G)$ is called *exact* if there is a Borel function $f : G \to \mathbb{T}$ such that $m(\varphi_1, \varphi_2) = (f(\varphi_1)f(\varphi_2))/f(\varphi_1\varphi_2)$ for φ_1, φ_2 in G. Equivalently, m is exact if any projective representation with multiplier m is equivalent to an ordinary representation. The set $M_0(G)$ of all exact multipliers on G form a subgroup of $M(G)$. Two multipliers m_1, m_2 are said to be equivalent if they belong to the same coset of $M_0(G)$. In other words, m_1 and m_2 are equivalent if there exist equivalent projective representations π_1, π_2 whose multipliers are m_1 and m_2 respectively. The quotient $M(G)/M_0(G)$ is denoted by $H^2(C, \mathbb{T})$ and is called the second cohomology group of G with respect to the trivial action of G on $\mathbb{T}$ (see [24] for the relevant group cohomology theory). For $m \in M(G)$, $[m] \in H^2(G, \mathbb{T})$ will denote the cohomology class containing m, i.e., $[\ \ ] : M(G) \to H^2(G, \mathbb{T})$ is the canonical homomorphism.

The following theorem from [8] (also see [9]) provides an explicit description of $H^2(G, \mathbb{T})$ for any connected semi-simple Lie group G.

Theorem 3.1. *Let G be a connected semi-simple Lie group. Then $H^2(G, \mathbb{T})$ is naturally isomorphic to the Pontryagin dual $\widehat{\pi^1(G)}$ of the fundamental group $\pi^1(G)$ of G.*

Explicitly, if $\tilde{G}$ is the universal cover of G and $\pi : \tilde{G} \to G$ is the covering map (so that the fundamental group $\pi^1(G)$ is naturally identified with the kernel Z of π) then choose a Borel section $s : G \to \tilde{G}$ for the covering map (i.e., s is a Borel function such that $\pi \circ s$ is the identity on G, and $s(1) = 1$). For $\chi \in \widehat{Z}$, define $m_\chi : G \times G \to \mathbb{T}$ by

$$m_\chi(x, y) = \chi(s(y)^{-1}s(x)^{-1}s(xy)), \ \ x, y \in G. \quad (3.4)$$

Then the main theorem in [8] shows that $\chi \mapsto [m_\chi]$ is an isomorphism from $\widehat{Z}$ onto $H^2(G, \mathbb{T})$ and this isomorphism is independent of the choice of the section s.

The following companion theorem from [8] shows that to find all the irreducible projective representations of a group G satisfying the hypotheses of Theorem 3.1, it suffices to find the ordinary irreducible representations of its universal cover $\tilde{G}$. Let Z be the kernel of the covering map from $\tilde{G}$ onto G. Let β be an ordinary unitary representation of $\tilde{G}$. Then we shall say that β is of *pure type* if there is a character χ of Z such that $\beta(z) = \chi(z)I$ for all z in Z. If we wish to emphasize the particular character which occurs here, we may also say that β is pure of type χ. Notice that, if β is irreducible then (as Z is central) by Schur's

Lemma β is necessarily of pure type. In terms of this definition, the second theorem in [8] says

Theorem 3.2. *Let G be a connected semi-simple Lie group and let $\tilde{G}$ be its universal cover. Then there is a natural bijection between (the equivalence classes of) projective unitary representations of G and (the equivalence classes of) ordinary unitary representations of pure type of $\tilde{G}$. Under this bijection, for each χ the projective representations of G with multiplier m_χ correspond to the representations of $\tilde{G}$ of pure type χ, and vice versa. Further, the irreducible projective representations of G correspond to the irreducible representations of $\tilde{G}$, and vice versa.*

Explicitly, if β is an ordinary representation of pure type χ of $\tilde{G}$ then define $f_\chi : \tilde{G} \to \mathbb{T}$ by $f_\chi(x) = \chi(x^{-1} \cdot s \circ \pi(x))$, $x \in \tilde{G}$. Define $\tilde{\alpha}$ on $\tilde{G}$ by $\tilde{\alpha}(x) = f_\chi(x)\beta(x)$. Then $\tilde{\alpha}$ is a projective representation of $\tilde{G}$ which is trivial on Z. Therefore there is a well-defined (and uniquely determined) projective representation α of G such that $\tilde{\alpha} = \alpha \circ \pi$. The multiplier associated with α is m_χ. The map $\beta \mapsto \alpha$ is the bijection mentioned in Theorem 3.2.

Finally, as was pointed out in [9], any projective representation (say with multiplier m) of a connected semi-simple Lie group can be written as a direct integral of irreducible projective representations (all with the same multiplier m) of the group. It follows, of course, that any multiplier of such a group arises from irreducible projective representations. It also shows that, in order to have a description of all the projective representations, it is sufficient to have a list of the irreducible ones and to know when two of them have identical multipliers. This is where Theorems 3.1 and 3.2 come in handy.

3.2 *The multipliers on Möb*

Notice that for any element φ of the Möbius group, φ' is a non-vanishing analytic function on $\bar{\mathbb{D}}$ and hence has a continuous logarithm on this closed disc. Let us fix, once for all, a Borel determination of these logarithms. More precisely, we fix a Borel function $(z, \varphi) \mapsto \log \varphi'(z)$ from $\bar{\mathbb{D}} \times$ Möb into $\mathbb{C}$ such that $\log \varphi'(z) \equiv 0$ for $\varphi = \mathrm{id}$. Now define $\arg \varphi'(z)$ to be the imaginary part of $\log \varphi'(z)$.

Define the Borel function $\mathbf{n}$: Möb × Möb $\to \mathbb{Z}$ by

$$\mathbf{n}(\varphi_1^{-1}, \varphi_2^{-1}) = \frac{1}{2\pi}(\arg(\varphi_2\varphi_1)'(0) - \arg \varphi_1'(0) - \arg \varphi_2'(\varphi_1(0))).$$

For any $\omega \in \mathbb{T}$, define m_ω : Möb × Möb $\to \mathbb{T}$ by

$$m_\omega(\varphi_1, \varphi_2) = \omega^{\mathbf{n}(\varphi_1, \varphi_2)}.$$

The following proposition is a special case of Theorem 3.1. Detailed proofs may be found in [9].

PROPOSITION 3.1

For $\omega \in \mathbb{T}$, m_ω is a multiplier of Möb. *It is trivial if and only if $\omega = 1$. Every multiplier on* Möb *is equivalent to m_ω for a uniquely determined ω in $\mathbb{T}$. In other words, $\omega \mapsto [m_\omega]$ is a group isomorphism between the circle group $\mathbb{T}$ and the second cohomology group H^2(Möb, $\mathbb{T}$).*

3.3 *The projective representations of the Möbius group*

Every projective representation of a connected semi-simple Lie group is a direct integral of irreducible projective representations (cf. [9], Theorem 3.1). Hence, for our purposes, it suffices to have a complete list of these irreducible representations of Möb. A complete list of the (ordinary) irreducible unitary representations of the universal cover of Möb was obtained by Bargmann (see [29] for instance). Since Möb is a semi-simple and connected Lie group, one may manufacture all the irreducible projective representations of Möb (with Bargmann's list as the starting point) via Theorem 3.2. Following [8] and [9], we proceed to describe the result. (*Warning*: Our parametrization of these representations differs somewhat from the one used by Bargmann and Sally. We have changed the parametrization in order to produce a unified description.)

For $n \in \mathbb{Z}$, let $f_n : \mathbb{T} \to \mathbb{T}$ be defined by $f_n(z) = z^n$. In all of the following examples, the Hilbert space $\mathcal{F}$ is spanned by an orthogonal set $\{f_n : n \in I\}$, where I is some subset of $\mathbb{Z}$. Thus the Hilbert space of functions is specified by the set I and $\{\|f_n\|, n \in I\}$. (In each case, $\|f_n\|$ behaves at worst like a polynomial in $|n|$ as $n \to \infty$, so that this really defines a space of function on $\mathbb{T}$.) For $\varphi \in$ Möb and complex parameters λ and μ, define the operator $R_{\lambda,\mu}(\varphi^{-1})$ on $\mathcal{F}$ by

$$(R_{\lambda,\mu}(\varphi^{-1})f)(z) = \varphi'(z)^{\lambda/2}|\varphi'(z)|^{\mu}(f(\varphi(z)), \quad z \in \mathbb{T}, \ f \in \mathcal{F}, \ \varphi \in \text{Möb}.$$

Here one defines $\varphi'(z)^{\lambda/2}$ as $\exp \lambda/2 \log \varphi'(z)$ using the previously fixed Borel determination of these logarithms.

Of course, there is no *a priori* guarantee that $R_{\lambda,\mu}(\varphi^{-1})$ is a unitary (or even bounded) operator. But, when it is unitary for every φ in Möb, it is easy to see that $R_{\lambda,\mu}$ is then a projective representation of Möb with associated multiplier m_ω, where $\omega = e^{i\pi\lambda}$. Thus the description of the representation is complete if we specify I, $\{\|f_n\|^2, n \in I\}$ and the two parameters λ, μ. It turns out that almost all the irreducible projective representations of Möb have this form.

In terms of these notations, here is the complete list of the irreducible projective unitary representations of Möb. (However, see the concluding remark of this section.)

- *Principal series representations* $P_{\lambda,s}$, $-1 < \lambda \leq 1$, s purely imaginary. Here $\lambda = \lambda$, $\mu = \frac{1-\lambda}{2} + s$, $I = Z$, $\|f_n\|^2 = 1$ for all n (so the space is $L^2(\mathbb{T})$).

- *Holomorphic discrete series representations* D_λ^+: Here $\lambda > 0$, $\mu = 0$, $I = \{n \in Z : n \geq 0\}$ and $\|f_n\|^2 = \frac{\Gamma(n+1)\Gamma(\lambda)}{\Gamma(n+\lambda)}$ for $n \geq 0$. For each f in the representation space there is an $\tilde{f}$, analytic in $\mathbb{D}$, such that f is the non-tangential boundary value of $\tilde{f}$. By the identification $f \leftrightarrow \tilde{f}$, the representation space may be identified with the functional Hilbert space $\mathcal{H}^{(\lambda)}$ of analytic functions on $\mathbb{D}$ with reproducing kernel $(1 - z\bar{w})^{-\lambda}$, $z, w \in \mathbb{D}$.

- *Anti-holomorphic discrete series representations* D_λ^-, $\lambda > 0$: D_λ^- may be defined as the composition of D_λ^+ with the automorphism $*$ of eq. (1.2): $D_\lambda^-(\varphi) = D_\lambda^+(\varphi^*)$, φ in Möb. This may be realized on a functional Hilbert space of anti-holomorphic functions on $\mathbb{D}$, in a natural way.

- *Complementary series representation* $C_{\lambda,\sigma}$, $-1 < \lambda < 1$, $0 < \sigma < \frac{1}{2}(1 - |\lambda|)$: Here $\lambda = \lambda$, $\mu = \frac{1}{2}(1 - \lambda) + \sigma$, $I = Z$, and

$$\|f_n\|^2 = \prod_{k=0}^{|n|-1} \frac{k \pm \frac{\lambda}{2} + \frac{1}{2} - \sigma}{k \pm \frac{\lambda}{2} + \frac{1}{2} + \sigma}, \quad n \in Z,$$

where one takes the upper or lower sign according as n is positive or negative.

Remark 3.1. (a) All these projective representation of Möb are irreducible with the sole exception of $P_{1,0}$ for which we have the decomposition $P_{1,0} = D_1^+ \oplus D_1^-$. (b) The multiplier associated with each of these representations is m_ω where $\omega = \mathrm{e}^{-i\pi\lambda}$ if the representation is in the anti-holomorphic discrete series, and $\omega = \mathrm{e}^{i\pi\lambda}$ otherwise. It follows that the multipliers associated with two representations π_1 and π_2 from this list are either identical or inequivalent. Further, if neither or both of π_1 and π_2 are from the anti-holomorphic discrete series, then their multipliers are identical iff their λ parameters differ by an even integer. In the contrary case (i.e., if exactly one of π_1 and π_2 is from the anti-holomorphic discrete series), then they have identical multipliers iff their λ parameters add to an even integer. This is Corollary 3.2 from [9]. Using this information, one can now describe all the projective representations of Möb (at least in principle).

4. Projective representations and homogeneous operators

If T is an operator on a Hilbert space $\mathcal{H}$ then a projective representation π of Möb on $\mathcal{H}$ is said to be associated with T if the spectrum of T is contained in $\bar{\mathbb{D}}$ and

$$\varphi(T) = \pi(\varphi)^* T \pi(\varphi) \tag{4.1}$$

for all elements φ of Möb. Clearly, if T has an associated representation then T is homogeneous. In the converse direction, we have

Theorem 4.1. *If T is an irreducible homogeneous operator then T has a projective representation of Möb associated with it. This projective representation is unique up to equivalence.*

We sketch a proof of Theorem 4.1 below. The details of the proof may be found in [9]. The existence part of this theorem was first proved in [23] using a powerful selection theorem. This result is the prime reason for our interest in projective unitary representations of Möb. It is also the basic tool in the classification program for the irreducible homogeneous operators which is now in progress.

Sketch of Proof. Notice that the scalar unitaries in $\mathcal{U}(\mathcal{H})$ form a copy of the circle group $\mathbb{T}$ in $\mathcal{U}(\mathcal{H})$. There exist Borel transversals E to this subgroup, i.e., Borel subsets E of $\mathcal{U}(\mathcal{H})$ which meet every coset of $\mathbb{T}$ in a singleton. Fix one such (in the Proof of Theorem 2.2 in [9], we present an explicit construction of such a transversal). For each element φ of Möb, let E_φ denote the set of all unitaries U in $\mathcal{U}(\mathcal{H})$ such that $U^*TU = \varphi(T)$. Since T is an irreducible homogeneous operator, Schur's Lemma implies that each E_φ is a coset of $\mathbb{T}$ in $\mathcal{U}(\mathcal{H})$. Define $\pi : \text{Möb} \to \mathcal{U}(\mathcal{H})$ by

$$\{\pi(\varphi)\} = E \cap E_\varphi.$$

It is easy to see that π, thus defined, is indeed a projective representation associated with T. Another appeal to Schur's Lemma shows that any representation associated with T must be equivalent to π. This completes the proof.

For any projective representation π of Möb, let $\pi^{\#}$ denote the projective representation of Möb obtained by composing π with the automorphism $*$ of Möb (cf. (1.2)). That is,

$$\pi^{\#}(\varphi) := \pi(\varphi^*), \quad \varphi \in \text{Möb}. \tag{4.2}$$

Clearly, if m is the multiplier of π, then $\bar{m}$ is the multiplier of $\pi^{\#}$. Also, from (1.3) it is more or less immediate that if π is associated with a homogeneous operator T then $\pi^{\#}$ is associated with the adjoint T^* of T. If, further, T is invertible, then $\pi^{\#}$ is associated with T^{-1} also.

4.1 *Classification of irreducible homogeneous operators*

Recall that an operator T on a Hilbert space $\mathcal{H}$ is said to be a *block shift* if there are non trivial subspaces V_n (indexed by all integers, all non-negative integers or all non-positive integers – accordingly T is called a bilateral, forward unilateral or backward unilateral block shift) such that $\mathcal{H}$ is the orthogonal direct sum of these subspaces and we have $T(V_n) \subseteq V_{n+1}$ for each index n (where, in the case of a backward block shift, we take $V_1 = \{0\}$). In [9] we present a proof (due to Ordower) of the somewhat surprising fact that in case T is an irreducible block shift, these subspaces V_n (which are called the blocks of T) are uniquely determined by T. This result lends substance to the following theorem.

For any connected semi-simple Lie group G takes a maximal compact subgroup $\mathbb{K}$ of G (it is unique up to conjugation). Let $\hat{\mathbb{K}}$ denote, as usual, the set of all irreducible (ordinary) unitary representation of $\mathbb{K}$ (modulo equivalence). Let us say that a projective representation π of G is *normalized* if $\pi|_{\mathbb{K}}$ is an ordinary representation of $\mathbb{K}$. (If $H^2(\mathbb{K}, \mathbb{T})$ is trivial, then it is easy to see that every projective representation of G is equivalent to a normalized representation). If π is normalized, then, for any $\chi \in \hat{\mathbb{K}}$, let V_χ denote the subspace of $\mathcal{H}_\pi$ (the space on which π acts) given by

$$V_\chi = \{v \in \mathcal{H}_\pi \ : \pi(k)v = \chi(k)v \ \forall k \in \mathbb{K}\}.$$

Clearly $\mathcal{H}_\pi$ is the orthogonal direct sum of the subspaces V_χ, $\chi \in \hat{\mathbb{K}}$. The subspace V_χ is called the $\mathbb{K}$-*isotypic subspace* of $\mathcal{H}_\pi$ of type χ.

In particular, for the group $G =$ Möb, we may take $\mathbb{K}$ to be the copy $\{\varphi_{\alpha,0} : \alpha \in \mathbb{T}\}$ of the circle group $\mathbb{T}$. ($\mathbb{K}$ may be identified with $\mathbb{T}$ via $\alpha \mapsto \varphi_{\alpha,0}$.) For π as above and $n \in \mathbb{Z}$, let $V_n(\pi)$ denote the $\mathbb{K}$-isotypic subspace corresponding to the character $\chi_n : z \mapsto z^{-n}$, $z \in \mathbb{T}$. With these notations, we have the following theorem from [9].

Theorem 4.2. *Any irreducible homogeneous operator is a block shift. Indeed, if T is such an operator, and π is a normalized projective representation associated with T then the blocks of T are precisely the non-trivial $\mathbb{K}$-isotypic subspaces of π.*

(Note that if T is an irreducible homogeneous operator, then by Theorem 4.1 there is a representation π associated with T. Since such a representation is determined only up to equivalence, we may replace π by a normalized representation equivalent to it. Then the above theorem applies.)

A block shift is called a *weighted shift* if its blocks are one-dimensional. In [9] we define a *simple representation* of Möb to be a normalized representation π such that (i) the set $\mathcal{T}(\pi) := \{n \in \mathbb{Z} : V_n(\pi) \neq \{0\}\}$ is connected (in an obvious sense) and (ii) for each $n \in \mathcal{T}(\pi)$, $V_n(\pi)$ is one dimensional. If T is an irreducible homogeneous weighted shift, then, by the uniqueness of its blocks and by Theorem 4.2, it follows that any normalized representation π associated with T is necessarily simple. Using the list of irreducible projective representations of Möb given in the previous section (along with Remark 3.1(b) following this list) one can determine all the simple representations of Möb. This is done in Theorem 3.3 of [9]. Namely, we have

Theorem 4.3. *Up to equivalence, the only simple projective unitary representations of* Möb *are its irreducible representations along with the representations* $D_\lambda^+ \oplus D_{2-\lambda}^-$, $0 < \lambda < 2$.

Since the representations associated with irreducible homogeneous shifts are simple, to complete a classification of these operators, it now suffices to take each of the representations π of Theorem 4.3 and determine all the homogeneous operators T associated with π. Given that Theorem 4.2 pinpoints the way in which such an operator T must act on the space of π, it is now a simple matter to complete the classification of these operators (at least it is simple in principle – finding the optimum path to this goal turns out to be a challenging task!). To complete a classification of all homogeneous weighted shifts (with non-zero weights – permitting zero weights would introduce uninteresting complications), one still needs to find the reducible homogeneous shifts. Notice that the technique outlined here fails in the reducible case since Theorem 4.1 does not apply. However, in Theorem 2.1 of [9], we were able to show that there is a unique reducible homogeneous shift with non-zero weights, namely the unweighted bilateral shift B. Indeed, if T is a reducible shift (with non-zero weights) such that the spectral radius of T is $= 1$, then it can be shown that $T^k = B^k$ for some positive integer k, and hence T^k is unitary. But Lemma 2.1 in [9] shows that if T is a homogeneous operator such that T^k is unitary, then T itself must be unitary. Clearly, B is the only unitary weighted shift. This shows that B is the only reducible homogeneous weighted shift with non-zero weights. When all this is put together, we have the main theorem of [9].

Theorem 4.4. *Up to unitary equivalence, the only homogeneous weighted shifts are the known ones* (*namely, the first five series of examples from the list in* §6).

Yet another link between homogeneous operators and projective representations of Möb occurs in [10]. Beginning with Theorem 2.3, in [10] we prove a product formula, involving a pair of projective representations, for the characteristic function of any irreducible homogeneous contraction. Namely we have

Theorem 4.5. *If T is an irreducible homogeneous contraction then its characteristic function* $\theta : \mathbb{D} \to \mathcal{B}(\mathcal{K}, \mathcal{L})$ *is given by*

$$\theta(z) = \pi(\varphi_z)^* C \sigma(\varphi_z), \quad z \in \mathbb{D}$$

where π and σ are two projective representations of Möb (*on the Hilbert spaces $\mathcal{L}$ and $\mathcal{K}$ respectively*) *with a common multiplier. Further, $C : \mathcal{K} \to \mathcal{L}$ is a pure contraction which intertwines $\sigma|_{\mathbb{K}}$ and $\pi|_{\mathbb{K}}$.*

Conversely, whenever π, σ are projective representations of Möb *with a common multiplier and C is a purely contractive intertwiner between $\sigma|_{\mathbb{K}}$ and $\pi|_{\mathbb{K}}$ such that the function θ defined by $\theta(z) = \pi(\varphi_z)^* C \sigma(\varphi_z)$ is analytic on $\mathbb{D}$, then θ is the characteristic function of a homogeneous cnu contraction (not necessarily irreducible).*

(Here φ_z is the involution in Möb which interchanges 0 and z. Also, $\mathbb{K} = \{\varphi \in$ Möb : $\varphi(0) = 0\}$ is the standard maximal compact subgroup of Möb.)

Sketch of Proof. Let θ be the characteristic function of an irreducible homogeneous cnu contraction T. For any φ in Möb look at the set

$$E_\varphi := \{(U, V) : U^*\theta(w)V = \theta(\varphi^{-1}(w))\ \forall w \in \mathbb{D}\} \subseteq \mathcal{U}(\mathcal{L}) \times \mathcal{U}(\mathcal{K}).$$

By Theorem 2.3, E_φ is non-empty for each φ. By Theorem 3.4 in [25], for $(U, V) \in E_\varphi$ there is a unitary operator $\tau(U, V)$ such that (i) $\tau(U, V)^* T \tau(U, V) = \varphi(T)$ and (ii) the restriction of $\tau(U, V)$ to $\mathcal{L}$ and $\mathcal{K}$ equal U and V respectively. Therefore, irreducibility of T implies that, for (U, V), (U', V') in E_φ, $\tau(U', V')^*\tau(U, V)$ is a scalar unitary. Hence E_φ is a coset of the subgroup S (isomorphic to the torus $\mathbb{T}^2$) of $\mathcal{U}(\mathcal{L}) \times \mathcal{U}(\mathcal{K})$ consisting of pairs of scalar unitaries. As in the proof of Theorem 4.1, it follows that there are projective unitary representations π and σ with a common multiplier (on the spaces $\mathcal{L}$ and $\mathcal{K}$ respectively) such that $(\pi(\varphi), \sigma(\varphi)) \in E_\varphi$ for all φ in Möb. So we have

$$\pi(\varphi)^*\theta(w)\sigma(\varphi) = \theta(\varphi^{-1}(w)), \quad w \in \mathbb{D},\ \varphi \in \text{Möb}. \tag{4.3}$$

Now, choose $\varphi = \varphi_z$ and evaluate both sides of (4.3) at $w = 0$ to find the claimed formula for θ with $C = \theta(0)$. Also, taking $w = 0$ and $\varphi \in \mathbb{K}$ in (4.3), one sees that C intertwines $\sigma|_{\mathbb{K}}$ and $\pi|_{\mathbb{K}}$.

For the converse, let $\theta(z) := \pi(\varphi_z)^* C \sigma(\varphi_z)$ be an analytic function. Since $C = \theta(0)$ is a pure contraction and $\theta(z)$ coincides with $\theta(0)$ for all z, θ is pure contraction valued. Hence θ is the characteristic function of a cnu contraction T. For $\varphi \in$ Möb and $w \in \mathbb{D}$, write $\varphi_w\varphi = k\varphi_z$ where $k \in \mathbb{K}$ and $z = (\varphi_w\varphi)^{-1}(0) = \varphi^{-1}(w)$. Then we have

$$\begin{aligned}
\pi(\phi)^*\theta(w)\sigma(\varphi) &= \pi(\varphi)^*\pi(\varphi_w)^* C\sigma(\varphi_w)\sigma(\varphi)\\
&= \pi(\varphi_w\varphi)^* C\sigma(\varphi_w\varphi)\\
&= \pi(k\varphi_z)^* C\sigma(k\varphi_z)\\
&= \pi(\varphi_z)^*\pi(k)^* C\sigma(k)\sigma(\varphi_z)\\
&= \pi(\varphi_z)^* C\sigma(\varphi_z)\\
&= \theta(\varphi^{-1}(w)).
\end{aligned}$$

(Here, for the second and fourth equality we have used the assumption that π and σ are projective representations with a common multiplier. For the penultimate equality, the assumption that C intertwines $\sigma|_{\mathbb{K}}$ and $\pi|_{\mathbb{K}}$ has been used.) Thus θ satisfies (4.3). Therefore $\theta \circ \varphi$ coincides with θ for all φ in Möb. Hence Theorem 2.3 implies that T is homogeneous.

5. Some constructions of homogeneous operators

Let us say that a projective representation π of Möb is a *multiplier representation* if it is concretely realized as follows. π acts on a Hilbert space $\mathcal{H}$ of E – valued functions on

Ω, where Ω is either $\mathbb{D}$ or $\mathbb{T}$ and E is a Hilbert space. The action of π on $\mathcal{H}$ is given by $(\pi(\varphi)f)(z) = c(\varphi, z)f(\varphi^{-1}z)$ for $z \in \Omega$, $f \in \mathcal{H}$, $\varphi \in$ Möb. Here c is a suitable Borel function from Möb $\times\ \Omega$ into the Borel group of invertible operators on E.

Theorem 5.1. *Let $\mathcal{H}$ be a Hilbert space of functions on Ω such that the operator T on $\mathcal{H}$ given by*

$$(Tf)(x) = xf(x),\ x \in \Omega,\ f \in \mathcal{H},$$

is bounded. Suppose there is a multiplier representation π of Möb *on $\mathcal{F}$. Then T is homogeneous and π is associated with T.*

This easy but basic construction is from Proposition 2.3 of [6]. To apply this theorem, we only need a good supply of what we have called multiplier representations of Möb. Notice that all the irreducible projective representations of Möb (as concretely presented in the previous section) are multiplier representations.

A second construction goes as follows. It is contained in Proposition 2.4 of [6].

Theorem 5.2. *Let T be a homogeneous operator on a Hilbert space $\mathcal{H}$ with associated representation π. Let $\mathcal{K}$ be a subspace of $\mathcal{H}$ which is invariant or co-invariant under both T and π. Then the compression of T to $\mathcal{K}$ is homogeneous. Further, the restriction of π to $\mathcal{K}$ is associated with this operator.*

A third construction (as yet unreported) goes as follows:

Theorem 5.3. *Let π be a projective representation of* Möb *associated with two homogeneous operators T_1 and T_2 on a Hilbert space $\mathcal{H}$. Let T denote the operator on $\mathcal{H} \oplus \mathcal{H}$ given by*

$$T = \begin{pmatrix} T_1 & T_1 - T_2 \\ 0 & T_2 \end{pmatrix}.$$

Then T is homogeneous with associated representation $\pi \oplus \pi$.

Sketch of proof. For φ in Möb, one verifies that

$$\varphi(T) = \begin{pmatrix} \varphi(T_1) & \varphi(T_1) - \varphi(T_2) \\ 0 & \varphi(T_2) \end{pmatrix}.$$

Hence it is clear that $\pi \oplus \pi$ is associated with T.

6. Examples of homogeneous operators

It would be tragic if we built up a huge theory of homogeneous operators only to find at the end that there are very few of them. Here are some examples to show that this is not going to happen.

- *The principal series example.* The unweighted bilateral shift B (i.e., the bilateral shift with weight sequence $w_n = 1$, $n = 0, \pm 1, \ldots$) is homogeneous. To see this, apply Theorem 5.1 to any of the principal series representations of Möb. By construction, all the principal series representations are associated with B.

The discrete series examples. For any real number $\lambda > 0$, the unilateral shift $M^{(\lambda)}$ with weight sequence $\sqrt{\frac{n+1}{n+\lambda}}, n = 0, 1, 2, \ldots$ is homogeneous. To see this, apply Theorem 5.1 to the discrete series representation D_λ^+.

$\lambda \geq 1$, $M^{(\lambda)}$ is a cnu contraction. For $\lambda = 1$, its characteristic function is the (constant) ction 0 – not very interesting! But for $\lambda > 1$ we proved the following formula for the racteristic function of $M^{(\lambda)}$ (cf. [7]).

eorem 6.1. *For $\lambda > 1$, the characteristic function of $M^{(\lambda)}$ coincides with the function given by*

$$\theta_\lambda(z) = (\lambda(\lambda-1))^{-1/2} D_{\lambda-1}^+(\varphi_z)^* \partial^* D_{\lambda+1}^+(\varphi_z), \quad z \in \mathbb{D},$$

ere ∂^* is the adjoint of the differentiation operator $\partial : \mathcal{H}^{(\lambda-1)} \to \mathcal{H}^{(\lambda+1)}$.

is theorem is, of course, an instance of the product formula in Theorem 4.5.

The anti-holomorphic discrete series examples. These are the adjoints $M^{(\lambda)*}$ of the operators in the previous family. The associated representation is D_λ^-.

vas shown in [22] that

eorem 6.2. *Up to unitary equivalence, the operators $M^{(\lambda)*}$, $\lambda > 0$ are the only homo-ıeous operators in the Cowen–Douglas class $B_1(\mathbb{D})$.*

This theorem was independently re-discovered by Wilkins in ([33], Theorem 4.1).

The complementary series examples. For any two real numbers a and b in the open unit interval (0, 1), the bilateral shift $K_{a,b}$ with weight sequence $\sqrt{\frac{n+a}{n+b}}, n = 0, \pm 1, \pm 2, \ldots,$ is homogeneous. To see this in case $0 < a < b < 1$, apply Theorem 5.1 to the complementary series representation $C_{\lambda,\sigma}$ with $\lambda = a + b - 1$ and $\sigma = (b-a)/2$. If $a = b$ then $K_{a,b} = B$ is homogeneous. If $0 < b < a < 1$ then $K_{a,b}$ is the adjoint inverse of the homogeneous operator $K_{b,a}$, and hence is homogeneous.

The constant characteristic examples. For any real number $\lambda > 0$, the bilateral shift B_λ with weight sequence $\ldots, 1, 1, 1, \lambda, 1, 1, 1, \ldots,$ (λ in the zeroth slot, 1 elsewhere) is homogeneous. Indeed, if $0 < \lambda < 1$ then B_λ is a cnu contraction with constant characteristic function $-\lambda$; hence it is homogeneous. Of course, $B_1 = B$ is also homogeneous. If $\lambda > 1$, B_λ is the inverse of the homogeneous operator B_μ with $\mu = \lambda^{-1}$, hence it is homogeneous. (In [6] we presented an unnecessarily convoluted argument to show that B_λ is homogeneous for $\lambda > 1$ as well.) It was shown in [6] that the representation $D_1^+ \oplus D_1^-$ is associated with each of the operators B_λ, $\lambda > 0$. (Recall that this is the only reducible representation in the principal series!)

In [6] we show that apart from the unweighted unilateral shift and its adjoint, the operators , $\lambda > 0$ are the only irreducible contractions with a constant characteristic function. fact,

ıeorem 6.3. *The only cnu contractions with a constant characteristic function are the ect integrals of the operators $M^{(1)}$, $M^{(1)*}$ and B_λ, $\lambda > 0$.*

Since all the constant characteristic examples are associated with a common representation, one might expect that the construction in Theorem 5.3 could be applied to any two of them to yield a plethora of new examples of homogeneous operators. Unfortunately, this is not the case. Indeed, it is not difficult to verify that for $\lambda \neq \mu$, the operator $\begin{pmatrix} B_\lambda & B_\lambda - B_\mu \\ 0 & B_\mu \end{pmatrix}$ is unitarily equivalent to $B_\sigma \oplus B_\delta$ where σ and δ are the eigenvalues of $(AA^*)^{1/2}$, $A = \begin{pmatrix} \lambda & \lambda - \mu \\ 0 & \mu \end{pmatrix}$.

Notice that the examples of homogeneous operators given so far are all weighted shifts. By Theorem 4.4, these are the only homogeneous weighted shifts with non-zero weights. Wilkins was the first to come up with examples of (irreducible) homogeneous operators which are not scalar shifts.

- *The generalized Wilkins examples.* Recall that for any real number $\lambda > 0$, $\mathcal{H}^{(\lambda)}$ denotes the Hilbert space of analytic functions on $\mathbb{D}$ with reproducing kernel $(z, w) \mapsto (1 - z\bar{w})^{-\lambda}$. (It is the Hilbert space on which the holomorphic discrete series representation D_λ^+ lives.) For any two real numbers $\lambda_1 > 0, \lambda_2 > 0$, and any positive integer k, view the tensor product $\mathcal{H}^{(\lambda_1)} \otimes \mathcal{H}^{(\lambda_2)}$ as a space of analytic functions on the bidisc $\mathbb{D} \times \mathbb{D}$. Look at the Hilbert space $V_k^{(\lambda_1, \lambda_2)} \subseteq \mathcal{H}^{(\lambda_1)} \otimes \mathcal{H}^{(\lambda_2)}$ defined as the orthocomplement of the subspace consisting of the functions vanishing to order k on the diagonal $\Delta = \{(z, z) : z \in \mathbb{D}\} \subseteq \mathbb{D} \times \mathbb{D}$. Finally define the generalized Wilkins operator $W_k^{(\lambda_1, \lambda_2)}$ as the compression to $V_k^{(\lambda_1, \lambda_2)}$ of the operator $M^{(\lambda_1)} \otimes I$ on $\mathcal{H}^{(\lambda_1)} \otimes \mathcal{H}^{(\lambda_2)}$. The subspace $V_k^{(\lambda_1, \lambda_2)}$ is co-invariant under the homogeneous operator $M^{(\lambda_1)} \otimes I$ as well as under the associated representation $D_{\lambda_1}^+ \otimes D_{\lambda_2}^+$. Therefore, by Theorem 5.2, $W_k^{(\lambda_1, \lambda_2)}$ is a homogeneous operator. For $k = 1$, $W_1^{(\lambda_1, \lambda_2)}$ is easily seen to be unitarily equivalent to $M^{(\lambda_1+\lambda_2)}$, see [7] and [14], for instance. But for $k \geq 2$, these are new examples.

The operator $W_k^{(\lambda_1, \lambda_2)}$ may alternatively be described as multiplication by the co-ordinate function z on the space of $\mathbb{C}^k$-valued analytic functions on $\mathbb{D}$ with reproducing kernel

$$(z, w) \mapsto (1 - z\bar{w})^{-\lambda_1} \left(\left(\partial^i \bar{\partial}^j (1 - z\bar{w})^{-\lambda_2}\right)\right)_{0 \leq i, j \leq k-1}.$$

(Here ∂ and $\bar{\partial}$ denote differentiation with respect to z and $\bar{w}$, respectively.) Indeed (with the obvious identification of Δ and $\mathbb{D}$) the map $f \mapsto (f, f', \ldots, f^{(k-1)})|_\Delta$ is easily seen to be a unitary between $V_k^{(\lambda_1, \lambda_2)}$ and this reproducing kernel Hilbert space intertwining $W_k^{(\lambda_1, \lambda_2)}$ and the multiplication operator on the latter space. (This is a particular instance of the *jet construction* discussed in [15].) Using this description, it is not hard to verify that the adjoint of $W_k^{(\lambda_1, \lambda_2)}$ is an operator in the Cowen–Douglas class $B_k(\mathbb{D})$. The following is (essentially) one of the main results in [34].

Theorem 6.4. *Up to unitary equivalence, the only irreducible homogeneous operators in the Cowen–Douglas class $B_2(\mathbb{D})$ are the adjoints of the operators $W_2^{(\lambda_1, \lambda_2)}$, $\lambda_1 > 0, \lambda_2 > 0$.*

This is not the description of these operators given in [34]. But it can be shown that Wilkin's operator $T_{\lambda,\varrho}^*$ is unitarily equivalent to the operator $W_2^{(\lambda_1, \lambda_2)}$ with $\lambda = \lambda_1 + \lambda_2 + 1$, $\varrho = (\lambda_1 + \lambda_2 + 1)/(\lambda_2 + 1)$. Indeed, though his reproducing kernel $H_{\lambda,\varrho}$ looks a little different from the kernel (with $k = 2$) displayed above, a calculation shows that these two kernels have the same normalization at the origin (cf. [12]), so that the corresponding multiplication operators are unitarily equivalent. However, it is hard to see how Wilkins arrived at his examples $T^*_{\lambda,\varrho}$ while the construction of the operators $W_k^{(\lambda_1,\lambda_2)}$ given above has a clear geometric meaning, particularly in view of Theorem 5.2. But, as of now, we know that the case $k = 2$ of this construction provides a complete list of the irreducible homogeneous operators in $B_2(\mathbb{D})$ only by comparing them with Wilkins' list – we have no independent explanation of this phenomenon.

Theorem 6.1 has the following generalization to some of the operators in this series. (Theorem 6.1 is the special case $k = 1$ of this theorem.)

Theorem 6.5. *For $k = 1, 2, \ldots$ and real numbers $\lambda > k$, the characteristic function of the operator $W_k^{(1,\lambda-k)}$ coincides with the inner analytic function $\theta_k^{(\lambda)} : \mathbb{D} \to \mathcal{B}(\mathcal{H}^{(\lambda+k)}, \mathcal{H}^{(\lambda-k)})$ given by*

$$\theta_k^{(\lambda)}(z) = c_{\lambda,k} D^+_{\lambda-k}(\varphi_z)^* \partial^{k*} D^+_{\lambda+k}(\varphi_z), \quad z \in \mathbb{D}.$$

Here ∂^{k*} is the adjoint of the k-times differentiation operator $\partial^k : \mathcal{H}^{(\lambda-k)} \to \mathcal{H}^{\lambda+k)}$ and $c_{\lambda,k} = \prod_{\ell=-(k-1)}^{k}(\lambda-\ell)^{-1/2}$.

Sketch of Proof. It is easy to check that $C := c_{\lambda,k}\partial^{k*}$ is a pure contraction intertwining the restrictions to $\mathbb{K}$ of $D^+_{\lambda+k}$ and $D^+_{\lambda-k}$. Since we already know (by Theorem 6.1) that θ_k^λ is an inner analytic function for $k = 1$, the recurrence formula

$$\theta_{k+1}^{(\lambda)} = \theta_1^{(\lambda-k)}\theta_{k-1}^{(\lambda)}\theta_1^{(\lambda+k)}$$

(for $k \geq 1$, $\lambda > k + 1$, with the interpretation that $\theta_0^{(\lambda)}$ denotes the constant function 1) shows that $\theta_k^{(\lambda)}$ is an inner analytic function on $\mathbb{D}$ for $\lambda > k$, $k = 1, 2, \ldots$. Hence it is the characteristic function of a cnu contraction T in the class $C_{\cdot 0}$. By Theorem 2.1, T is the compression to $\mathcal{M}^\perp$ of the multiplication operator on $H^{(1)} \otimes \mathcal{H}^{(\lambda-k)}$, where $\mathcal{M}$ is the invariant subspace corresponding to this inner function. But one can verify that $\mathcal{M}$ is the subspace consisting of the functions vanishing to order k on the diagonal. Therefore $T = W_k^{(1,\lambda-k)}$.

- *Some perturbations of the discrete series examples.* Let $\mathcal{H}$ be a Hilbert space with orthonormal basis $\{f_k : k = 0, 1, \ldots\} \cup \{h_{k,\ell} : k = 0, \pm 1, \pm 2, \ldots\}$. For any three strictly positive real numbers λ, μ and δ, let $M^{(\lambda)}[\mu, \delta]$ be the operator on $\mathcal{H}$ given by

$$M^{(\lambda)}[\mu,\delta]f_k = \sqrt{\frac{k+1}{k+\lambda+1}}f_{k+1} + \sqrt{\frac{\delta}{k+\lambda+1}}h_{1,k+1},$$

$$M^{(\lambda)}[\mu,\delta]h_{0,\ell} = \mu h_{1,\ell},$$

and

$$M^{(\lambda)}[\mu,\delta]h_{k,\ell} = h_{k+1,\ell}, \quad \text{for } k \geq 1.$$

An application of Theorem 2.2 to the operators $M^{(\lambda)}$ in conjunction with an analytic continuation argument shows that these operators are homogeneous. This was observed in [7].

- *The normal atom.* Define the operator N on $L^2(\mathbb{D})$ by $(Nf)(z) = zf(z)$, $z \in \mathbb{D}$, $f \in L^2(\mathbb{D})$. The discrete series representation D_2^+ naturally lifts to a representation of Möb on $L^2(\mathbb{D})$. Applying Theorem 5.1 to this representation yields the homogeneity of N.

Using spectral theory, it is easy to see that the operators B and N are the only homogeneous normal operators of multiplicity one. In consequence, we have

Theorem 6.6. *Every normal homogeneous operator is a direct sum of (countably many) copies of B and N.*

Let us define an *atomic homogeneous operator* to be a homogeneous operator which can not be written as the direct sum of two homogeneous operators. Trivially, irreducible homogeneous operators are atomic. As an immediate consequence of Theorem 6.6, we have

COROLLARY 6.1

B and N are atomic (but reducible) homogeneous operators.

N is a cnu contraction. Its characteristic function was given in [7].

Theorem 6.7. *The characteristic function $\theta_N : \mathbb{D} \to \mathcal{B}(L^2(\mathbb{D}))$ of the operator N is given by the formula*

$$(\theta_N(z)f)(w) = -\varphi_w(z)f(w), \quad z, w \in \mathbb{D}, \quad f \in L^2(\mathbb{D}).$$

(Here, as before, φ_w is the involution in Möb which interchanges 0 and w.)

The usual transition formula between cartesian and polar coordinates shows that $L^2(\mathbb{D}) = L^2(\mathbb{T}) \otimes L^2([0, 1], r\,dr)$. Since B may be represented as multiplication by the coordinate function on $L^2(\mathbb{T})$, it follows that the normal atom N is related to the other normal atom B by $N = B \otimes C$ where C is multiplication by the coordinate function on $L^2([0, 1], r\,dr)$. Clearly C is a positive contraction. Let $\{f_n : n \geq 0\}$ be the orthonormal basis of $L^2([0, 1], r\,dr)$ obtained by Gram–Schmidt orthogonalization of the sequence $\{r \mapsto r^n : n \geq 0\}$. (Except for scaling, f_n is given in terms of classical Jacobi polynomials by $x \mapsto P_n^{(0,1)}(2x - 1)$, cf. [31].) Then the theory of orthogonal polynomials shows that (with respect to this orthonormal basis) C is a tri-diagonal operator. Thus we have

Theorem 6.8. *Up to unitary equivalence, we have $N = B \otimes C$ where the positive contraction C is given on a Hilbert space with orthonormal basis $\{f_n : n \geq 0\}$ by the formula*

$$Cf_n = a_n f_{n-1} + b_n f_n + a_{n+1} f_{n+1}, \quad n = 0, 1, 2, \ldots$$

where $(f_{-1} = 0)$ and the constants a_n, b_n are given by

$$a_n = \frac{\sqrt{n(n+1)}}{4n+2}, \quad b_n = \frac{2(n+1)^2}{(2n+1)(2n+3)}, \quad n \geq 0.$$

7. Open questions

7.1 *Classification*

The primary question in this area is, of course, the classification of homogeneous operators up to unitary equivalence. Theorem 4.4 is a beginning in this direction. We expect that the same methodology will permit us to classify all the homogeneous operators in the Cowen–Douglas classes $B_k(\mathbb{D})$, $k = 1, 2, \ldots$. Work on this project has already begun. More generally, though there seem to be considerable difficulties involved, it is conceivable that extension of the same techniques will eventually classify all irreducible homogeneous operators. But, depending as it does on Theorem 4.1, this technique draws a blank when it comes to classifying reducible homogeneous operators. In particular, we do not know how to approach the following questions.

Question 1. Is every homogeneous operator a direct integral of atomic homogeneous operators?

Question 2. Are B and N the only atomic homogeneous operators which are not irreducible?

We have seen that the homogeneous operator N can be written as $N = B \otimes C$. In this connection, we can ask:

Question 3. Find all homogeneous operators of the form $B \otimes X$. More generally, find all homogeneous operators which have a homogeneous operator as a 'tensor factor'.

Another possible approach towards the classification of irreducible homogeneous contractions could be via Theorem 4.5. (Notice that any irreducible operator is automatically cnu.) Namely, given any two projective representations π and σ of Möb having a common multiplier, we can seek to determine the class $C(\pi, \sigma)$ of all operators $C : \mathcal{H}_\sigma \to \mathcal{H}_\pi$ such that (i) C intertwines $\sigma|_{\mathbb{K}}$ and $\pi|_{\mathbb{K}}$ and (ii) the function $z \mapsto \pi(\varphi_z)^* C \sigma(\varphi_z)$ is analytic on $\mathbb{D}$. Clearly $C(\pi, \sigma)$ is a subspace of $\mathcal{B}(\mathcal{H}_\sigma, \mathcal{H}_\pi)$, and Theorem 4.5 says that any pure contraction in this subspace yields a homogeneous operator. Further, this method yields all irreducible homogeneous contractions as one runs over all π and σ. This approach is almost totally unexplored. We have only observed that, up to multiplication by scalars, the homogeneous characteristic functions listed in Theorem 6.5 are the only ones in which both π and σ are holomorphic discrete series representations. (But the trivial operation of multiplying the characteristic function by scalars correspond to a highly non-trivial operation at the level of the operator. This operation was explored in [7].) So a natural question is:

Question 4. Determine $C(\pi, \sigma)$ at least for irreducible projective representations π and σ (with a common multiplier).

Note that Theorem 6.5 gives the product formula for the characteristic function of $W_k^{(\lambda_1, \lambda_2)}$ for $\lambda_1 = 1$. But for $W_k^{(\lambda_1, \lambda_2)}$ to be a contraction it is sufficient (though not necessary) to have $\lambda_1 \geq 1$. So on a more modest vein, we may ask:

Question 5. What is the (explicit) product formula for the characteristic functions of the operators $W_k^{(\lambda_1, \lambda_2)}$ for $\lambda_1 > 1$?

Recall that a cnu contraction T is said to be in the class C_{11} if for every nonzero vector x, $\lim_{n\to\infty} T^n x \neq 0$ and $\lim_{n\to\infty} T^{*n} x \neq 0$. In [19], Kerchy asks:

Question 6. Does every homogeneous contraction in the class C_{11} have a constant characteristic function?

7.2 *Möbius bounded and polynomially bounded operators*

Recall from [30] that a Hilbert space operator T is said to be *Möbius bounded* if the family $\{\varphi(T) : \varphi \in \text{Möb}\}$ is uniformly bounded in norm. Clearly homogeneous operators are Möbius bounded, but the converse is false. In [30], Shields proved:

Theorem 7.1. *If T is a Möbius bounded operator then $\|T^m\| = O(m)$ as $m \to \infty$.*

Sketch of proof. Say $\|\varphi(T)\| \leq c$ for $\varphi \in$ Möb. For any $\varphi \in$ Möb, we have an expansion $\varphi(z) = \sum_{m=0}^{\infty} a_m z^m$, valid in the closed unit disc. Hence,

$$a_m T^m = \int_{\mathbb{T}} \varphi(\alpha T)\alpha^{-m}\, d\alpha,$$

where the integral is with respect to the normalized Haar measure on $\mathbb{T}$. Therefore we get the estimate $|a_m|\|T^m\| \leq c$ for all m. Choosing $\varphi = \varphi_{1,\beta}$, we see that for $m \geq 1$, $|a_m| = (1-r^2)r^{m-1}$ where $r = |\beta|$. The optimal choice $r = \sqrt{(m-1)/(m+1)}$ gives $|a_m| = O(1/m)$ and hence $\|T^m\| = O(m)$.

On the basis of this Theorem and some examples, we may pose:

Conjecture. For any Möbius bounded operator T, we have $\|T^m\| = O(m^{1/2})$ as $m \to \infty$.

In [30], Shields already asked if this is true. This question has remained unanswered for more than twenty years. One possible reason for its intractability may be the difficulty involved in finding non-trivial examples of Möbius bounded operators. (Contractions are Möbius bounded by von Neumann's inequality, but these trivially satisfy Shield's conjecture.) As already mentioned, non-contractive homogeneous operators provide non-trivial examples. For the homogeneous operator $T = M^{(\lambda)}$ with $\lambda < 1$, we have $\|T^m\| = \sqrt{\frac{\Gamma(\lambda)\Gamma(m+1)}{\Gamma(m+\lambda)}}$ and hence (by Sterling's formula) $\|T^m\| \sim c m^{(1-\lambda)/2}$ with $c = \Gamma(\lambda)^{1/2}$. Thus the above conjecture, if true, is close to best possible (in the sense that the exponent $1/2$ in this conjecture cannot be replaced by a smaller constant). An analogous calculation with the complementary series examples $C(a, b)$ (with $0 < a \neq b < 1$) leads to a similar conclusion. This leads us to ask:

Question 7. Is the conjecture made above true at least for homogeneous operators T?

(It is conceivable that the operators $T_{\lambda,s}$ introduced below contain counter examples to Shield's conjecture in its full generality.)

Recall that an operator T, whose spectrum is contained in $\bar{\mathbb{D}}$, is said to be *polynomially bounded* if there is a constant $c > 0$ such that $\|p(T)\| \leq c$ for all polynomial maps $p : \mathbb{D} \to \mathbb{D}$. (von Neumann's inequality says that this holds with $c = 1$ iff T is a contraction.) Clearly, if T is similar to a contraction then T is polynomially bounded. Halmos asked if the converse is true, i.e., whether every polynomially bounded operator is similar to a contraction. In [28], Pisier constructed a counter-example to this conjecture. (Also see [13] for a streamlined version of this counter-example.) However, one may still hope that the Halmos conjecture is still true of some 'nice' classes of operators. In particular, we ask

Question 8. Is every polynomially bounded homogeneous operator similar to a contraction? For that matter, is there any polynomially bounded (even power bounded) homogeneous operator which is not a contraction?

Notice that the discrete series examples show that homogeneous operators (though Möbius bounded) need not even be power bounded. So certainly they need not be polynomially bounded.

7.3 *Invariant subspaces*

If T is a homogeneous operator with associated representation π, then for each invariant subspace $\mathcal{M}$ of T and each $\varphi \in$ Möb, $\pi(\varphi)(\mathcal{M})$ is again T-invariant. Thus Möb acts on the lattice of T-invariant subspaces via π. We wonder if this fact can be exploited to explore the structure of this lattice. Further, if T is a cnu contraction, then the Sz-Nagy–Foias theory gives a natural correspondence between the invariant subspaces of T and the 'regular factorizations' of its characteristic function (cf. [25]). Since we have nice explicit formulae for the characteristic functions of the homogeneous contractions $M(\lambda)$, $\lambda > 1$, may be these formulae can be exploited to shed light on the structure of the corresponding lattices.

Recall that Beurling's theorem describes the lattice of invariant subspaces of $M^{(1)}$ in terms of inner functions. Recently, it was found ([18] and [1]) that certain partial analogues of this theorem are valid for the Bergman shift $M^{(2)}$ as well. We may ask:

Question 9. Do the theorems of Hedenmalm and Aleman *et al* generalize to the family $M^{(\lambda)}$, $\lambda \geq 1$ of homogeneous unilateral shifts?

7.4 *Generalizations of homogeneity*

In the definition of homogeneous operators, one may replace unitary equivalence by similarity. Formally, we define a *weakly homogeneous operator* to be an operator T such that (i) the spectrum of T is contained in $\bar{\mathbb{D}}$ and (ii) $\varphi(T)$ is similar to T for every φ in Möb. Of course, every operator which is similar to a homogeneous operator is weakly homogeneous. In [11] it was asked if the converse is true. It is not – as one can see from the following examples:

Example 1. Take $\mathcal{H} = L^2(\mathbb{T})$ and, for any real number in the range $-1 < \lambda \leq 1$ and any complex number s with $\mathrm{Im}(s) > 0$, define $P_{\lambda,s}$: Möb $\to \mathcal{B}(\mathcal{H})$ by

$$P_{\lambda,s}(\varphi^{-1})f = \varphi'^{\lambda/2}|\varphi'|^{(1-\lambda)/2+s} f \circ \varphi, \quad f \in \mathcal{H}.$$

For purely imaginary s, these are just the principal series unitary projective representations discussed earlier. For s outside the imaginary axis, $P_{\lambda,s}$ is not unitary valued. But, formally, it still satisfies the condition (3.1) with $m = m_\omega$, $\omega = e^{i\pi\lambda}$. In consequence, $P_{\lambda,s}$ is an invertible operator valued function on Möb.

For λ and s as above, let $T_{\lambda,s}$ denote the bilateral shift on $L^2(\mathbb{T})$ with weight sequence

$$\frac{n + (1+\lambda)/2 + s}{n + (1+\lambda)/2 - s}, \; n \in \mathbb{Z}.$$

When s is purely imaginary, these weights are unimodular and hence $T_{\lambda,s}$ is unitarily equivalent to the unweighted bilateral shift B. In [9] it is shown that, in this case the principal series representation $P_{\lambda,s}$ is associated with $T_{\lambda,s}$ as well as to B. That is, we have

$$\varphi(T_{\lambda,s}) = P_{\lambda,s}(\varphi)^{-1} T_{\lambda,s} P_{\lambda,s}(\varphi) \tag{7.1}$$

for purely imaginary s. By analytic continuation, it follows that eq. (7.1) holds for all complex numbers s. Thus $T_{\lambda,s}$ is weakly homogeneous for $\operatorname{Im}(s) > 0$. It is easy to see that $\|T_{\lambda,s}^m\| \geq \|T_{\lambda,s}^m f_0\| \geq \frac{|\Gamma(m+a)\Gamma(b)|}{|\Gamma(m+b)\Gamma(a)|}$ where $a = (1+\lambda)/2 + s$, $b = (1+\lambda)/2 - s$ and f_0 is the constant function 1. Hence by Sterling's formula, we get

$$\|T_{\lambda,s}^m\| \geq cm^{2\operatorname{Re}(s)}$$

for all large m (and some constant $c > 0$). If $T_{\lambda,s}$ were similar to a homogeneous operator, it would be Möbius bounded and hence by Theorem 7.1 we would get $\|T_{\lambda,s}^m\| = O(m)$ which contradicts the above estimate when $\operatorname{Re}(s) > 1/2$. Therefore we have

Theorem 7.2. *The operators $T_{\lambda,s}$ is weakly homogeneous for all λ, s as above. However, for* $\operatorname{Re}(s) > 1/2$, *this operator is not Möbius bounded and hence is not similar to any homogeneous operator.*

Example 2 (due to Ordower). For any homogeneous operator T, say on the Hilbert space $\mathcal{H}$, let $\tilde{T}$ denote the operator $\begin{pmatrix} T & I \\ 0 & T \end{pmatrix}$. For any φ in a sufficiently small neighbourhood of the identity, $\varphi(\tilde{T})$ makes sense and one verifies that $\varphi(\tilde{T}) = \begin{pmatrix} \varphi(T) & \varphi'(T) \\ 0 & \varphi(T) \end{pmatrix}$. If U is a unitary on $\mathcal{H}$ such that $\varphi(T) = U^*TU$ then an easy computation shows that the operator $L = U\varphi'(T)^{1/2} \oplus U\varphi'(T)^{-1/2}$ satisfies $L\tilde{T}L^{-1} = \varphi(T)$. Thus $\varphi(\tilde{T})$ is similar to $\tilde{T}$ for all φ in a small neighbourhood. Therefore an obvious extension of Theorem 1.1 shows that $\tilde{T}$ is weakly homogeneous. Since $\|\varphi(\tilde{T})\| \geq \|\varphi'(T)\|$ and since the family φ', $\varphi \in$ Möb is not uniformly bounded on the spectrum of T, it follows that $\tilde{T}$ is not Möbius bounded. Therefore we have

Theorem 7.3. *For any homogeneous operator T, the operator $\tilde{T}$ is weakly homogeneous but not Möbius bounded. Therefore this operator is not similar to any homogeneous operator.*

These two classes of examples indicate that the right question to ask is

Question 10. Is it true that every Möbius bounded weakly homogeneous operator is similar to a homogeneous operator?

For purely imaginary s, the homogeneous operators $T_{\lambda,s}$ and B share the common associated representation $P_{\lambda,s}$; hence one may apply the construction in Theorem 5.3 to this pair. We now ask

Question 11. Is the resulting homogeneous operator atomic? Is it irreducible? More generally, are there instances where Theorem 5.3 lead to atomic homogeneous operators?

Another direction of generalization is to replace the group Möb by some subgroup G. For any such G, one might say that an operator T is G-homogeneous if $\varphi(T)$ is unitarily equivalent to T for all 'sufficiently small' φ in G. (If G is connected, the analogue of Theorem 1.1 holds.) The case $G = \mathbb{K}$ has been studied under the name of 'circularly symmetric operators'. See, for instance, [17] and [3]. Notice that if S is a circularly symmetric operator then so is $S \otimes T$ for any operator T – showing that this is a rather weak notion and no satisfactory classification can be expected when the group G is so small. A more interesting possibility is to take G to be a Fuchsian group. (Recall that a closed subgroup of Möb is said to be Fuchsian if it acts discontinuously on $\mathbb{D}$.) Fuchsian homogeneity was briefly studied by Wilkins in [33]. He examines the nature of the representations (if any) associated with such an operator.

Another interesting generalization is to introduce a notion of homogeneity for commuting tuples of operators. Recall that a bounded domain Ω in $\mathbb{C}^d$ is said to be a *bounded symmetric domain* if, for each $\omega \in \Omega$, there is a bi-holomorphic involution of Ω which has ω as an isolated fixed point. Such a domain is called irreducible if it cannot be written as the cartesian product of two bounded symmetric domains. The irreducible bounded symmetric domains are completely classified modulo biholomorphic equivalence (see [2] or [16] for instance) – they include the unit ball $I_{m,n}$ in the Banach space of all $m \times n$ matrices (with operator norm). Let G_Ω denote the connected component of the identity in the group of all bi-holomorphic automorphisms of an irreducible bounded symmetric domain Ω. If $\mathbf{T} = (T_1, \ldots, T_d)$ is a commuting d-tuple of operators then one may say that $\mathbf{T}$ is homogeneous if, for all 'sufficiently small' $\varphi \in G_\Omega$, $\varphi(\mathbf{T})$ is (jointly) unitarily equivalent to $\mathbf{T}$. (Of course, this notion depends on the choice of Ω – for most values of d there are several choices – so, to be precise, one ought to speak of Ω-homogeneity). Theorem 1.1 generalizes to show that, in this setting, the Taylor spectrum of $\mathbf{T}$ is contained in $\bar{\Omega}$ (and is a G_Ω-invariant closed subset thereof). Also, if $\mathbf{T}$ is an irreducible homogeneous tuple (in the sense that its components have no common non-trivial reducing subspace), then Theorem 4.1 generalizes to yield a projective representation of G_Ω associated with it. Therefore, many of the techniques employed in the single variable case have their several variable counterparts. But these are yet to be systematically investigated. One difficulty is that for $d \geq 2$, the (projective) representation theory of G_Ω (which is a semi-simple Lie group) is not as well understood as in the case $\Omega = \mathbb{D}$. But this also has the potential advantage that when (and if) this theory of homogeneous operator tuples is investigated in depth, the operator theory is likely to have significant impact on the representation theory.

With each domain Ω as above is associated a kernel B_Ω (called the Bergman kernel) which is the reproducing kernel of the Hilbert space of all square integrable (with respect to Lebesgue measure) analytic functions on Ω. The Wallach set $W = W_\Omega$ of Ω is the set of all $\lambda > 0$ such that $B_\Omega^{\lambda/g}$ is (a non-negative definite kernel and hence) the reproducing kernel of a Hilbert space $\mathcal{H}^{(\lambda)}(\Omega)$. (Here g is an invariant of the domain Ω called its genus, cf. [2].) It is well-known that the Wallach set W can be written as a disjoint union $W_d \cup W_c$ where the 'discrete' part W_d is a finite set (consisting of r points, where the 'rank' r of Ω is the number of orbits into which the topological boundary of Ω is broken by the action of G_Ω) and the 'continuous' part W_c is a semi-infinite interval.

The constant functions are always in $\mathcal{H}^{(\lambda)}(\Omega)$ but, for $\lambda \in W_d$, $\mathcal{H}^{(\lambda)}(\Omega)$ does not contain all the analytic polynomial functions on Ω. It follows that for $\lambda \in W_d$ multiplication by the co-ordinate functions do not define bounded operators on $\mathcal{H}^{(\lambda)}(\Omega)$. However, it was conjectured in [4] that for $\lambda \in W_c$, the d-tuple $\mathbf{M}^{(\lambda)}$ of multiplication by the d co-ordinates is bounded. (In [5], this conjecture was proved in the cases $\Omega = I_{m,n}$. In general, it is known that for sufficiently large λ the norm on $\mathcal{H}^{(\lambda)}(\Omega)$ is defined by a finite measure on $\bar{\Omega}$, so that this tuple is certainly bounded in these cases.) Assuming this conjecture, the operator tuples $\mathbf{M}^{(\lambda)}$, $\lambda \in W_c$, constitute examples of homogeneous tuples – this is in consequence of the obvious extension of Theorem 5.1 to tuples. In [4] it was shown that the Taylor spectrum of this tuple is $\bar{\Omega}$ and

Theorem 7.4. *Up to unitary equivalence, the adjoints of the tuples* $\mathbf{M}^{(\lambda)}$, $\lambda \in W_c$, *are the only homogeneous tuples in the Cowen–Douglas class* $B_1(\Omega)$.

For what values of $\lambda \in W_c$ is the tuple $\mathbf{M}^{(\lambda)}$ sub-normal? This is equivalent to asking for the values of λ for which the norm on $\mathcal{H}^{(\lambda)}(\Omega)$ is defined by a measure. In [4] we conjecture a precise answer. Again, the special case $\Omega = I_{m,n}$ of this conjecture was proved in [5].

Regarding homogeneous tuples, an obvious meta-question to be asked is

Question 12. Formulate appropriate generalizations to tuples of all the questions we asked before of single homogeneous operators – and answer them!

A d-tuple $\mathbf{T}$ on the Hilbert space $\mathcal{H}$ is said to be *completely contractive with respect to* Ω if for every polynomial map $P : \Omega \to I_{m,n}$, $P(\mathbf{T})$ is contractive when viewed as an operator from $\mathcal{H} \otimes \mathbb{C}^n$ to $\mathcal{H} \otimes \mathbb{C}^m$. $\mathbf{T}$ is called *contractive with respect to* Ω if this holds in the case $m = n = 1$. In general one may ask whether contractivity implies complete contractivity. In general the answer is 'no' for all $d \geq 5$ [27]. However one has a positive answer in the case $\Omega = \mathbb{D}$. But an affirmative answer (for special classes of tuples) would be interesting because complete contractivity is tantamount to existence of nice dilations which make the tuple in question tractable. For instance, we have an affirmative answer for subnormal tuples. We ask

Question 13. Is every contractive homogeneous tuple completely contractive?

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ultiwavelet packets and frame packets of $L^2(\mathbb{R}^d)$

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Abstract. The orthonormal basis generated by a wavelet of $L^2(\mathbb{R})$ has poor frequency localization. To overcome this disadvantage Coifman, Meyer, and Wickerhauser constructed wavelet packets. We extend this concept to the higher dimensions where we consider arbitrary dilation matrices. The resulting basis of $L^2(\mathbb{R}^d)$ is called the multiwavelet packet basis. The concept of wavelet frame packet is also generalized to this setting. Further, we show how to construct various orthonormal bases of $L^2(\mathbb{R}^d)$ from the multiwavelet packets.

Keywords. Wavelet; wavelet packets; frame packets; dilation matrix.

Introduction

onsider an orthonormal wavelet of $L^2(\mathbb{R})$. At the jth resolution level, the orthonormal sis $\{\psi_{jk} : j, k \in \mathbb{Z}\}$ generated by the wavelet has a frequency localization proportional 2^j. For example, if the wavelet ψ is band-limited (i.e., $\hat{\psi}$ is compactly supported), then e measure of the support of $(\psi_{jk})^\wedge$ is 2^j times the measure of the support of $\hat{\psi}$, since

$$(\psi_{jk})^\wedge(\xi) = 2^{-j/2}\hat{\psi}(2^{-j}\xi)e^{-i2^{-j}k\xi}, \quad j, k \in \mathbb{Z},$$

here

$$\psi_{jk} = 2^{j/2}\psi(2^j \cdot - k), \quad j, k \in \mathbb{Z}.$$

when j is large, the wavelet bases have poor frequency localization. Better frequency calization can be achieved by a suitable construction starting from an MRA wavelet basis.

Let $\{V_j : j \in \mathbb{Z}\}$ be an MRA of $L^2(\mathbb{R})$ with corresponding scaling function φ and avelet ψ. Let W_j be the corresponding wavelet subspaces: $W_j = \overline{sp}\{\psi_{jk} : k \in \mathbb{Z}\}$. In e construction of a wavelet from an MRA, essentially the space V_1 was split into two thogonal components V_0 and W_0. Note that V_1 is the closure of the linear span of the nctions $\{2^{\frac{1}{2}}\varphi(2 \cdot - k) : k \in \mathbb{Z}\}$, whereas V_0 and W_0 are respectively the closure of the an of $\{\varphi(\cdot - k) : k\}$ and $\{\psi(\cdot - k) : k\}$. Since $\varphi(2 \cdot - k) = \varphi\left(2(\cdot - \frac{k}{2})\right)$, we see that e above procedure splits the half-integer translates of a function into integer translates of o functions.

In fact, the splitting is not confined to V_1 alone: we can choose to split W_j, which is the an of $\{\psi(2^j \cdot - k) : k\} = \{\psi\left(2^j(\cdot - \frac{k}{2^j})\right) : k\}$, to get two functions whose $2^{-(j-1)}k$ anslates will span the same space W_j. Repeating the splitting procedure j times, we get

2^j functions whose integer translates alone span the space W_j. If we apply this to each W_j, then the resulting basis of $L^2(\mathbb{R})$, which will consist of integer translates of a countable number of functions (instead of all dilations and translations of the wavelet ψ), will give us a better frequency localization. This basis is called 'wavelet packet basis'. The concept of wavelet packet was introduced by Coifman, Meyer and Wickerhauser [6, 7]. For a nice exposition of wavelet packets of $L^2(\mathbb{R})$ with dilation 2, see [11].

The concept of wavelet packet was subsequently generalized to $\mathbb{R}^d$ by taking tensor products [5]. The non-tensor product version is due to Shen [16]. Other notable generalizations are the biorthogonal wavelet packets [4], non-orthogonal version of wavelet packets [3], the wavelet frame packets [2] on $\mathbb{R}$ for dilation 2, and the orthogonal, biorthogonal and frame packets on $\mathbb{R}^d$ by Long and Chen [13] for the dyadic dilation.

In this article we generalize these concepts to $\mathbb{R}^d$ for arbitrary dilation matrices and we will not restrict ourselves to one scaling function: we consider the case of those MRAs for which the central space is generated by several scaling functions.

DEFINITION 1.1

A $d \times d$ matrix A is said to be a dilation matrix for $\mathbb{R}^d$ if

(i) $A(\mathbb{Z}^d) \subset \mathbb{Z}^d$ and

(ii) all eigenvalues λ of A satisfy $|\lambda| > 1$.

Property (i) implies that A has integer entries and hence $|\det A|$ is an integer, and (ii) says that $|\det A|$ is greater than 1. Let $B = A^t$, the transpose of A and $a = |\det A| = |\det B|$.

Considering $\mathbb{Z}^d$ as an additive group, we see that $A\mathbb{Z}^d$ is a normal subgroup of $\mathbb{Z}^d$. So we can form the cosets of $A\mathbb{Z}^d$ in $\mathbb{Z}^d$. It is a well-known fact that the number of distinct cosets of $A\mathbb{Z}^d$ in $\mathbb{Z}^d$ is equal to $a = |\det A|$ ([10, 17]). A subset of $\mathbb{Z}^d$ which consists of exactly one element from each of the a cosets of $A\mathbb{Z}^d$ in $\mathbb{Z}^d$ will be called a *set of digits* for the dilation matrix A. Therefore, if K_A is a set of digits for A, then we can write

$$\mathbb{Z}^d = \bigcup_{\mu \in K_A} (A\mathbb{Z}^d + \mu),$$

where $\{A\mathbb{Z}^d + \mu : \mu \in K_A\}$ are pairwise disjoint. A set of digits for A need not be a set of digits for its transpose. For example, for the dilation matrix $M = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$ of $\mathbb{R}^2$, the set $\left\{\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\}$ is a set of digits for M but not for M^t. It is easy to see that if K is a set of digits for A, then so is $K - \mu$, where $\mu \in K$. Therefore, we can assume, without loss of generality, that $0 \in K$.

The notion of a multiresolution analysis can be extended to $L^2(\mathbb{R}^d)$ by replacing the dyadic dilation by a dilation matrix and allowing the resolution spaces to be spanned by more than one scaling function.

DEFINITION 1.2

A sequence $\{V_j : j \in \mathbb{Z}\}$ of closed subspaces of $L^2(\mathbb{R}^d)$ will be called a multiresolution analysis (MRA) of $L^2(\mathbb{R}^d)$ of multiplicity L associated with the dilation matrix A if the following conditions are satisfied:

M1) $V_j \subset V_{j+1}$ for all $j \in \mathbb{Z}$

M2) $\cup_{j\in\mathbb{Z}} V_j$ is dense in $L^2(\mathbb{R}^d)$ and $\cap_{j\in\mathbb{Z}} V_j = \{0\}$

M3) $f \in V_j$ if and only if $f(A\cdot) \in V_{j+1}$

M4) there exist L functions $\{\varphi_1, \varphi_2, \ldots, \varphi_L\}$ in V_0, called the *scaling functions*, such that the system of functions $\{\varphi_l(\cdot - k) : 1 \le l \le L,\ k \in \mathbb{Z}^d\}$ forms an orthonormal basis for V_0.

The concept of multiplicity was introduced by Hervé [12] in his Ph.D. thesis.

Since $\{\varphi_l(\cdot - k) : 1 \le l \le L,\ k \in \mathbb{Z}^d\}$ is an orthonormal basis of V_0, it follows from roperty (M3) that $\{a^{j/2}\varphi_l(A^j \cdot - k) : 1 \le l \le L,\ k \in \mathbb{Z}^d\}$ is an orthonormal basis of V_j. Observe that if $f \in L^2(\mathbb{R}^d)$, then

$$\left(a^{j/2} f(A^j \cdot - k)\right)^\wedge(\xi) = a^{-j/2} e^{-i\langle B^{-j}\xi, k\rangle} \hat{f}(B^{-j}\xi), \quad \xi \in \mathbb{R}^d,\ k \in \mathbb{Z}^d.$$

The Fourier transform of a function $f \in L^1(\mathbb{R}^d)$ is defined by

$$\mathcal{F} f(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-i\langle \xi, x\rangle} dx, \quad \xi \in \mathbb{R}^d.$$

To define the Fourier transform for functions of $L^2(\mathbb{R}^d)$, the operator $\mathcal{F}$ is extended from $^1 \cap L^2(\mathbb{R}^d)$, which is dense in $L^2(\mathbb{R}^d)$ in the L^2-norm, to the whole of $L^2(\mathbb{R}^d)$. For this efinition of the Fourier transform, Plancherel theorem takes the form

$$\langle f, g\rangle = \frac{1}{(2\pi)^d} \langle \hat{f}, \hat{g}\rangle; \quad f, g \in L^2(\mathbb{R}^d).$$

First of all we will prove a lemma, the splitting lemma (see [8]), which is essential for the onstruction of wavelet packets. We need the following facts for the proof of the splitting emma.

a) Let $\mathbb{T}^d = [-\pi, \pi]^d$ and $f \in L^1(\mathbb{R}^d)$. Since $\mathbb{R}^d = \cup_{k\in\mathbb{Z}^d}(\mathbb{T}^d + 2k\pi)$, we can write

$$\int_{\mathbb{R}^d} f(x) dx = \int_{\mathbb{T}^d} \left\{ \sum_{k\in\mathbb{Z}^d} f(x + 2k\pi) \right\} dx. \tag{1}$$

b) Let $\{s_k : k \in \mathbb{Z}^d\} \in l^1(\mathbb{Z}^d)$ and K_B be a set of digits for the dilation matrix B. As $\mathbb{Z}^d$ can be decomposed as $\mathbb{Z}^d = \cup_{\mu\in K_B}(B\mathbb{Z}^d + \mu)$, we can write

$$\sum_{k\in\mathbb{Z}^d} s_k = \sum_{\mu\in K_B} \sum_{k\in\mathbb{Z}^d} s_{\mu + Bk}. \tag{2}$$

c) Let K_B be a set of digits for B. Define

$$Q_0 = \bigcup_{\mu\in K_B} B^{-1}(\mathbb{T}^d + 2\mu\pi).$$

Since K_B is a set of digits for B, the set Q_0 satisfies $\cup_{k\in\mathbb{Z}^d}(Q_0 + 2k\pi) = \mathbb{R}^d$. This fact, together with $|Q_0| = (2\pi)^d$, implies that $\{Q_0 + 2k\pi : k \in \mathbb{Z}^d\}$ is a pairwise disjoint collection (see Lemma 1 of [10]). Therefore,

$$\int_{\mathbb{R}^d} f(x) dx = \int_{Q_0} \left\{ \sum_{k\in\mathbb{Z}^d} f(x + 2k\pi) \right\} dx, \quad \text{for } f \in L^1(\mathbb{R}^d). \tag{3}$$

A function f is said to be $2\pi\mathbb{Z}^d$-periodic if $f(x + 2k\pi) = f(x)$ for all $k \in \mathbb{Z}^d$ and for a.e. $x \in \mathbb{R}^d$.

2. The splitting lemma

Let $\{\varphi_l : 1 \le l \le L\}$ be functions in $L^2(\mathbb{R}^d)$ such that $\{\varphi_l(\cdot - k) : 1 \le l \le L, k \in \mathbb{Z}^d\}$ is an orthonormal system. Let $V = \overline{sp}\{a^{1/2}\varphi_l(A \cdot - k) : l, k\}$. For $1 \le l, j \le L$ and $0 \le r \le a-1$, suppose that there exist sequences $\{h^r_{ljk} : k \in \mathbb{Z}^d\} \in l^2(\mathbb{Z}^d)$. Define

$$f^r_l(x) = \sum_{j=1}^{L} \sum_{k\in\mathbb{Z}^d} h^r_{ljk} a^{1/2} \varphi_j(Ax - k). \tag{4}$$

Taking Fourier transform of both sides

$$\begin{aligned} \hat{f}^r_l(\xi) &= \sum_{j=1}^{L} \sum_{k\in\mathbb{Z}^d} h^r_{ljk} a^{-1/2} \mathrm{e}^{-i\langle B^{-1}\xi, k\rangle} \hat{\varphi}_j(B^{-1}\xi) \\ &= \sum_{j=1}^{L} h^r_{lj}(B^{-1}\xi)\hat{\varphi}_j(B^{-1}\xi), \end{aligned} \tag{5}$$

where

$$h^r_{lj}(\xi) = \sum_{k\in\mathbb{Z}^d} a^{-1/2} h^r_{ljk} \mathrm{e}^{-i\langle \xi, k\rangle}, \quad 1 \le l, j \le L,\ 0 \le r \le a-1, \tag{6}$$

and h^r_{lj} is $2\pi\mathbb{Z}^d$-periodic and is in $L^2(\mathbb{T}^d)$. Now, for $0 \le r \le a-1$, define the $L \times L$ matrices

$$H_r(\xi) = \left(h^r_{lj}(\xi)\right)_{1\le l, j \le L}. \tag{7}$$

By denoting

$$\Phi(x) = (\varphi_1(x), \ldots, \varphi_L(x))^t \tag{8}$$

$$\hat{\Phi}(\xi) = \left(\hat{\varphi}_1(\xi), \ldots, \hat{\varphi}_L(\xi)\right)^t, \tag{9}$$

we can write (5) as

$$\hat{F}_r(\xi) = H_r(B^{-1}\xi)\hat{\Phi}(B^{-1}\xi), \quad 0 \le r \le a-1, \tag{10}$$

where $F_r(x) = (f^r_1(x), f^r_2(x), \ldots, f^r_L(x))^t$ and $\hat{F}_r(\xi) = (\hat{f}^r_1(\xi), \hat{f}^r_2(\xi), \ldots, \hat{f}^r_L(\xi))^t$.

The following well-known lemma characterizes the orthonormality of the system $\{\varphi_l(\cdot - k) : 1 \le l \le L, k \in \mathbb{Z}^d\}$. We give a proof for the sake of completeness.

Lemma 2.1. *The system* $\{\varphi_l(\cdot - k)\colon 1 \le l \le L, k \in \mathbb{Z}^d\}$ *is orthonormal if and only if*

$$\sum_{k\in\mathbb{Z}^d} \hat{\varphi}_j(\xi + 2k\pi)\overline{\hat{\varphi}_l(\xi + 2k\pi)} = \delta_{jl}, \quad 1 \le j, l \le L.$$

Proof. Suppose that the system $\{\varphi_l(\cdot - k) : 1 \le l \le L, k \in \mathbb{Z}^d\}$ is orthonormal. Note that $\langle \varphi_j(\cdot - p), \varphi_l(\cdot - q)\rangle = \langle \varphi_j, \varphi_l(\cdot - (q-p))\rangle$ for $1 \le j, l \le L$ and $p, q \in \mathbb{Z}^d$. Now

$$\begin{aligned} \delta_{jl}\delta_{0p} &= \langle \varphi_j, \varphi_l(\cdot - p)\rangle = \frac{1}{(2\pi)^d}\langle \hat{\varphi}_j, (\varphi_l(\cdot - p))^\wedge\rangle \\ &= \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} \hat{\varphi}_j(\xi)\overline{\hat{\varphi}_l(\xi)}\mathrm{e}^{i\langle p, \xi\rangle}\mathrm{d}\xi \\ &= \frac{1}{(2\pi)^d}\int_{\mathbb{T}^d} \left\{\sum_{k\in\mathbb{Z}^d} \hat{\varphi}_j(\xi + 2k\pi)\overline{\hat{\varphi}_l(\xi + 2k\pi)}\right\}\mathrm{e}^{i\langle p, \xi\rangle}\mathrm{d}\xi, \quad \text{by (1)}. \end{aligned}$$

Therefore, the $2\pi\mathbb{Z}^d$-periodic function $G_{jl}(\xi) = \sum_{k\in\mathbb{Z}^d} \hat{\varphi}_j(\xi+2k\pi)\overline{\hat{\varphi}_l(\xi+2k\pi)}$ has Fourier coefficients $\hat{G}_{jl}(-p) = \delta_{jl}\delta_{0p}$, $p \in \mathbb{Z}^d$ which implies that $G_{jl} = \delta_{jl}$ a.e. By reversing the above steps we can prove the converse. □

Let $M^*(\xi)$ be the conjugate transpose of the matrix $M(\xi)$ and I_L denote the identity matrix of order L.

Lemma 2.2. (*The splitting lemma*) *Let* $\{\varphi_l : 1 \le l \le L\}$ *be functions in* $L^2(\mathbb{R}^d)$ *such that the system* $\{a^{1/2}\varphi_j(A\cdot -k) : 1 \le j \le L,\ k \in \mathbb{Z}^d\}$ *is orthonormal. Let* V *be its closed linear span. Let* K *be a set of digits for* B*. Also let* f_l^r, H_r *be as above. Then*

$$\{f_l^r(\cdot - k) : 0 \le r \le a-1,\ 1 \le l \le L,\ k \in \mathbb{Z}^d\}$$

is an orthonormal system if and only if

$$\sum_{\mu\in K} H_r(\xi + 2B^{-1}\mu\pi)H_s^*(\xi + 2B^{-1}\mu\pi) = \delta_{rs}I_L, \quad 0 \le r, s \le a-1. \quad (11)$$

Moreover, $\{f_l^r(\cdot - k) : 0 \le r \le a-1,\ 1 \le l \le L,\ k \in \mathbb{Z}^d\}$ *is an orthonormal basis of* V *whenever it is orthonormal.*

Proof. For $1 \le l, j \le L, 0 \le r, s \le a-1$ and $p \in \mathbb{Z}^d$, we have

$$\begin{aligned}
&\left\langle f_j^r, f_l^s(\cdot - p)\right\rangle \\
&= \frac{1}{(2\pi)^d}\left\langle (f_j^r)^\wedge, (f_l^s(\cdot - p))^\wedge\right\rangle \\
&= \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} (f_j^r)^\wedge(\xi)\overline{(f_l^s)^\wedge(\xi)\mathrm{e}^{-i\langle p,\xi\rangle}}\,\mathrm{d}\xi \\
&= \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} \sum_{m=1}^{L}\sum_{n=1}^{L} h_{jm}^r(B^{-1}\xi)\overline{h_{ln}^s(B^{-1}\xi)}\hat{\varphi}_m(B^{-1}\xi)\overline{\hat{\varphi}_n(B^{-1}\xi)}\mathrm{e}^{i\langle p,\xi\rangle}\mathrm{d}\xi \\
&\hspace{20em} \text{(by (5))} \\
&= \frac{1}{(2\pi)^d}\int_{\mathbb{T}^d} \sum_{k\in\mathbb{Z}^d}\sum_{m=1}^{L}\sum_{n=1}^{L} \Big\{h_{jm}^r(B^{-1}(\xi+2k\pi))\overline{h_{ln}^s(B^{-1}(\xi+2k\pi))} \\
&\qquad \cdot\hat{\varphi}_m(B^{-1}(\xi+2k\pi))\overline{\hat{\varphi}_n(B^{-1}(\xi+2k\pi))}\Big\}\,\mathrm{e}^{i\langle p,\xi+2k\pi\rangle}\mathrm{d}\xi \quad \text{(by (1))} \\
&= \frac{1}{(2\pi)^d}\int_{\mathbb{T}^d} \sum_{\mu\in K}\sum_{m=1}^{L}\sum_{n=1}^{L} h_{jm}^r(B^{-1}\xi + 2B^{-1}\mu\pi)\overline{h_{ln}^s(B^{-1}\xi + 2B^{-1}\mu\pi)} \\
&\qquad \cdot\left\{\sum_{k\in\mathbb{Z}^d} \hat{\varphi}_m(B^{-1}(\xi+2\mu\pi)+2k\pi)\overline{\hat{\varphi}_n(B^{-1}(\xi+2\mu\pi)+2k\pi)}\right\}\mathrm{e}^{i\langle p,\xi\rangle}\mathrm{d}\xi \\
&\hspace{20em} \text{(by (2))} \\
&= \frac{1}{(2\pi)^d}\int_{\mathbb{T}^d} \sum_{\mu\in K}\sum_{m=1}^{L}\sum_{n=1}^{L} h_{jm}^r(B^{-1}\xi + 2B^{-1}\mu\pi)\overline{h_{ln}^s(B^{-1}\xi + 2B^{-1}\mu\pi)} \\
&\hspace{14em} \cdot\delta_{mn}\mathrm{e}^{i\langle p,\xi\rangle}\mathrm{d}\xi \quad \text{(by Lemma 2.1)} \\
&= \frac{1}{(2\pi)^d}\int_{\mathbb{T}^d} \left\{\sum_{\mu\in K}\sum_{m=1}^{L} h_{jm}^r(B^{-1}\xi + 2B^{-1}\mu\pi)\overline{h_{lm}^s(B^{-1}\xi + 2B^{-1}\mu\pi)}\right\}\mathrm{e}^{i\langle p,\xi\rangle}\mathrm{d}\xi.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\left\langle f_j^r, f_l^s(\cdot - p)\right\rangle &= \delta_{rs}\delta_{jl}\delta_{0p} \\
\Leftrightarrow \sum_{\mu\in K}\sum_{m=1}^{L} h_{jm}^r(B^{-1}\xi + 2B^{-1}\mu\pi)\overline{h_{lm}^s(B^{-1}\xi + 2B^{-1}\mu\pi)} &= \delta_{rs}\delta_{jl} \text{ for a.e. } \xi \in \mathbb{R}^d \\
\Leftrightarrow \sum_{\mu\in K}\sum_{m=1}^{L} h_{jm}^r(\xi + 2B^{-1}\mu\pi)\overline{h_{lm}^s(\xi + 2B^{-1}\mu\pi)} &= \delta_{rs}\delta_{jl} \text{ for a.e. } \xi \in \mathbb{R}^d \\
\Leftrightarrow \sum_{\mu\in K} H_r(\xi + 2B^{-1}\mu\pi)H_s^*(\xi + 2B^{-1}\mu\pi) &= \delta_{rs}I_L \text{ for a.e. } \xi \in \mathbb{R}^d.
\end{aligned}$$

We have proved the first part of the lemma.

Now assume that $\{f_l^r(\cdot - k) : 0 \le r \le a-1,\ 1 \le l \le L,\ k \in \mathbb{Z}^d\}$ is an orthonormal system. We want to show that this is an orthonormal basis of V. Let $f \in V$. So there exists $\{c_{jp} : p \in \mathbb{Z}^d\} \in l^2(Z^d)$, $1 \le j \le L$ such that

$$f(x) = \sum_{j=1}^{L}\sum_{p\in\mathbb{Z}^d} c_{jp}a^{1/2}\varphi_j(Ax - p).$$

Assume that $f \perp f_l^r(\cdot - k)$ for all r, l, k.

Claim. $f = 0$.

For all r, l, k such that $0 \le r \le a-1, 1 \le l \le L, k \in \mathbb{Z}^d$, we have

$$\begin{aligned}
0 &= \left\langle f_l^r(\cdot - k), f\right\rangle = \left\langle f_l^r(\cdot - k), \sum_{j=1}^{L}\sum_{p\in\mathbb{Z}^d} c_{jp}a^{1/2}\varphi_j(A\cdot - p)\right\rangle \\
&= \frac{1}{(2\pi)^d}\left\langle \left(f_l^r(\cdot - k)\right)^{\wedge}, \left(\sum_{j=1}^{L}\sum_{p\in\mathbb{Z}^d} c_{jp}a^{1/2}\varphi_j(A\cdot - p)\right)^{\wedge}\right\rangle \\
&= \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} (f_l^r)^{\wedge}(\xi)\mathrm{e}^{-i\langle k,\xi\rangle}\sum_{j=1}^{L}\sum_{p\in\mathbb{Z}^d}\overline{c_{jp}}a^{-1/2}\mathrm{e}^{i\langle B^{-1}\xi, p\rangle}\overline{\hat{\varphi}_j(B^{-1}\xi)}\mathrm{d}\xi \\
&= \frac{a^{-1/2}}{(2\pi)^d}\int_{\mathbb{R}^d}\sum_{m=1}^{L} h_{lm}^r(B^{-1}\xi)\hat{\varphi}_m(B^{-1}\xi)\mathrm{e}^{-i\langle k,\xi\rangle}\sum_{j=1}^{L}\sum_{p\in\mathbb{Z}^d}\overline{c_{jp}}\mathrm{e}^{i\langle B^{-1}\xi, p\rangle}\overline{\hat{\varphi}_j(B^{-1}\xi)}\mathrm{d}\xi \\
&\qquad\qquad\text{(by (5))} \\
&= \frac{a^{1/2}}{(2\pi)^d}\int_{\mathbb{R}^d}\sum_{m=1}^{L} h_{lm}^r(\xi)\hat{\varphi}_m(\xi)\sum_{j=1}^{L}\sum_{p\in\mathbb{Z}^d}\overline{c_{jp}}\overline{\hat{\varphi}_j(\xi)}\mathrm{e}^{-i\langle k,B\xi\rangle}\mathrm{e}^{i\langle p,\xi\rangle}\mathrm{d}\xi \quad (\xi \to B\xi) \\
&= \frac{a^{1/2}}{(2\pi)^d}\int_{Q_0}\sum_{q\in\mathbb{Z}^d}\sum_{m=1}^{L} h_{lm}^r(\xi + 2q\pi)\hat{\varphi}_m(\xi + 2q\pi) \\
&\quad \cdot\sum_{j=1}^{L}\sum_{p\in\mathbb{Z}^d}\overline{c_{jp}}\overline{\hat{\varphi}_j(\xi + 2q\pi)}\mathrm{e}^{-i\langle k, B(\xi+2q\pi)\rangle}\mathrm{e}^{i\langle p, \xi+2q\pi\rangle}\mathrm{d}\xi \quad \text{(by (3))} \\
&= \frac{a^{1/2}}{(2\pi)^d}\int_{Q_0}\sum_{m=1}^{L}\sum_{j=1}^{L}\sum_{p\in\mathbb{Z}^d} h_{lm}^r(\xi)\overline{c_{jp}}\left\{\sum_{q\in\mathbb{Z}^d}\hat{\varphi}_m(\xi + 2q\pi)\overline{\hat{\varphi}_j(\xi + 2q\pi)}\right\} \\
&\qquad \cdot\mathrm{e}^{-i\langle k, B\xi\rangle}\mathrm{e}^{i\langle p, \xi\rangle}\mathrm{d}\xi
\end{aligned}$$

$$= \frac{a^{1/2}}{(2\pi)^d}\int_{Q_0}\sum_{m=1}^{L}\sum_{p\in\mathbb{Z}^d} h^r_{lm}(\xi)\overline{c_{mp}}\mathrm{e}^{-i\langle k,B\xi\rangle}\mathrm{e}^{i\langle p,\xi\rangle}\mathrm{d}\xi \quad \text{(by Lemma 2.1)}$$

$$= \frac{a^{1/2}}{(2\pi)^d}\sum_{\mu\in K}\int_{B^{-1}(\mathbb{T}^d+2\mu\pi)}\sum_{m=1}^{L}\sum_{p\in\mathbb{Z}^d} h^r_{lm}(\xi)\overline{c_{mp}}\mathrm{e}^{-i\langle k,B\xi\rangle}\mathrm{e}^{i\langle p,\xi\rangle}\mathrm{d}\xi$$

$$= \frac{a^{1/2}}{(2\pi)^d}\sum_{\mu\in K}\int_{B^{-1}\mathbb{T}^d}\sum_{m=1}^{L}\sum_{p\in\mathbb{Z}^d} h^r_{lm}(\xi+2B^{-1}\mu\pi)\overline{c_{mp}}\mathrm{e}^{-i\langle k,B(\xi+2B^{-1}\mu\pi)\rangle}$$
$$\cdot\mathrm{e}^{i\langle p,\xi+2B^{-1}\mu\pi\rangle}\mathrm{d}\xi$$

$$= \frac{a^{1/2}}{(2\pi)^d}\int_{B^{-1}\mathbb{T}^d}\left\{\sum_{\mu\in K}\sum_{m=1}^{L}\sum_{p\in\mathbb{Z}^d} h^r_{lm}(\xi+2B^{-1}\mu\pi)\overline{c_{mp}}\mathrm{e}^{i\langle p,\xi+2B^{-1}\mu\pi\rangle}\right\}$$
$$\cdot\mathrm{e}^{-i\langle k,B\xi\rangle}\mathrm{d}\xi.$$

ce $\left\{\frac{a^{1/2}}{(2\pi)^d}\mathrm{e}^{-i\langle k,B\cdot\rangle} : k\in\mathbb{Z}^d\right\}$ is an orthonormal basis for $L^2(B^{-1}\mathbb{T}^d)$, the above equa-
s give

$$\sum_{\mu\in K}\sum_{m=1}^{L}\sum_{p\in\mathbb{Z}^d}\overline{c_{mp}}\mathrm{e}^{i\langle\xi+2B^{-1}\mu\pi,p\rangle}h^r_{lm}(\xi+2B^{-1}\mu\pi) = 0 \text{ a.e.} \quad \text{for all } r,\ l.$$

$m = 1, 2, \ldots, L$, define

$$C_m(\xi) = \sum_{p\in\mathbb{Z}^d} c_{mp}\mathrm{e}^{-i\langle\xi,p\rangle}. \tag{12}$$

ve have

$$\sum_{K}\sum_{m=1}^{L}\overline{C_m(\xi+2B^{-1}\mu\pi)}h^r_{lm}(\xi+2B^{-1}\mu\pi) = 0,\ \ 0\le r\le a-1,\ \ 1\le l\le L. \tag{13}$$

ations (11) are equivalent to saying that for $0\le r\le a-1,\ 1\le l\le L$ and for a.e.
$\mathbb{R}^d$, the vectors

$$\left(h^r_{lm}(\xi+2B^{-1}\mu\pi) : 1\le m\le L,\ \mu\in K\right)$$

mutually orthogonal and each has norm 1, considered as a vector in the aL-dimensional
ce $\mathbb{C}^{aL}$, so that they form an orthonormal basis for $\mathbb{C}^{aL}$. Equation (13) says that the
tor

$$\left(C_m(\xi+2B^{-1}\mu\pi) : 1\le m\le L,\ \mu\in K\right) \tag{14}$$

rthogonal to each member of the above orthonormal basis of $\mathbb{C}^{aL}$. Hence, the vector
ne expression (14) is zero. In particular, $C_m(\xi) = 0$, for all m, $1\le m\le L$. That is,
$= 0,\ 1\le m\le L,\ p\in\mathbb{Z}^d$. Therefore, $f = 0$. This ends the proof. □

he splitting lemma can be used to decompose an arbitrary Hilbert space into mutually
ogonal subspaces, as in [7]. We will use the following corollary later.

COROLLARY 2.3

Let $\{E_{lk} : 1 \le l \le L,\ k \in \mathbb{Z}^d\}$ *be an orthonormal basis of a separable Hilbert space* $\mathcal{H}$. *Let* $H_r,\ 0 \le r \le a-1$ *be as above and satisfy* (11). *Define*

$$F^r_{lk} = \sum_{m=1}^{L} \sum_{p \in \mathbb{Z}^d} h^r_{l,m,p-Ak} E_{mp}; \qquad 0 \le r \le a-1,\ 1 \le l \le L,\ k \in \mathbb{Z}^d.$$

Then $\{F^r_{lk} : 1 \le l \le L,\ k \in \mathbb{Z}^d\}$ *is an orthonormal basis for its closed linear span* $\mathcal{H}^r$ *and* $\mathcal{H} = \oplus_{r=0}^{a-1} \mathcal{H}^r$.

Proof. Let $\varphi_1, \varphi_2, \ldots, \varphi_L$ be functions in $L^2(\mathbb{R}^d)$ such that $\{\varphi_l(\cdot - k) : 1 \le l \le L, k \in \mathbb{Z}^d\}$ is an orthonormal system. Let $V = \overline{sp}\{a^{1/2}\varphi_l(A \cdot -k) : l,\ k\}$. Define a linear operator T from the Hilbert space V to $\mathcal{H}$ by $T(a^{1/2}\varphi_l(A \cdot -k)) = E_{l,k}$. Let f^r_l are as in (4). Then, $T(f^r_l(\cdot - k)) = F^r_{l,k}$. Now the corollary follows from the splitting lemma. □

3. Construction of multiwavelet packets

Let $\{V_j : j \in \mathbb{Z}\}$ be an MRA of $L^2(\mathbb{R}^d)$ of multiplicity L associated with the dilation matrix A. Let $\{\varphi_l : 1 \le l \le L\}$ be the scaling functions. Since φ_l, $1 \le l \le L$ are in $V_0 \subset V_1$ and $\{a^{1/2}\varphi_j(A \cdot -k) : 1 \le j \le L, k \in \mathbb{Z}^d\}$ forms an orthonormal basis of V_1, there exist $\{h_{ljk} : k \in \mathbb{Z}^d\} \in l^2(\mathbb{Z}^d)$ for $1 \le l, j \le L$ such that

$$\varphi_l(x) = \sum_{j=1}^{L} \sum_{k \in \mathbb{Z}^d} h_{ljk} a^{1/2} \varphi_j(Ax - k).$$

Taking Fourier transform, we get

$$\begin{aligned} \hat{\varphi}_l(\xi) &= \sum_{j=1}^{L} \sum_{k \in \mathbb{Z}^d} h_{ljk} a^{-1/2} e^{-i\langle B^{-1}\xi, k\rangle} \hat{\varphi}_j(B^{-1}\xi) \\ &= \sum_{j=1}^{L} h_{lj}(B^{-1}\xi)\hat{\varphi}_j(B^{-1}\xi), \end{aligned} \tag{15}$$

where $h_{lj}(\xi) = \sum_{k \in \mathbb{Z}^d} a^{-1/2} h_{ljk} e^{-i\langle \xi, k\rangle}$, and h_{lj} is $2\pi\mathbb{Z}^d$-periodic and is in $L^2(\mathbb{T}^d)$. Let $H_0(\xi)$ be the $L \times L$ matrix defined by

$$H_0(\xi) = \Big((h_{lj}(\xi)\Big)_{1 \le l, j \le L}.$$

We will call H_0 the *low-pass filter matrix.* Rewriting (15) in the vector notations (8) and (9), we have

$$\hat{\Phi}(\xi) = H_0(B^{-1}\xi)\hat{\Phi}(B^{-1}\xi). \tag{16}$$

Let W_j be the wavelet subspaces, the orthogonal complement of V_j in V_{j+1}:

$$W_j = V_{j+1} \ominus V_j.$$

Properties (M1) and (M3) of Definition 1.2 now imply that

$$W_j \perp W_{j'}, \quad j \ne j'$$

and

$$f \in W_j \Leftrightarrow f(A^{-j}\cdot) \in W_0. \tag{17}$$

Moreover, by (M2), $L^2(\mathbb{R}^d)$ can be decomposed into orthogonal direct sums as

$$\begin{aligned} L^2(\mathbb{R}^d) &= \bigoplus_{j\in\mathbb{Z}} W_j && (18)\\ &= V_0 \oplus \left(\bigoplus_{j\geq 0} W_j\right). && (19)\end{aligned}$$

By Lemma 2.1 and eq. (15), we have (for $1 \leq l, j \leq L$)

$$\begin{aligned} \delta_{jl} &= \sum_{k\in\mathbb{Z}^d} \hat{\varphi}_j(\xi + 2k\pi)\overline{\hat{\varphi}_l(\xi + 2k\pi)}\\ &= \sum_{k\in\mathbb{Z}^d} \left\{\sum_{m=1}^{L} h_{jm}(B^{-1}(\xi + 2k\pi))\hat{\varphi}_m(B^{-1}(\xi + 2k\pi))\right\}\\ &\quad \cdot\left\{\sum_{n=1}^{L} \overline{h_{ln}(B^{-1}(\xi + 2k\pi))\hat{\varphi}_n(B^{-1}(\xi + 2k\pi))}\right\}.\end{aligned}$$

Now, using (2), we have

$$\begin{aligned} \delta_{jl} &= \sum_{\mu\in K_B}\sum_{m=1}^{L}\sum_{n=1}^{L} h_{jm}(B^{-1}\xi + 2B^{-1}\mu\pi)\overline{h_{ln}(B^{-1}\xi + 2B^{-1}\mu\pi)}\\ &\quad \cdot \sum_{k\in\mathbb{Z}^d}\left\{\hat{\varphi}_m(B^{-1}(\xi + 2\mu\pi) + 2k\pi)\overline{\hat{\varphi}_n(B^{-1}(\xi + 2\mu\pi) + 2k\pi)}\right\},\end{aligned}$$

where K_B is a set of digits for B. Using Lemma 2.1 again, we get

$$\delta_{jl} = \sum_{\mu\in K_B}\sum_{m=1}^{L} h_{jm}(B^{-1}\xi + 2B^{-1}\mu\pi)\overline{h_{lm}(B^{-1}\xi + 2B^{-1}\mu\pi)}. \tag{20}$$

This is equivalent to saying that

$$\sum_{\mu\in K_B} H_0(\xi + 2B^{-1}\mu\pi)H_0^*(\xi + 2B^{-1}\mu\pi) = I_L \quad \text{for a.e. } \xi.$$

Equation (20) is also equivalent to the orthonormality of the vectors

$$\left(h_{lj}(\xi + 2B^{-1}\mu\pi) : \ 1 \leq j \leq L,\ \mu \in K_B\right), \qquad 1 \leq l \leq L,\ \xi \in \mathbb{T}^d.$$

These L orthonormal vectors in the aL-dimensional space $\mathbb{C}^{aL}$ can be completed, by Gram–Schmidt orthonormalization process, to produce an orthonormal basis for $\mathbb{C}^{aL}$. Let us denote the new vectors by

$$\left(h_{lj}^r(\xi + 2B^{-1}\mu\pi) : \ 1 \leq j \leq L,\ \mu \in K_B\right), \qquad 1 \leq l \leq L,\ 1 \leq r \leq a-1,\ \xi \in \mathbb{T}^d,$$

and extend the functions h_{lj}^r $(1 \leq r \leq a-1,\ 1 \leq l, j \leq L)$ $2\pi\mathbb{Z}^d$-periodically (see [9] for the one-dimensional dyadic dilation). Denoting by $H_r(\xi)$, $1 \leq r \leq a-1$ the $L \times L$ matrix

$$\left(h_{lj}^r(\xi)\right)_{1\leq l,j\leq L},$$

we have

$$\sum_{\mu\in K_B} H_r(\xi+2B^{-1}\mu\pi)H_s^*(\xi+2B^{-1}\mu\pi)=\delta_{rs}I_L \quad \text{for a.e. } \xi.$$

Now, for $1\le r\le a-1,\ 1\le l\le L$, define

$$\hat{f}_l^r(\xi)=\sum_{j=1}^{L} h_{lj}^r(B^{-1}\xi)\hat{\varphi}_j(B^{-1}\xi). \tag{21}$$

Since h_{lj}^r are $2\pi\mathbb{Z}^d$-periodic, there exist $\{h_{ljk}^r : k\in\mathbb{Z}^d\}\in l^2(\mathbb{Z}^d)$ such that

$$h_{lj}^r(\xi)=\sum_{k\in\mathbb{Z}^d} a^{-1/2}h_{ljk}^r e^{-i\langle\xi,k\rangle}.$$

Now, applying the splitting lemma to V_1, we see that $\{f_l^r(\cdot-k) : 0\le r\le a-1,\ 1\le l\le L, k\in\mathbb{Z}^d\}$ is an orthonormal basis for V_1. We use the convention $\varphi_l=f_l^0$, $1\le l\le L$ with $h_{lj}=h_{lj}^0$ and $h_{ljk}=h_{ljk}^0$. The decomposition $V_1=V_0\oplus W_0$, and the fact that $\{f_l^0(\cdot-k) : 1\le l\le L,\ k\in\mathbb{Z}^d\}$ is an orthonormal basis of V_0, imply that

$$\{f_l^r(\cdot-k) : 1\le r\le a-1,\ 1\le l\le L,\ k\in\mathbb{Z}^d\}$$

is an orthonormal basis for W_0. By (17) and (18), we see that

$$\{a^{j/2}f_l^r(A^j\cdot-k) : 1\le r\le a-1,\ 1\le l\le L,\ j\in\mathbb{Z},\ k\in\mathbb{Z}^d\}$$

is an orthonormal basis for $L^2(\mathbb{R}^d)$. This basis is called the *multiwavelet basis* and the functions $\{f_l^r : 1\le r\le a-1,\ 1\le l\le L\}$ are the *multiwavelets* associated with the MRA $\{V_j : j\in\mathbb{Z}\}$ of multiplicity L. For $0\le r\le a-1$, by denoting $F_r(x)=\left(f_1^r(x),f_2^r(x),\ldots,f_L^r(x)\right)^t$ and $\hat{F}_r(\xi)=\left(\hat{f}_1^r(\xi),\hat{f}_2^r(\xi),\ldots,\hat{f}_L^r(\xi)\right)^t$, we can write (16) and (21) as

$$\hat{F}_r(\xi)=H_r(B^{-1}\xi)\hat{\Phi}(B^{-1}\xi),\quad 0\le r\le a-1. \tag{22}$$

This equation is known as the *scaling relation* satisfied by the scaling functions ($r=0$) and the multiwavelets ($1\le r\le a-1$).

As we observed, applying splitting lemma to the space $V_1=\overline{sp}\{a^{1/2}\varphi_l(A\cdot-k) : 1\le l\le L,\ k\in\mathbb{Z}^d\}$, we get the functions f_l^r, $0\le r\le a-1$, $1\le l\le L$. Now, for any $n\in\mathbb{N}_0=\mathbb{N}\cup\{0\}$, we define f_l^n, $1\le l\le L$ recursively as follows. Suppose that f_l^r, $r\in\mathbb{N}_0$, $1\le l\le L$ are defined already. Then define

$$f_l^{s+ar}(x)=\sum_{j=1}^{L}\sum_{k\in\mathbb{Z}^d} h_{ljk}^s a^{1/2}f_j^r(Ax-k);\ 0\le s\le a-1,\ 1\le l\le L. \tag{23}$$

Taking Fourier transform

$$(f_l^{s+ar})^\wedge(\xi)=\sum_{j=1}^{L} h_{lj}^s(B^{-1}\xi)(f_j^r)^\wedge(B^{-1}\xi). \tag{24}$$

In vector notation, (24) can be written as

$$(F_{s+ar})^\wedge(\xi)=H_s(B^{-1}\xi)\hat{F}_r(B^{-1}\xi). \tag{25}$$

ote that (23) defines f_l^n for every non-negative integer n and every l such that $1 \le l \le L$. bserve that $f_l^0 = \varphi_l$, $1 \le l \le L$ are the scaling functions and f_l^r, $1 \le r \le a-1, 1 \le$ $\le L$ are the multiwavelets. So this definition is consistent with the scaling relation (22) tisfied by the scaling functions and the multiwavelets.

EFINITION 3.1

he functions $\{f_l^n : n \ge 0,\ 1 \le l \le L\}$ as defined above will be called the *basic multi-avelet packets* corresponding to the MRA $\{V_j : j \in \mathbb{Z}\}$ of $L^2(\mathbb{R}^d)$ of multiplicity L ssociated with the dilation A.

he Fourier transforms of the multiwavelet packets

ur aim is to find an expression for the Fourier transform of the basic multiwavelet packets terms of the Fourier transform of the scaling functions. For an integer $n \ge 1$, we consider e unique *'a-adic expansion'* (i.e., expansion in the base a):

$$n = \mu_1 + \mu_2 a + \mu_3 a^2 + \cdots + \mu_j a^{j-1}, \tag{26}$$

here $0 \le \mu_i \le a-1$ for all $i = 1, 2, \ldots, j$ and $\mu_j \ne 0$.

If n can be expressed as in (26) then we will say n has a-adic length j. We claim that if has length j and has expansion (26), then

$$\hat{F}_n(\xi) = H_{\mu_1}(B^{-1}\xi)H_{\mu_2}(B^{-2}\xi)\cdots H_{\mu_j}(B^{-j}\xi)\hat{\Phi}(B^{-j}\xi), \tag{27}$$

that $(f_l^n)^\wedge(\xi)$ is the lth component of the column vector in the right hand side of (27). e will prove the claim by induction.

From (22) we see that the claim is true for all n of length 1. Assume it for length j. Then n integer m of a-adic length $j+1$ is of the form $m = \mu + an$, where $0 \le \mu \le a-1$ and has length j. Suppose n has the expansion (26). Then from (25) and(27), we have

$$\begin{aligned} (F_m)^\wedge(\xi) &= (F_{\mu+an})^\wedge(\xi) \\ &= H_\mu(B^{-1}\xi)\hat{F}_n(B^{-1}\xi) \\ &= H_\mu(B^{-1}\xi)H_{\mu_1}(B^{-2}\xi)\cdots H_{\mu_j}(B^{-(j+1)}\xi)\hat{\Phi}(B^{-(j+1)}\xi). \end{aligned}$$

ince $m = \mu + an = \mu + \mu_1 a + \mu_2 a^2 + \cdots + \mu_j a^j$, $\hat{F}_m(\xi)$ has the desired form. Hence, e induction is complete.

The first theorem regarding the multiwavelet packets is the following.

heorem 3.2. *Let $\{f_l^n : n \ge 0,\ 1 \le l \le L\}$ be the basic multiwavelet packets constructed bove. Then*

(i) $\{f_l^n(\cdot - k) : a^j \le n \le a^{j+1} - 1,\ \ 1 \le l \le L,\ \ k \in \mathbb{Z}^d\}$ *is an orthonormal basis of* $W_j,\ j \ge 0$.

ii) $\{f_l^n(\cdot - k)\ :\ 0 \le n \le a^j - 1,\ \ 1 \le l \le L,\ \ k \in \mathbb{Z}^d\}$ *is an orthonormal basis of* $V_j,\ j \ge 0$.

ii) $\{f_l^n(\cdot - k) : n \ge 0,\ \ 1 \le l \le L,\ \ k \in \mathbb{Z}^d\}$ *is an orthonormal basis of* $L^2(\mathbb{R}^d)$.

Proof. Since $\{f_l^n : 1 \leq n \leq a-1,\ \ 1 \leq l \leq L\}$ are the multiwavelets, their $\mathbb{Z}^d$-translates form an orthonormal basis for W_0. So (i) is verified for $j = 0$. Assume for j. We will prove for $j+1$. By assumption, the functions $\{f_l^n(\cdot - k) : a^j \leq n \leq a^{j+1}-1,$ $1 \leq l \leq L,\ \ k \in \mathbb{Z}^d\}$ is an orthonormal basis of W_j. Since $f \in W_j \Leftrightarrow f(A\cdot) \in W_{j+1}$, the system of functions

$$\{a^{1/2} f_l^n(A \cdot -k) : a^j \leq n \leq a^{j+1}-1,\ \ 1 \leq l \leq L,\ \ k \in \mathbb{Z}^d\}$$

is an orthonormal basis of W_{j+1}. Let

$$E_n = \overline{sp}\{a^{1/2} f_l^n(A \cdot -k) :\ 1 \leq l \leq L,\ \ k \in \mathbb{Z}^d\}.$$

Hence,

$$W_{j+1} = \bigoplus_{n=a^j}^{a^{j+1}-1} E_n. \tag{28}$$

Applying the splitting lemma to E_n, we get the functions

$$g_l^{n,r}(x) = \sum_{m=1}^{L} \sum_{k \in \mathbb{Z}^d} h_{lmk}^r a^{1/2} f_m^n(Ax-k) \quad (0 \leq r \leq a-1,\ \ 1 \leq l \leq L) \tag{29}$$

so that $\{g_l^{n,r}(\cdot - k) :\ 0 \leq r \leq a-1,\ \ 1 \leq l \leq L,\ \ k \in \mathbb{Z}^d\}$ is an orthonormal basis of E_n. But by (23), we have

$$g_l^{n,r} = f_l^{r+an}.$$

This fact, together with (28), shows that

$$\begin{aligned}
&\{f_l^{r+an}(\cdot - k) :\ 0 \leq r \leq a-1,\ \ 1 \leq l \leq L,\ \ k \in \mathbb{Z}^d,\ a^j \leq n \leq a^{j+1}-1\} \\
&= \ \{f_l^n(\cdot - k) : a^{j+1} \leq n \leq a^{j+2}-1,\ \ 1 \leq l \leq L,\ \ k \in \mathbb{Z}^d\}
\end{aligned}$$

is an orthonormal basis of W_{j+1}. So (i) is proved. Item (ii) follows from the observation that $V_j = V_0 \oplus W_0 \oplus \cdots \oplus W_{j-1}$ and (iii) follows from the fact that $\overline{\cup V_j} = L^2(\mathbb{R}^d)$. □

4. Construction of orthonormal bases from the multiwavelet packets

We now take *all* dilations by the matrix A and *all* $\mathbb{Z}^d$-translations of the basic multiwavelet packet functions.

DEFINITION 4.1

Let $\{f_l^n : n \geq 0,\ \ 1 \leq l \leq L\}$ be the basic multiwavelet packets. The collection of functions

$$\mathcal{P} = \{a^{j/2} f_l^n(A^j \cdot -k) : n \geq 0,\ \ 1 \leq l \leq L,\ \ j \in \mathbb{Z},\ \ k \in \mathbb{Z}^d\}$$

will be called the '*general multiwavelet packets*' associated with the MRA $\{V_j\}$ of $L^2(\mathbb{R}^d)$ of multiplicity L.

Remark 4.2. Obviously the collection $\mathcal{P}$ is overcomplete in $L^2(\mathbb{R}^d)$. For example

(i) The subcollection with $j = 0,\ n \geq 0,\ 1 \leq l \leq L,\ k \in \mathbb{Z}^d$ gives us the basic multiwavelet packet basis constructed in the previous section.

(ii) The subcollection with $n = 1, 2, \ldots, a-1;\ 1 \leq l \leq L,\ j \in \mathbb{Z},\ k \in \mathbb{Z}^d$ is the usual multiwavelet basis.

So it will be interesting to find out other subcollections of $\mathcal{P}$ which form orthonormal bases for $L^2(\mathbb{R}^d)$.

For $n \geq 0$ and $j \in \mathbb{Z}$, define the subspaces

$$U_j^n = \overline{sp}\{a^{j/2} f_l^n(A^j \cdot - k) : 1 \leq l \leq L,\ k \in \mathbb{Z}^d\}. \tag{30}$$

Observe that

$$U_j^0 = V_j \quad \text{and} \quad \bigoplus_{r=1}^{a-1} U_j^r = W_j, \quad j \in \mathbb{Z}.$$

Hence, the orthogonal decomposition $V_{j+1} = V_j \oplus W_j$ can be written as

$$U_{j+1}^0 = \bigoplus_{r=0}^{a-1} U_j^r.$$

We can generalize this fact to other values of n.

PROPOSITION 4.3

For $n \geq 0$ and $j \in \mathbb{Z}$, we have

$$U_{j+1}^n = \bigoplus_{r=0}^{a-1} U_j^{an+r}. \tag{31}$$

Proof. By definition

$$U_{j+1}^n = \overline{sp}\left\{a^{\frac{j+1}{2}} f_l^n(A^{j+1} \cdot - k) : 1 \leq l \leq L,\ k \in \mathbb{Z}^d\right\}.$$

Let

$$E_{l,k}(x) = a^{\frac{j+1}{2}} f_l^n(A^{j+1} \cdot - k), \quad \text{for } 1 \leq l \leq L,\ k \in \mathbb{Z}^d.$$

Then $\{E_{l,k} : 1 \leq l \leq L,\ k \in \mathbb{Z}^d\}$ is an orthonormal basis of the Hilbert space U_{j+1}^n. For $0 \leq r \leq a-1$, let

$$F_{l,k}^r(x) = \sum_{m=1}^{L} \sum_{\beta \in \mathbb{Z}^d} h_{l,m,\beta - Ak}^r E_{m,\beta}(x), \quad 1 \leq l \leq L,\ k \in \mathbb{Z}^d,$$

and

$$\mathcal{H}^r = \overline{sp}\{F_{l,k}^r : 1 \leq l \leq L,\ k \in \mathbb{Z}^d\}.$$

Then, by Corollary 2.3 we have

$$U_{j+1}^n = \bigoplus_{r=0}^{a-1} \mathcal{H}^r.$$

Now

$$\begin{aligned}
F^r_{l,k}(x) &= \sum_{m=1}^{L} \sum_{\beta\in\mathbb{Z}^d} h^r_{l,m,\beta-Ak} E_{m,\beta}(x) \\
&= \sum_{m=1}^{L} \sum_{\alpha\in\mathbb{Z}^d} h^r_{l,m,\alpha} E_{m,Ak+\alpha}(x) \\
&= \sum_{m=1}^{L} \sum_{\alpha\in\mathbb{Z}^d} h^r_{l,m,\alpha} a^{\frac{j+1}{2}} f^n_m(A^{j+1}x - Ak - \alpha) \\
&= a^{\frac{j}{2}} \sum_{m=1}^{L} \sum_{\alpha\in\mathbb{Z}^d} h^r_{l,m,\alpha} a^{\frac{1}{2}} f^n_m \Big(A(A^j x - k) - \alpha\Big) \\
&= a^{\frac{j}{2}} f_l^{an+r}(A^j x - k), \quad \text{by (23).}
\end{aligned}$$

Therefore,

$$\mathcal{H}^r = U_j^{an+r}$$

and

$$U^n_{j+1} = \bigoplus_{r=0}^{a-1} U_j^{an+r}.$$

Using Proposition 4.3 we can get various decompositions of the wavelet subspa W_j, $j \geq 0$, which in turn will give rise to various orthonormal bases of $L^2(\mathbb{R}^d)$.

Theorem 4.4. *Let* $j \geq 0$. *Then, we have*

$$\begin{aligned}
W_j &= \bigoplus_{r=1}^{a-1} U^r_j \\
W_j &= \bigoplus_{r=a}^{a^2-1} U^r_{j-1} \\
&\ \ \vdots \\
W_j &= \bigoplus_{r=a^l}^{a^{l+1}-1} U^r_{j-l}, \quad l \leq j \\
W_j &= \bigoplus_{r=a^j}^{a^{j+1}-1} U^r_0, \qquad (
\end{aligned}$$

where U^n_j *is defined in* (30).

Proof. Since $W_j = \bigoplus_{r=1}^{a-1} U^r_j$, we can apply Proposition 4.3 repeatedly to get (32).

Theorem 4.4 can be used to construct many orthonormal bases of $L^2(\mathbb{R}^d)$. We have following orthogonal decomposition (see (19)):

$$L^2(\mathbb{R}^d) = V_0 \oplus W_0 \oplus W_1 \oplus W_2 \oplus \cdots$$

For each $j \geq 0$, we can choose any of the decompositions of W_j described in (32). For example, if we do not want to decompose any W_j, then we have the usual multiwavelet decomposition. On the other hand, if we prefer the last decomposition in (32) for each W_j, then we get the multiwavelet packet decomposition. There are other decompositions as well. Observe that in (32), the lower index of U_j^n's are decreased by 1 in each successive step. If we keep some of these spaces fixed and choose to decompose others by using (31), then we get decompositions of W_j which do not appear in (32). So there is certain interplay between the indices $n \in \mathbb{N}_0$ and $j \in \mathbb{Z}$.

Let S be a subset of $\mathbb{N}_0 \times \mathbb{Z}$, where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Our aim is to characterize those S for which the collection

$$\mathcal{P}_S = \left\{a^{\frac{j}{2}} f_l^n(A^j \cdot -k) : \ 1 \leq l \leq L, \ \ k \in \mathbb{Z}^d, \ (n, j) \in S\right\}$$

will be an orthonormal basis of $L^2(\mathbb{R}^d)$. In other words, we want to find out those subsets S of $\mathbb{N}_0 \times \mathbb{Z}$ for which

$$\bigoplus_{(n,j)\in S} U_j^n = L^2(\mathbb{R}^d). \tag{33}$$

By using (31) repeatedly, we have

$$\begin{aligned} U_j^n &= \bigoplus_{r=0}^{a-1} U_{j-1}^{an+r} \\ &= \bigoplus_{r=an}^{a(n+1)-1} U_{j-1}^r = \bigoplus_{r=an}^{a(n+1)-1} \left[\bigoplus_{s=0}^{a-1} U_{j-2}^{ar+s}\right] \\ &= \bigoplus_{r=a^2n}^{a^2(n+1)-1} U_{j-2}^r = \cdots = \bigoplus_{r=a^jn}^{a^j(n+1)-1} U_0^r. \end{aligned} \tag{34}$$

Let $I_{n,j} = \{r \in \mathbb{N}_0 : a^jn \leq r \leq a^j(n+1) - 1\}$. Hence,

$$U_j^n = \bigoplus_{r\in I_{n,j}} U_0^r. \tag{35}$$

That is,

$$\bigoplus_{(n,j)\in S} U_j^n = \bigoplus_{(n,j)\in S} \ \bigoplus_{r\in I_{(n,j)}} U_0^r.$$

But we have already proved in Theorem 3.2 that

$$L^2(\mathbb{R}^d) = \bigoplus_{r\in\mathbb{N}_0} U_0^r.$$

Thus, for (33) to be true, it is necessary and sufficient that $\{I_{n,j} : (n, j) \in S\}$ is a partition of $\mathbb{N}_0$. We say $\{A_l : l \in I\}$ is a partition of $\mathbb{N}_0$ if $A_l \subset \mathbb{N}_0$, A_l's are pairwise disjoint, and $\bigcup_{l\in I} A_l = \mathbb{N}_0$. We summarize the above discussion in the following theorem.

Theorem 4.5. *Let $\{f_l^n : n \geq 0, \ \ 1 \leq l \leq L\}$ be the basic multiwavelet packets and $S \subset \mathbb{N}_0 \times \mathbb{Z}$. Then the collection of functions*

$$\left\{a^{\frac{j}{2}} f_l^n(A^j \cdot -k) : \ 1 \leq l \leq L, \ \ k \in \mathbb{Z}^d, \ (n, j) \in S\right\}$$

is an orthonormal basis of $L^2(\mathbb{R}^d)$ if and only if $\{I_{n,j} : (n, j) \in S\}$ is a partition of $\mathbb{N}_0$.

5. Wavelet frame packets

Let $\mathcal{H}$ be a separable Hilbert space. A sequence $\{x_k : k \in \mathbb{Z}\}$ of $\mathcal{H}$ is said to be a frame for $\mathcal{H}$ if there exist constants C_1 and C_2, $0 < C_1 \le C_2 < \infty$ such that for all $x \in \mathcal{H}$,

$$C_1\|x\|^2 \le \sum_{k\in\mathbb{Z}} |\langle x, x_k\rangle|^2 \le C_2\|x\|^2. \tag{36}$$

The largest C_1 and the smallest C_2 for which (36) holds are called the frame bounds.

Suppose that $\Phi = \left\{\varphi^1, \varphi^2, \ldots, \varphi^N\right\} \subset L^2(\mathbb{R}^d)$ such that $\{\varphi^l(\cdot - k) : 1 \le l \le N, k \in \mathbb{Z}^d\}$ is a frame for its closed linear span $S(\Phi)$. Let $\psi^1, \psi^2, \ldots, \psi^N$ be elements in $S(\Phi)$ so that each ψ^j is a linear combination of $\varphi^l(\cdot - k)$; $1 \le l \le L$, $k \in \mathbb{Z}^d$. A natural question to ask is the following: when can we say that $\{\psi^j(\cdot - k) : 1 \le j \le N, k \in \mathbb{Z}^d\}$ is also a frame for $S(\Phi)$?

If $\psi^j \in S(\Phi)$, then there exists $\left\{p_{jlk} : k \in \mathbb{Z}^d\right\}$ in $l^2(\mathbb{Z}^d)$ such that

$$\psi^j(x) = \sum_{l=1}^{N} \sum_{k\in\mathbb{Z}^d} p_{jlk}\varphi^l(x-k).$$

In terms of Fourier transform

$$\begin{aligned}\hat{\psi}^j(\xi) &= \sum_{l=1}^{N} \sum_{k\in\mathbb{Z}^d} p_{jlk} e^{-i\langle k,\xi\rangle}\hat{\varphi}^l(\xi)\\ &= \sum_{l=1}^{N} P_{jl}(\xi)\hat{\varphi}^l(\xi) \quad (1 \le j \le N),\end{aligned} \tag{37}$$

where $P_{jl}(\xi) = \sum_{k\in\mathbb{Z}^d} p_{jlk} e^{-i\langle k,\xi\rangle}$. Let $P(\xi)$ be the $N \times N$ matrix:

$$P(\xi) = \left(P_{jl}(\xi)\right)_{1\le j,l\le N}.$$

Let S and T be two positive definite matrices of order N. We say $S \le T$ if $\langle x, Sx\rangle \le \langle x, Tx\rangle$ for all $x \in \mathbb{R}^N$. The following lemma is the generalization of Lemma 3.1 in [2].

Lemma 5.1. *Let φ^l, ψ^l for $1 \le l \le N$, and $P(\xi)$ be as above. Suppose that there exist constants C_1 and C_2, $0 < C_1 \le C_2 < \infty$ such that*

$$C_1 I \le P^*(\xi)P(\xi) \le C_2 I \quad \text{for a.e. } \xi \in \mathbb{T}^d. \tag{38}$$

Then, for all $f \in L^2(\mathbb{R}^d)$, we have

$$C_1\sum_{l=1}^{N}\sum_{k\in\mathbb{Z}^d}\left|\langle f, \varphi^l(\cdot - k)\rangle\right|^2 \le \sum_{l=1}^{N}\sum_{k\in\mathbb{Z}^d}\left|\langle f, \psi^l(\cdot - k)\rangle\right|^2 \le C_2\sum_{l=1}^{N}\sum_{k\in\mathbb{Z}^d}\left|\langle f, \varphi^l(\cdot - k)\rangle\right|^2. \tag{39}$$

Let A be a dilation matrix, $B = A^t$ and $a = |\det A| = |\det B|$. Let

$$K_A = \{\alpha_0, \alpha_1, \ldots, \alpha_{a-1}\} \tag{40}$$

and

$$K_B = \{\beta_0, \beta_1, \ldots, \beta_{a-1}\} \tag{41}$$

e fixed sets of digits for A and B respectively. For $0 \le r, s \le a-1$ and $1 \le l, j \le L$, efine for a.e. ξ,

$$\mathcal{E}_{lj}^{rs}(\xi) = \delta_{lj} a^{-\frac{1}{2}} \mathrm{e}^{-i\langle \xi + 2B^{-1}\beta_s \pi, \alpha_r \rangle}. \tag{42}$$

et

$$E^{rs}(\xi) = \left(\mathcal{E}_{lj}^{rs}(\xi) \right)_{1 \le l, j \le L} \tag{43}$$

nd

$$E(\xi) = \left(E^{rs}(\xi) \right)_{0 \le r, s \le a-1}. \tag{44}$$

o $E(\xi)$ is block matrix with a blocks in each row and each column, and each block is square matrix of order L, so that $E(\xi)$ is a square matrix of order aL. We have the ollowing lemma which will be useful for the splitting trick for frames.

emma 5.2. (i) *If* $\nu \in K_A$, *then* $\sum\limits_{\mu \in K_B} \mathrm{e}^{-i2\pi\langle B^{-1}\mu, \nu \rangle} = a\delta_{0\nu}$.
i) *The matrix* $E(\xi)$, *defined in* (44), *is unitary.*

roof. Item (i) is the orthogonal relation for the characters of the finite group $\mathbb{Z}^d / B\mathbb{Z}^d$ (see 14]). Observe that the mapping

$$\mu + B\mathbb{Z}^d \mapsto \mathrm{e}^{-i2\pi\langle B^{-1}\mu, \nu \rangle}, \quad \nu \in K_A$$

s a character of the (finite) coset group $\mathbb{Z}^d / B\mathbb{Z}^d$. If $\nu = 0$ (i.e., if $\nu \in A\mathbb{Z}^d$), then here is nothing to prove. Suppose that $\nu \neq 0$, then there exists a $\mu' \in K_B$ such that $^{-i2\pi\langle B^{-1}\mu', \nu \rangle} \neq 1$. Since K_B is a set of digits for B, so is $K_B - \mu'$. Hence,

$$\sum_{\mu \in K_B} \mathrm{e}^{-i2\pi\langle B^{-1}(\mu - \mu'), \nu \rangle} = \sum_{\mu \in K_B} \mathrm{e}^{-i2\pi\langle B^{-1}\mu, \nu \rangle}. \tag{45}$$

low

$$\begin{aligned} \sum_{\mu \in K_B} \mathrm{e}^{-i2\pi\langle B^{-1}\mu, \nu \rangle} &= \mathrm{e}^{-i2\pi\langle B^{-1}\mu', \nu \rangle} \cdot \sum_{\mu \in K_B} \mathrm{e}^{-i2\pi\langle B^{-1}(\mu - \mu'), \nu \rangle} \\ &= \mathrm{e}^{-i2\pi\langle B^{-1}\mu', \nu \rangle} \cdot \sum_{\mu \in K_B} \mathrm{e}^{-i2\pi\langle B^{-1}\mu, \nu \rangle}, \quad \text{by (45)}. \end{aligned}$$

herefore,

$$\sum_{\mu \in K_B} \mathrm{e}^{-i2\pi\langle B^{-1}\mu, \nu \rangle} = 0, \quad \text{since } \mathrm{e}^{-i2\pi\langle B^{-1}\mu', \nu \rangle} \neq 1.$$

o prove (ii), observe that the (r, s)th block of the matrix $E(\xi)E^*(\xi)$ is

$$\sum_{t=0}^{a-1} E^{rt}(\xi) \left(E^{ts}(\xi) \right)^*.$$

he (l, j)th entry in this block is

$$\sum_{t=0}^{a-1} \sum_{m=1}^{L} \mathcal{E}_{lm}^{rt}(\xi) \left(\mathcal{E}_{mj}^{ts}(\xi) \right)^*$$

$$\begin{aligned}
&= \sum_{t=0}^{a-1}\sum_{m=1}^{L} \delta_{lm} a^{-1/2} e^{-i\langle \xi+2B^{-1}\beta_t\pi,\alpha_r\rangle} \cdot \delta_{jm} a^{-1/2} e^{i\langle \xi+2B^{-1}\beta_t\pi,\alpha_s\rangle}\\
&= \sum_{m=1}^{L} \delta_{lm}\delta_{jm} \sum_{t=0}^{a-1} a^{-1} e^{-i\langle \xi+2B^{-1}\beta_t\pi,\alpha_r-\alpha_s\rangle}\\
&= \sum_{m=1}^{L} \delta_{lm}\delta_{jm}\delta_{rs}, \quad \text{(by (i) of the lemma)}\\
&= \delta_{lj}\delta_{rs}.
\end{aligned}$$

This proves that $E(\xi)E^*(\xi) = I$. Similarly, $E^*(\xi)E(\xi) = I$. Therefore, $E(\xi)$ is a unitary matrix. □

6. Splitting lemma for frame packets

Let $\{\varphi_l : 1 \le l \le L\}$ be functions in $L^2(\mathbb{R}^d)$ such that $\{\varphi_l(\cdot - k) : 1 \le l \le L, k \in \mathbb{Z}^d\}$ is a frame for its closed linear span V. For $0 \le r \le a-1$ and $1 \le l \le L$, suppose that there exist sequences $\{h^r_{ljk} : k \in \mathbb{Z}^d\} \in l^2(\mathbb{Z}^d)$. Define f^r_l as in (4) and (5). That is,

$$f^r_l(x) = \sum_{j=1}^{L}\sum_{k\in\mathbb{Z}^d} h^r_{ljk} a^{1/2}\varphi_j(Ax-k). \tag{46}$$

Let $H_r(\xi)$ be the matrix defined in (7). Let K_A and K_B be respectively fixed sets of digits for A and B as in (40) and (41). Let $H(\xi)$ be the matrix

$$H(\xi) = \Big(H_r(\xi + 2B^{-1}\beta_s\pi)\Big)_{0\le r,s\le a-1}. \tag{47}$$

$H(\xi)$ is a block matrix with a blocks in each row and each column, and each block is of order L so that $H(\xi)$ is a square matrix of order aL. Assume that there exist constants C_1 and C_2, $0 < C_1 \le C_2 < \infty$ such that

$$C_1 I \le H^*(\xi)H(\xi) \le C_2 I \quad \text{for a.e. } \xi \in \mathbb{T}^d. \tag{48}$$

We can write f^r_l as

$$\begin{aligned}
f^r_l(x) &= \sum_{j=1}^{L}\sum_{k\in\mathbb{Z}^d} h^r_{ljk} a^{1/2}\varphi_j(Ax-k)\\
&= \sum_{j=1}^{L}\sum_{s=0}^{a-1}\sum_{k\in\mathbb{Z}^d} h^r_{l,j,\alpha_s+Ak} a^{1/2}\varphi_j(Ax-\alpha_s-Ak), \text{ by (2)}\\
&= \sum_{j=1}^{L}\sum_{s=0}^{a-1}\sum_{k\in\mathbb{Z}^d} h^r_{l,j,\alpha_s+Ak}\varphi^{(s)}_j(x-k),
\end{aligned}$$

where

$$\varphi^{(s)}_j(x) = a^{1/2}\varphi_j(Ax-\alpha_s), \qquad 0 \le s \le a-1. \tag{49}$$

Taking Fourier transform, we obtain

$$\begin{aligned}
(f^r_l)^\wedge(\xi) &= \sum_{j=1}^{L}\sum_{s=0}^{a-1}\sum_{k\in\mathbb{Z}^d} h^r_{l,j,\alpha_s+Ak} e^{-i\langle\xi,k\rangle}(\varphi^{(s)}_j)^\wedge(\xi)\\
&= \sum_{j=1}^{L}\sum_{s=0}^{a-1} p^{rs}_{lj}(\xi)(\varphi^{(s)}_j)^\wedge(\xi),
\end{aligned}$$

here $p_{lj}^{rs}(\xi) = \sum_{k\in\mathbb{Z}^d} h^r_{l,j,\alpha_s+Ak}\mathrm{e}^{-i\langle \xi,k\rangle}$. Define

$$P^{rs}(\xi) = \left(p_{lj}^{rs}(\xi)\right)_{1\le l,j\le L} \tag{50}$$

nd

$$P(\xi) = \left(P^{rs}(\xi)\right)_{0\le r,s\le a-1}. \tag{51}$$

laim.

$$H(\xi) = P(B\xi)E(\xi), \tag{52}$$

here $E(\xi)$ is defined in (42)–(44).

roof of the claim. The (r,s)th block of the matrix $P(B\xi)E(\xi)$ is the matrix

$$\sum_{t=0}^{a-1} P^{rt}(B\xi)E^{ts}(\xi).$$

he (l, j)th entry in this block is equal to

$$\begin{aligned}&\sum_{t=0}^{a-1}\sum_{m=1}^{L} p_{lm}^{rt}(B\xi)\mathcal{E}_{mj}^{ts}(\xi)\\ &= \sum_{t=0}^{a-1}\sum_{m=1}^{L}\sum_{k\in\mathbb{Z}^d} h^r_{l,m,\alpha_t+Ak}\mathrm{e}^{-i\langle B\xi,k\rangle}\delta_{mj}a^{-1/2}\mathrm{e}^{-i\langle \xi+2B^{-1}\beta_s\pi,\alpha_t\rangle}\\ &= \sum_{t=0}^{a-1}\sum_{k\in\mathbb{Z}^d} h^r_{l,j,\alpha_t+Ak}\mathrm{e}^{-i\langle B\xi,k\rangle}a^{-1/2}\mathrm{e}^{-i\langle \xi+2B^{-1}\beta_s\pi,\alpha_t\rangle}.\end{aligned}$$

Now, the (l, j)th entry in the (r,s)th block of $H(\xi)$ is

$$\begin{aligned}h^r_{lj}(\xi+2B^{-1}\beta_s\pi) &= a^{-1/2}\sum_{k\in\mathbb{Z}^d} h^r_{ljk}\mathrm{e}^{-i\langle \xi+2B^{-1}\beta_s\pi,k\rangle}\\ &= a^{-1/2}\sum_{t=0}^{a-1}\sum_{k\in\mathbb{Z}^d} h^r_{l,j,\alpha_t+Ak}\mathrm{e}^{-i\langle \xi+2B^{-1}\beta_s\pi,\alpha_t+Ak\rangle}, \quad \text{by (2)}\\ &= a^{-1/2}\sum_{t=0}^{a-1}\sum_{k\in\mathbb{Z}^d} h^r_{l,j,\alpha_t+Ak}\mathrm{e}^{-i\langle \xi+2B^{-1}\beta_s\pi,\alpha_t\rangle}\cdot\mathrm{e}^{-i\langle B\xi,k\rangle}.\end{aligned}$$

So the claim is proved. In particular, we have

$$H^*(\xi)H(\xi) = E^*(\xi)P^*(B\xi)P(B\xi)E(\xi). \tag{53}$$

Since $E(\xi)$ is unitary by Lemma 5.2, $H^*(\xi)H(\xi)$ and $P^*(B\xi)P(B\xi)$ are similar matrices. Let $\lambda(\xi)$ and $\Lambda(\xi)$ respectively be the minimal and maximal eigenvalues of the positive definite matrix $H^*(\xi)H(\xi)$, and let $\lambda = \inf_{\xi}\lambda(\xi)$ and $\Lambda = \sup_{\xi}\Lambda(\xi)$. (It is clear from (52) that $\lambda(\xi)$ and $\Lambda(\xi)$ are $2\pi\mathbb{Z}^d$-periodic functions.) Suppose $0<\lambda\le\Lambda<\infty$. Then we have, by (48) (in the sense of positive definite matrices),

$$\lambda I \le H^*(\xi)H(\xi) \le \Lambda I \quad \text{for a.e. } \xi\in\mathbb{T}^d$$

which is equivalent to

$$\lambda I \leq P^*(\xi)P(\xi) \leq \Lambda I \quad \text{for a.e. } \xi \in \mathbb{T}^d.$$

Then by Lemma 5.1, for all $g \in L^2(\mathbb{R}^d)$, we have

$$\begin{aligned} \lambda \sum_{s=0}^{a-1}\sum_{l=1}^{L}\sum_{k\in\mathbb{Z}^d} \left|\left\langle g, \varphi_l^{(s)}(\cdot - k)\right\rangle\right|^2 &\leq \sum_{s=0}^{a-1}\sum_{l=1}^{L}\sum_{k\in\mathbb{Z}^d} |\langle g, f_l^s(\cdot - k)\rangle|^2 \\ &\leq \Lambda \sum_{s=0}^{a-1}\sum_{l=1}^{L}\sum_{k\in\mathbb{Z}^d} \left|\left\langle g, \varphi_l^{(s)}(\cdot - k)\right\rangle\right|^2, \quad (54) \end{aligned}$$

where $\varphi_l^{(s)}$ is defined in (49). Since

$$\sum_{l=1}^{L}\sum_{k\in\mathbb{Z}^d} |\langle g, a^{1/2}\varphi_l(A\cdot - k)\rangle|^2 = \sum_{s=0}^{a-1}\sum_{l=1}^{L}\sum_{k\in\mathbb{Z}^d} \left|\left\langle g, \varphi_l^{(s)}(\cdot - k)\right\rangle\right|^2, \quad (55)$$

which follows from (49), inequality (54) can be written as

$$\begin{aligned} \lambda \sum_{l=1}^{L}\sum_{k\in\mathbb{Z}^d} |\langle g, a^{1/2}\varphi_l(A\cdot - k)\rangle|^2 &\leq \sum_{s=0}^{a-1}\sum_{l=1}^{L}\sum_{k\in\mathbb{Z}^d} |\langle g, f_l^s(\cdot - k)\rangle|^2 \\ &\leq \Lambda \sum_{l=1}^{L}\sum_{k\in\mathbb{Z}^d} |\langle g, a^{1/2}\varphi_l(A\cdot - k)\rangle|^2. \quad (56) \end{aligned}$$

This is the splitting trick for frames: the $A^{-1}\mathbb{Z}^d$-translates of the L dilated functions $\varphi_l(A\cdot)$, $1 \leq l \leq L$, are 'decomposed' into $\mathbb{Z}^d$-translates of the aL functions f_l^s, $0 \leq s \leq a-1$, $1 \leq l \leq L$.

We now apply the splitting trick to the functions $\{f_l^s : 1 \leq l \leq L\}$ for each s, $0 \leq s \leq a-1$ to obtain

$$\begin{aligned} \lambda \sum_{l=1}^{L}\sum_{k\in\mathbb{Z}^d} |\langle g, a^{1/2}f_l^s(A\cdot - k)\rangle|^2 &\leq \sum_{r=0}^{a-1}\sum_{l=1}^{L}\sum_{k\in\mathbb{Z}^d} |\langle g, f_l^{s,r}(\cdot - k)\rangle|^2 \\ &\leq \Lambda \sum_{l=1}^{L}\sum_{k\in\mathbb{Z}^d} |\langle g, a^{1/2}f_l^s(A\cdot - k)\rangle|^2, \quad (57) \end{aligned}$$

where $f_l^{s,r}$, $0 \leq r \leq a-1$ are defined as in (46) (f_l^s now replaces φ_l):

$$f_l^{s,r}(x) = \sum_{j=1}^{L}\sum_{k\in\mathbb{Z}^d} h_{ljk}^s a^{1/2} f_j^r(Ax - k); \quad 0 \leq s \leq a-1, \ 1 \leq l \leq L. \quad (58)$$

Summing (57) over $0 \leq s \leq a-1$, we have

$$\begin{aligned} \lambda \sum_{s=0}^{a-1}\sum_{l=1}^{L}\sum_{k\in\mathbb{Z}^d} |\langle g, a^{1/2}f_l^s(A\cdot - k)\rangle|^2 &\leq \sum_{s=0}^{a-1}\sum_{r=0}^{a-1}\sum_{l=1}^{L}\sum_{k\in\mathbb{Z}^d} |\langle g, f_l^{s,r}(\cdot - k)\rangle|^2 \\ &\leq \Lambda \sum_{s=0}^{a-1}\sum_{l=1}^{L}\sum_{k\in\mathbb{Z}^d} |\langle g, a^{1/2}f_l^s(A\cdot - k)\rangle|^2. \end{aligned}$$

Using (56), we obtain

$$\begin{aligned} \lambda^2 \sum_{l=1}^{L}\sum_{k\in\mathbb{Z}^d} |\langle g, a^{2/2}\varphi_l(A^2\cdot - k)\rangle|^2 &\leq \sum_{s=0}^{a-1}\sum_{r=0}^{a-1}\sum_{l=1}^{L}\sum_{k\in\mathbb{Z}^d} |\langle g, f_l^{s,r}(\cdot - k)\rangle|^2 \\ &\leq \Lambda^2 \sum_{l=1}^{L}\sum_{k\in\mathbb{Z}^d} |\langle g, a^{2/2}\varphi_l(A^2\cdot - k)\rangle|^2. \quad (59) \end{aligned}$$

Now as in the case of orthonormal wavelet packets, we can define f_l^n, for each $n \geq 0$ and $1 \leq l \leq L$ (see (23) and (27)). In order to ensure that f_l^n are in $L^2(\mathbb{R}^d)$, it is sufficient to assume that all the entries in the matrix $H(\xi)$, defined in (47), are bounded functions. Comparing (58) and (23), we see that

$$\{f_l^{s,r} : 0 \leq r, s \leq a-1\} = \{f_l^{s+ar} : 0 \leq r, s \leq a-1\} = \{f_l^n : 0 \leq n \leq a^2-1\}.$$

So (59) can be written as

$$\begin{aligned}\lambda^2 \sum_{l=1}^{L} \sum_{k \in \mathbb{Z}^d} \left|\langle g, a^{2/2}\varphi_l(A^2 \cdot -k)\rangle\right|^2 &\leq \sum_{n=0}^{a^2-1} \sum_{l=1}^{L} \sum_{k \in \mathbb{Z}^d} \left|\langle g, f_l^n(\cdot - k)\rangle\right|^2 \\ &\leq \Lambda^2 \sum_{l=1}^{L} \sum_{k \in \mathbb{Z}^d} \left|\langle g, a^{2/2}\varphi_l(A^2 \cdot -k)\rangle\right|^2.\end{aligned}$$

By induction, we get for each $j \geq 1$,

$$\begin{aligned}\lambda^j \sum_{l=1}^{L} \sum_{k \in \mathbb{Z}^d} \left|\langle g, a^{j/2}\varphi_l(A^j \cdot -k)\rangle\right|^2 &\leq \sum_{n=0}^{a^j-1} \sum_{l=1}^{L} \sum_{k \in \mathbb{Z}^d} \left|\langle g, f_l^n(\cdot - k)\rangle\right|^2 \\ &\leq \Lambda^j \sum_{l=1}^{L} \sum_{k \in \mathbb{Z}^d} \left|\langle g, a^{j/2}\varphi_l(A^j \cdot -k)\rangle\right|^2. \qquad (60)\end{aligned}$$

We summarize the above discussion in the following theorem.

Note. $\{f_l^n : n \geq 0,\ 1 \leq l \leq L\}$ will be called the wavelet frame packets.

Theorem 6.1. *Let* $\{\varphi_l : 1 \leq l \leq L\} \subset L^2(\mathbb{R}^d)$ *be such that* $\{\varphi_l(\cdot - k) : 1 \leq l \leq L, k \in \mathbb{Z}^d\}$ *is a frame for its closed linear span* V_0, *with frame bounds* C_1 *and* C_2. *Let* $H(\xi)$, $H_r(\xi)$, λ *and* Λ *be as above. Assume that all entries of* $H_r(\xi + 2B^{-1}\beta_s\pi)$ *are bounded measurable functions such that* $0 < \lambda \leq \Lambda < \infty$. *Let* $\{f_l^n : n \geq 0, 1 \leq l \leq L\}$ *be the wavelet frame packets and let* $V_j = \{f : f(A^{-j}\cdot) \in V_0\}$. *Then for all* $j \geq 0$, *the system of functions*

$$\{f_l^n(\cdot - k) : 0 \leq n \leq a^j - 1,\ 1 \leq l \leq L,\ k \in \mathbb{Z}^d\}$$

is a frame of V_j *with frame bounds* $\lambda^j C_1$ *and* $\Lambda^j C_2$.

Proof. Since $\{\varphi_l(\cdot - k) :\ 1 \leq l \leq L,\ k \in \mathbb{Z}^d\}$ is a frame of V_0 with frame bounds C_1 and C_2, it is clear that for all j

$$\{a^{j/2}\varphi_l(A^j \cdot -k) :\ 1 \leq l \leq L,\ k \in \mathbb{Z}^d\}$$

is a frame of V_j with the same bounds. So from (60), we have

$$\lambda^j C_1 \|g\|^2 \leq \sum_{n=0}^{a^j-1} \sum_{l=1}^{L} \sum_{k \in \mathbb{Z}^d} \left|\langle g, f_l^n(\cdot - k)\rangle\right|^2 \leq \Lambda^j C_2 \|g\|^2 \quad \text{for all } g \in V_j. \qquad (61)$$

□

In Theorem 3.2 we proved that the basic multiwavelet packets form an orthonormal basis for $L^2(\mathbb{R}^d) = \overline{\cup V_j}$. An analogous result holds for the wavelet frame packets if the matrix $H(\xi)$, defined in (47), is unitary.

Before proving this result let us observe how the space $\overline{\cup_{j\geq 0}V_j}$ looks like. Let $V_0 = \overline{sp}\{\varphi_l(\cdot - k) : 1 \leq l \leq L,\, k \in \mathbb{Z}^d\}$, $V_j = \{f : f(A^{-j}\cdot) \in V_0\}$ and $V_j \subset V_{j+1}$. Let $W = \cup V_j$. Then it is easy to check that $f \in W \Rightarrow f(\cdot - A^{-j}k) \in W$ for all $j \in \mathbb{Z}$ and $k \in \mathbb{Z}^d$. We claim that elements of the form $A^{-j}k$ are dense in $\mathbb{R}^d$. For $K = \{k_1, k_2, \ldots, k_a\}$ a set of digits for A, define the set

$$Q = Q(A, K) = \left\{x \in \mathbb{R}^d : x = \sum_{j\geq 1} A^{-j}\epsilon_j;\, \epsilon_j \in K\right\}.$$

In the above representation of x, ϵ_j's need not be distinct. We have

$$\|A^{-j}x\| \leq C\alpha^j\|x\|, \quad x \in \mathbb{R}^d,$$

where C is a constant and $0 < \alpha < 1$ (see [17], Chapter 5). Therefore, the series that defines x is convergent. For $x = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d$, $\|x\| = (|x_1|^2 + |x_2|^2 + \cdots + |x_d|^2)^{\frac{1}{2}}$. The set Q satisfies the following properties (see [10]):

(i) $Q = \cup_{i=1}^{a} A^{-1}(Q + k_i)$
(ii) $\cup_{k\in\mathbb{Z}}(Q + k) = \mathbb{R}^d$
(iii) Q is compact.

Let $\epsilon > 0$ and $y \in Q$. We first show that there exist $j \in \mathbb{Z}$ and $k \in \mathbb{Z}^d$ such that $\|y - A^{-j}k\| < \epsilon$. From (i) we have

$$\begin{aligned} Q &= \bigcup_{i=1}^{a} A^{-1}(Q + k_i) \\ &= \bigcup_{i=1}^{a} A^{-1}\left[\bigcup_{m=1}^{a} A^{-1}(Q + k_m) + k_i\right] \\ &= \bigcup_{i=1}^{a}\bigcup_{m=1}^{a}(A^{-2}Q + A^{-2}k_m + A^{-1}k_l). \end{aligned}$$

Hence, for any $j \geq 1$ and any $y \in Q$, there exist $y_j \in Q$ and $l_1, l_2, \ldots, l_j \in K$ such that

$$y = A^{-j}y_j + A^{-j}l_j + A^{-(j-1)}l_{j-1} + \cdots + A^{-1}l_1.$$

Therefore,

$$\begin{aligned} \|y - A^{-j}\{l_j + Al_{j-1} + \cdots + A^{j-1}l_1\}\| &= \|A^{-j}y_j\| \\ &\leq C\alpha^j\|y_j\| \\ &\leq C'\alpha^j \quad \text{(as } Q \text{ is compact)} \\ &< \epsilon, \quad \text{choosing } j \text{ suitably.} \end{aligned}$$

Now if $y \in \mathbb{R}^d$, then by (ii) $y = y_0 + p$ for some $y_0 \in Q$ and $p \in \mathbb{Z}^d$. For $y_0 \in Q$, there exist $j \geq 0$ and $k \in \mathbb{Z}^d$ such that $\|y_0 - A^{-j}k\| < \epsilon$. That is,

$$\|y_0 + p - A^{-j}(k + A^j p)\| < \epsilon$$

$$\Rightarrow \|y - A^{-j}(k + A^j p)\| < \epsilon.$$

So the claim is proved.

We have proved that W is invariant under translations by $A^{-j}k$ and these elements are dense in $\mathbb{R}^d$. Therefore, $\overline{W}$ is a closed translation invariant subspace of $L^2(\mathbb{R}^d)$. Hence, $\overline{W} = L^2_E(\mathbb{R}^d)$ for some $E \subset \mathbb{R}^d$ (see [15]), where

$$L^2_E(\mathbb{R}^d) = \{f \in L^2(\mathbb{R}^d) : \operatorname{supp} \hat{f} \subset E\}.$$

Now let

$$E_0 = \bigcup_{l=1}^{L} \bigcup_{j \ge 0} B^j(\operatorname{supp} \hat{\varphi}_l).$$

Claim. $E = E_0$ a.e.

To prove the claim we will follow [1], Theorem 4.3. Since $\varphi_l(A^j \cdot) \in V_j \subset \overline{W}$, the function $(\varphi_l(A^j \cdot))^\wedge = \frac{1}{a^j}\hat{\varphi}_l(B^{-j}\cdot) \in \widehat{\overline{W}} = \{\hat{f} : f \in \overline{W}\}$. Therefore, $B^j(\operatorname{supp} \hat{\varphi}_l) = \operatorname{supp}\left(\frac{1}{a^j}\hat{\varphi}_l(B^{-j}\cdot)\right) \subset E$ for all $j \ge 0$ and $1 \le l \le L$, which implies that $E_0 \subset E$. Let $E_1 = E \setminus E_0$. We have

$$f \in V_j \Leftrightarrow \hat{f} = \sum_{l=1}^{L} m_l(B^{-j}\xi)\hat{\varphi}_l(B^{-j}\xi), \tag{62}$$

for some $2\pi\mathbb{Z}^d$-periodic functions $m_l \in L^2(\mathbb{T}^d)$. Hence, (62) implies that $\hat{f} = 0$ on E_1 for all $f \in V_j$ and hence, for all $f \in \cup V_j = W$. Taking closure, we obtain that $\hat{f} = 0$ on E_1 for all $f \in \overline{W}$. But $\overline{W}$ is the set of all functions whose Fourier transform is supported in E. Since $E_1 \subset E$, we get that $E_1 = \emptyset$ a.e. Therefore, $E = E_0$ a.e. □

Theorem 6.2. *Let* $\{\varphi_l(\cdot - k) : 1 \le l \le L,\ k \in \mathbb{Z}^d\} \subset L^2(\mathbb{R}^d)$ *be a frame for its closed linear span* V_0, *with frame bounds* C_1 *and* C_2 *and let* $V_0 \subset V_1$, *where* $V_j = \{f : f(A^{-j}\cdot) \in V_0\}$. *Assume that* $H(\xi)$ *is unitary for a.e.* ξ. *Then* $\{f_l^n(\cdot - k) : n \ge 0,\ 1 \le l \le L,\ k \in \mathbb{Z}^d\}$ *is a frame for the space* $\overline{\cup_{j\ge 0} V_j}$ *with the same frame bounds.*

More generally, let $S = \{(n, j) \in \mathbb{N}_0 \times \mathbb{Z}\}$ *be such that* $\bigcup_{(n,j)\in S} I_{n,j}$ *is a partition of* $\mathbb{N}_0$. *Then the collection of functions* $\{a^{j/2} f_l^n(A^j \cdot - k) : 1 \le l \le L,\ (n, j) \in S,\ k \in \mathbb{Z}^d\}$ *is a frame for* $\overline{\cup_{j\ge 0} V_j}$ *with the same bounds* C_1 *and* C_2.

Proof. Since $H(\xi)$ is unitary, $\lambda = \Lambda = 1$ so that the inequalities in (60) are equalities, and from (61) we have

$$C_1\|g\|^2 \le \sum_{n=0}^{a^j-1} \sum_{l=1}^{L} \sum_{k\in\mathbb{Z}^d} \left|\langle g, f_l^n(\cdot - k)\rangle\right|^2 \le C_2\|g\|^2 \quad \text{for all } g \in V_j. \tag{63}$$

Now let $h \in \overline{\cup_{j\ge 0} V_j}$. Then there exists $h_j \in V_j$ such that $h_j \to h$ as $j \to \infty$. Fix j, then for $j < j'$, we have from (63)

$$\sum_{n=0}^{a^j-1} \sum_{l=1}^{L} \sum_{k\in\mathbb{Z}^d} \left|\langle h_{j'}, f_l^n(\cdot - k)\rangle\right|^2 \le C_2\|h_{j'}\|^2.$$

Letting $j' \to \infty$ first and then $j \to \infty$, we have for all $h \in \overline{\cup_{j\ge 0} V_j}$

$$\sum_{n\ge 0} \sum_{l=1}^{L} \sum_{k\in\mathbb{Z}^d} \left|\langle h, f_l^n(\cdot - k)\rangle\right|^2 \le C_2\|h\|^2. \tag{64}$$

To get the reverse inequality we again use (63):

$$\begin{aligned}C_1\|h_j\|^2 &\le \sum_{n=0}^{a^j-1}\sum_{l=1}^{L}\sum_{k\in\mathbb{Z}^d}\left|\langle h_j, f_l^n(\cdot-k)\rangle\right|^2\\ &= \sum_{n=0}^{a^j-1}\sum_{l=1}^{L}\sum_{k\in\mathbb{Z}^d}\left|\langle h_j-h, f_l^n(\cdot-k)\rangle+\langle h, f_l^n(\cdot-k)\rangle\right|^2.\end{aligned}$$

Therefore,

$$\begin{aligned}C_1^{1/2}\|h_j\| &\le \Big(\sum_{n=0}^{a^j-1}\sum_{l=1}^{L}\sum_{k\in\mathbb{Z}^d}\left|\langle h_j-h, f_l^n(\cdot-k)\rangle\right|^2\Big)^{\frac12}\\ &\quad+\Big(\sum_{n=0}^{a^j-1}\sum_{l=1}^{L}\sum_{k\in\mathbb{Z}^d}\left|\langle h, f_l^n(\cdot-k)\rangle\right|^2\Big)^{\frac12}\\ &\le C_2^{1/2}\|h_j-h\|+\Big(\sum_{n=0}^{a^j-1}\sum_{l=1}^{L}\sum_{k\in\mathbb{Z}^d}\left|\langle h, f_l^n(\cdot-k)\rangle\right|^2\Big)^{\frac12}, \qquad \text{by (64).}\end{aligned}$$

Taking $j\to\infty$, we get

$$C_1\|h\|^2\le\sum_{n\ge0}\sum_{l=1}^{L}\sum_{k\in\mathbb{Z}^d}\left|\langle h, f_l^n(\cdot-k)\rangle\right|^2$$

for all $h\in\overline{\cup V_j}$. So the first part is proved.

Now let $U_j^n=\overline{sp}\{a^{j/2}f_l^n(A^j\cdot-k):\ 1\le l\le L,\ \ k\in\mathbb{Z}^d\}$. Then we can prove as in the orthogonal case (see (35)) that

$$U_j^n=\bigoplus_{r\in I_{n,j}}U_0^r,$$

where $\bigoplus$ is just a direct sum not necessarily orthogonal, and $I_{n,j}=\{r\in\mathbb{N}_0: a^jn\le r\le a^j(n+1)-1\}$. Now, since $H(\xi)$ is unitary, we have $\lambda=\Lambda=1$ and hence (57) is an equality. Therefore,

$$\sum_{l=1}^{L}\sum_{k\in\mathbb{Z}^d}\left|\langle g, a^{1/2}f_l^n(A\cdot-k)\rangle\right|^2=\sum_{r=0}^{a-1}\sum_{l=1}^{L}\sum_{k\in\mathbb{Z}^d}\left|\langle g, f_l^{an+r}(\cdot-k)\rangle\right|^2.$$

From this we get

$$\begin{aligned}\sum_{l=1}^{L}\sum_{k\in\mathbb{Z}^d}\left|\langle g, a^{2/2}f_l^n(A^2\cdot-k)\rangle\right|^2 &=\sum_{t=0}^{a-1}\sum_{r=0}^{a-1}\sum_{l=1}^{L}\sum_{k\in\mathbb{Z}^d}\left|\left\langle g, f_l^{a(an+r)+t}(\cdot-k)\right\rangle\right|^2\\ &=\sum_{r=a^2n}^{a^2(n+1)-1}\sum_{l=1}^{L}\sum_{k\in\mathbb{Z}^d}\left|\langle g, f_l^r(\cdot-k)\rangle\right|^2.\end{aligned}$$

Similarly,

$$\begin{aligned}\sum_{l=1}^{L}\sum_{k\in\mathbb{Z}^d}\left|\langle g, a^{j/2}f_l^n(A^j\cdot-k)\rangle\right|^2 &=\sum_{r=a^jn}^{a^j(n+1)-1}\sum_{l=1}^{L}\sum_{k\in\mathbb{Z}^d}\left|\langle g, f_l^r(\cdot-k)\rangle\right|^2\\ &=\sum_{r\in I_{n,j}}\sum_{l=1}^{L}\sum_{k\in\mathbb{Z}^d}\left|\langle g, f_l^r(\cdot-k)\rangle\right|^2. \qquad (65)\end{aligned}$$

From the first part of the theorem, we have for all $f \in \overline{\cup V_j}$

$$C_1\|f\|^2 \leq \sum_{n\geq 0}\sum_{l=1}^{L}\sum_{k\in\mathbb{Z}^d} |\langle f, f_l^n(\cdot - k)\rangle|^2 \leq C_2\|f\|^2.$$

But, the set S is such that $\bigcup_{(n,j)\in S} I_{n,j} = \mathbb{N}_0$. Therefore,

$$C_1\|f\|^2 \leq \sum_{(n,j)\in S}\sum_{r\in I_{n,j}}\sum_{l=1}^{L}\sum_{k\in\mathbb{Z}^d} |\langle f, f_l^r(\cdot - k)\rangle|^2 \leq C_2\|f\|^2.$$

Using (65), we get

$$C_1\|f\|^2 \leq \sum_{(n,j)\in S}\sum_{l=1}^{L}\sum_{k\in\mathbb{Z}^d} |\langle f, a^{j/2} f_l^n(A^j\cdot - k)\rangle|^2 \leq C_2\|f\|^2$$

for all $f \in \overline{\cup V_j}$. This completes the proof of the theorem. □

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A variational principle for vector equilibrium problems

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Abstract. A variational principle is described and analysed for the solutions of vector equilibrium problems.

Keywords. Vector equlibrium problem; variational principle; P-convexity; $P-\psi$-monotonicity.

1. Introduction

Throughout this paper, X is a real topological vector space; $K \subset X$ be a nonempty, closed and convex set; (Y, P) be a real ordered topological vector space with a partial order $\leq_P$ induced by a solid, pointed, closed and convex cone P with apex at origin, thus

$$x \leq_P y \iff y - x \in P \quad \forall x, y \in Y.$$

If int$P$ denotes the topological interior of the cone P, then weak ordering, say $\not\leq_{\mathrm{int}P}$ (or $\not\geq_{\mathrm{int}P}$), on Y is defined by

$$x \not\leq_{\mathrm{int}P} y \text{ or } y \not\geq_{\mathrm{int}P} x \iff y - x \notin \mathrm{int}P \quad \forall x, y \in Y.$$

Let $f : X \times X \longrightarrow Y$ be a mapping with $f(x, x) = 0 \ \forall x \in X$, then vector equilibrium problem (for short, VEP(f, K)) is to find $x \in K$ such that

$$f(x, y) \not\leq_{\mathrm{int}P} 0, \quad \forall x, y \in K.$$

VEP(f, K) has been studied by Kazmi [K2]. VEP(f, K) includes as special cases, vector optimization problems, vector variational inequalities, vector variational-like inequalities, vector complementarity problems, etc., see Kazmi [K2] and the references therein.

If $Y = R$, $P = R_+$, then VEP(f, K) reduces to the scalar equilibrium problem [B-O1, B-O2] of finding $x \in K$ such that

$$f(x, y) \geq 0, \quad \forall y \in K.$$

In this paper, we shall describe and analyse a variational principle for the solutions of VEP(f, K).

The construction of variational principles is of interest both theoretically and in practice. Conceptually, it is of singnificance to know that there is a mapping defined on X which is optimized precisely at the solutions of VEP(f, K). In practice, it is of importance because it allows one to use the highly developed theory of numerical optimization to numerically approximate, and compute solutions of these problems.

More precisely, following the terminology of Auchmuty [A], we say that a variational principle holds for VEP(f, K), if there exists a mapping $F : K \longrightarrow Y$ depending on the data of VEP(f, K) but not on its solution set, such that the solution set of VEP(f, K) coincides with the solution set of the vector maximization problem (for short, VMP(f, K))

$$\max_{\mathrm{int}P} F(x), \quad \text{subject to } x \in K.$$

If $f(x, y) = \langle \phi'(x), \eta(y, x) \rangle$, where $\eta : K \times K \longrightarrow X$ is a continuous function and $\phi : K \longrightarrow Y$ is Fréchet (or linear Gateaux) differentiable and P-convex mapping, the x is a solution of VEP(f, K) if and only if x is a solution of

$$\min_{\mathrm{int}P} \phi(x), \quad \text{subject to } x \in K,$$

see Kazmi [K1]. For related work, see [K3, K-A].

Thus, setting $F = -\phi$, a variational principle for VEP(f, K) holds.

Now, consider the case:

$$f(x, y) := g(x, y) + h(x, y),$$

where $g, h : K \times K \longrightarrow Y$ are nonlinear mappings, then VEP(f, K) becomes:

(VEP($g + h$, K)), find $x \in K$ such that $g(x, y) + h(x, y) \not\leq_{\mathrm{int}P} 0, \quad \forall y \in K$.

We shall use the following concepts and result:

The mapping g is called P-monotone if and only if

$$g(x, y) \leq_P -g(y, x), \quad \forall x, y \in K.$$

A mapping $T : X \longrightarrow Y$ is called P-convex if and only if for each pair $x, y \in K$ and $\lambda \in [0, 1]$,

$$T(\lambda x + (1 - \lambda) y) \leq_P \lambda T(x) + (1 - \lambda) T(y).$$

Note that if $g(x, y) = \langle \phi'(x), y - x \rangle$ where $\phi : X \longrightarrow Y$ be P-convex and linear Gateaux differentiable, then g is P-monotone since $\phi'(\cdot)$ is P-monotone.

Lemma 1 [C]. *Let (Y, P) be an ordered topological vector space with a solid, pointed, closed and convex cone P. Then* $\forall x, y \in X$, *we have*

$$y \leq_P x \text{ and } y \not\leq_{\mathrm{int}P} 0 \text{ imply } x \not\leq_{\mathrm{int}P} 0.$$

Finally, in order to formulate our variational principle we introduce a perturbation mapping $\psi(\cdot, \cdot) : K \times K \longrightarrow Y$ which satisfies for all $x, y \in K$:

(i) $0 \leq_P \psi(x, y)$,

(ii) $\psi(x, x) = 0$,

(iii) $\psi(x, \lambda y + (1 - \lambda) x) = o(\lambda), \quad \lambda \in [0, 1]$.

Let us indicate some possible choices for $\psi(\cdot, \cdot)$: $K \times K \longrightarrow Y$ satisfying properties (i)–(iii) above. Clearly, the choice $\psi(\cdot, \cdot) = 0$ is always possible. Next let $\phi(\cdot, \cdot)$: $K \times K \longrightarrow Y$ be P-convex in the second argument, and $\forall x \in K$, let $\phi(\cdot, \cdot)$ be Gateaux differentiable at x with Gateaux differential $\phi'(x, \cdot) \in L(X, Y)$ where $L(X, Y)$ is a space of all linear bounded functionals from X to Y. Set

$$\psi(x, y) = \phi(x, y) - \phi(x, x) - \langle \phi'(x, x), y - x \rangle,$$

n $\psi(\cdot,\cdot)$ satisfies properties (i)–(iii). In particular if $\phi(\cdot)\colon K \longrightarrow Y$ be P-convex, and teaux differentiable, then we may choose

$$\psi(x,y)=\phi(y)-\phi(x)-\langle\phi'(x),y-x\rangle.$$

ally, if $\psi(\cdot,\cdot)\colon K\times K \longrightarrow R\bigcup\{\infty\}$, where K is a subset of normed linear space X, n we may choose $\psi(x,y)=\alpha\|y-x\|^2$, for $\alpha>0$, which satisfies (i)–(iii).

Now, we define a mapping $G: K \longrightarrow Y$ by means of

$$G(x):=\inf\{-g(y,x)+h(x,y)+\psi(x,y): y\in K\}, \tag{1}$$

l we associate to VEP$(g+h,\,K)$ the following vector maximization problem:

$$\mathrm{VMP}(g+h,\psi,K):\ \max_{\mathrm{int}P}\{G(x): x\in K\}.$$

remark that the mapping $G(\cdot)$ generalizes the gap function used in connection with iational inequalities, see Herker and Pang [H-P], and the references therein.

From $-g(x,x)+h(x,x)+\psi(x,x)=0$ follows

$$G(x)\le_P 0 \quad \forall x\in K. \tag{2}$$

also define the following concept.

Let ψ satisfy (i)–(iii), the mapping g is called $P-\psi$-monotone if and only if

$$g(x,y)\le_P \psi(x,y)-g(y,x), \quad \forall x,y\in K.$$

$\psi(x,y)=0$, $\forall x,y\in K$, then $P-\psi$-monotone mapping becomes P-monotone.

Results

st we prove the following results:

eorem 2. *Let the following assumptions hold:*

The mapping g satisfies: $g(x,x)=0$, $\forall x\in K$; g is P-monotone; $\forall x,y\in K$, the mapping $\lambda\in[0,1]\longrightarrow g(\lambda y+(1-\lambda)x,y)$ is continuous at 0_+; g is P-convex in the second argument.

The mapping h satisfies: $h(x,x)=0$, $\forall x\in K$; h is P-convex in the second argument.

en VEP$(g+h,\,K)$ *and the problem of finding $x\in K$ such that*

$$G(x)\not\le_{\mathrm{int}P} 0, \tag{3}$$

h have the same solution set.

oof. Let x be a solution of VEP$(g+h,\,K)$, that is,

$$g(x,y)+h(x,y)\not\le_{\mathrm{int}P} 0, \quad \forall y\in K. \tag{4}$$

ce g is P-monotone, $\forall x,y\in K$, we have

$$(g(x,y)+h(x,y))-(-g(y,x)+h(x,y))\le_P 0. \tag{5}$$

om Lemma 1, eqs (4) and (5), it follows that

$$-g(y,x)+h(x,y)\not\le_{\mathrm{int}P} 0, \quad \forall y\in K$$

or,

$$-g(y,x)+h(x,y)\in W:=Y\setminus(-\mathrm{int}P),\quad \forall y\in K. \tag{6}$$

Since $\psi(x,y)\in P$, we have

$$-g(y,x)+h(x,y)+\psi(x,y)\in W+P\subset W,\quad \forall y\in K,$$

which implies

$$\inf\{-g(y,x)+h(x,y)+\psi(x,y)\ :\ y\in K\}\not\leq_{\mathrm{int}P} 0,$$

that is,

$$G(x)\not\leq_{\mathrm{int}P} 0.$$

Conversely, let x be a solution of problem (3). Then by the definition of $G(\cdot)$, we have

$$-g(y,x)+h(x,y)+\psi(x,y)\not\leq_{\mathrm{int}P} 0,\quad \forall y\in K.$$

Fix $y\in K$ arbitrarily, let $x_\lambda:=\lambda y+(1-\lambda)x,\ \lambda\in]0,1],\ x_\lambda\in K$ as K is convex, and hence the above inequality becomes

$$-g(x_\lambda,x)+h(x,x_\lambda)+\psi(x,x_\lambda)\not\leq_{\mathrm{int}P} 0. \tag{7}$$

Since g is P-convex in the second argument, we have

$$\begin{aligned} 0 &= g(x_\lambda,x_\lambda)\\ &\leq_P \lambda g(x_\lambda,y)+(1-\lambda)g(x_\lambda,x)\\ -(1-\lambda)g(x_\lambda,x) &\leq_P \lambda g(x_\lambda,y). \end{aligned}$$

By using preceding inequality and the properties of cone P, we have

$$\begin{aligned} &(1-\lambda)(-g(x_\lambda,x)+h(x,x_\lambda)+\psi(x,x_\lambda))\\ &\quad\leq_P \lambda g(x_\lambda,y)+(1-\lambda)(h(x,x_\lambda)+\psi(x,x_\lambda))\\ &\quad\leq_P \lambda g(x_\lambda,y)+(1-\lambda)(h(x,y)+o(\lambda)), \end{aligned}$$

using the properties of h and ψ.

Since $(1-\lambda)>0$, after dividing the preceding inequality by $(1-\lambda)>0$, we have, from Lemma 1, (7) and the resultant inequality,

$$\frac{\lambda}{(1-\lambda)}g(x_\lambda,y)+\lambda h(x,y)+o(\lambda)\not\leq_{\mathrm{int}P} 0\in W.$$

After dividing the preceding inclusion by $\lambda>0$, letting $\lambda\downarrow 0$ and hence $x_\lambda\longrightarrow x\in K$, and then by hemicontinuity of g and closedness of W, we have

$$g(x,y)+h(x,y)\not\leq_{\mathrm{int}P} 0,\quad \forall y\in K.$$

This completes the proof.

Theorem 3. *Let all the assumptions of Theorem 2 except P-monotonicity of g hold. Let g be $P-\psi$-monotone, then* VEP($g+h$, K) *and problem* (3) *both have the same solution set.*

Proof. Let x be a solution of VEP($g+h$, K), that is,

$$g(x,y)+h(x,y)\not\leq_{\text{int}P} 0, \quad \forall y \in K. \tag{8}$$

Since g is $P-\psi$-monotone, $\forall x, y \in K$, we have

$$(g(x,y)+h(x,y)) - (-g(y,x)+h(x,y)+\psi(x,y)) \leq_P 0. \tag{9}$$

From Lemma 1, (8) and (9), it follows that

$$-g(y,x)+h(x,y)+\psi(x,y)\not\leq_{\text{int}P} 0, \quad \forall y \in K,$$

which implies

$$\inf\{-g(y,x)+h(x,y)+\psi(x,y) \ : \ y \in K\}\not\leq_{\text{int}P} 0.$$

Converse part of theorem is just same as the converse part of Theorem 2. This completes the proof.

Now, on combining Theorem 2 (or Theorem 3) with inequality (2), we have the following variational principle for VEP($g+h$, K).

Theorem 4. *Let the assumptions of Theorem* 2 *and inequality* (2) *hold.* x *is a solution of* VEP($g+h$, K) *if and only if* $G(x)=0$. *If the solution set of* VEP($g+h$, K) *is nonempty, then the solution sets of* VEP($g+h$, K) *and* VMP($g+h$, ψ, K) *coincide.*

Proof. If x is a solution of VEP($g+h$, K) then, by Theorem 2,

$$G(x)\not\leq_{\text{int}P} 0.$$

From (2),

$$G(x) \leq_P 0.$$

These above inequalities imply that $G(x)=0$. Next, if $G(x)=0$ then by definition of $G(\cdot)$, we have

$$0 \not\geq_{\text{int}P} 0 \leq_P -g(y,x)+h(x,y)+\psi(x,y), \quad \forall y \in K.$$

By Lemma 1, it follows that

$$-g(y,x)+h(x,y)+\psi(x,y)\not\leq_{\text{int}P} 0 \quad \forall y \in K.$$

Follow the same lines of converse part of Theorem 2, we can have that x is a solution of VEP($g+h$, K). This proves the first part of the theorem. If x is a solution of VEP($g+h$, K), then $G(x)=0$, and from inequality (2) follows that x is a solution of VMP($g+h$, ψ, K). Then all solutions of VMP($g+h$, ψ, K) must satisfy $G(x)=0$, and therefore are in the solution set of VEP($g+h$, K). This completes the proof.

We remark that the variational principle described in this paper is a generalization of variational principles described by Blum and Oettli [B-O1, B-O2], and Auchmuty [A].

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On a generalized Hankel type convolution of generalized functions

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Abstract. The classical generalized Hankel type convolution are defined and extended to a class of generalized functions. Algebraic properties of the convolution are explained and the existence and significance of an identity element are discussed.

Keywords. Generalized Hankel type transformation; Parserval relation; generalized functions (distributions); convolution.

1. Introduction

The fact that there is no simple expression for the product $J_\mu[x^\nu]J_\mu[y^\nu]$ in the sense that there is a simple expression $e^{i(\nu x+y)}$ for the product $e^{i\nu x}e^{iy}$ means that there is no simple Faltung or convolution theorem for the Hankel transform corresponding to well known transforms like Laplace, Fourier transform and so on.

Hankel-convolution operation has been defined in the classical sense by [4] and [2]. We consider here the generalized Hankel type convolution and an extension of that definition to a class of generalized functions analogous to that introduced by [11, 5] and [6]. This extension has useful applications, when dealing with continuous linear systems which can be characterized by a Hankel convolutional representation; such systems, which we may call 'generalized Hankel translation invariant continuous linear systems', may thereafter be considered when developing sampling expansions for inverse generalized Hankel type transforms of distributions of compact support on the positive half line of which the work of [8] is a particular case.

2. Notation and preliminary results

We use the following definition for the classical generalized Hankel type transform of order $\mu \geq -1/2$.

$$(h_{\mu,\nu}f)(\tau) = F(\tau) = \nu\tau^{-1}\int_0^\infty (x\tau)^\nu J_\mu[(x\tau)^\nu]f(x)\mathrm{d}x, \tag{1}$$

$$f(x) = h_{\mu,\nu}^{-1}[F](x) = \nu x^{-1}\int_0^\infty (x\tau)^\nu J_\mu[(x\tau)^\nu]F(\tau)\mathrm{d}\tau. \tag{2}$$

The transform pair (1) and (2) has been extended to certain spaces of generalized functions in [6] by kernel method and in [5] by mixed Parseval equation (a new adjoint method).

We begin with a brief review of the essential results obtained by [5] for the generalized Hankel type transform of generalized functions.

Lemma 2.1. *If $f(x)$ is of bounded variation and $x^{\nu/2}f(x) \in L^1(0,\infty)$ then the direct transform is well defined by* (1), *and the inversion formula* (2) *holds almost everywhere in a neighbourhood of every point $y = x > 0$.*

Lemma 2.2. *For $f(x)$ and $G(x)$ satisfying the conditions of Lemma* 2.1 *we have the Parseval relation*

$$\int_0^\infty xf(x)g(x)\mathrm{d}x = \int_0^\infty \tau F(\tau)G(\tau)\mathrm{d}\tau. \tag{3}$$

Finally we shall need results involving the linear differential operator $N_{\mu,\nu}$, $\mu \geq -1/2$, defined by

$$N_{\mu,\nu}[f(x)] = x^{\nu\mu}Dx^{-\nu\mu-\nu+1}f(x) \tag{4}$$

and the Bessel type differential operator of order μ, Δ, defined by

$$\Delta[f(x)] = x^{\nu-1}\Delta_1 x^{-\nu+1} = x^{-\nu-\nu\mu}Dx^{2\nu\mu+1}Dx^{-\nu\mu-\nu+1}, \tag{5}$$

where $\Delta_1 = x^{-\nu\mu-2\nu+1}Dx^{2\nu\mu+1}Dx^{-\nu\mu} = \Delta_{1,x}$ and D stands for the usual differential operator.

PROPOSITION 2.3

If $xf(x) \to 0$ as $x \to \infty$ where $f(x)$ is sufficiently smooth $h_{\mu,\nu}$-transformable function, then integration by parts shows that

$$h_{\mu+1,\nu}N_{\mu,\nu}[f](\tau) = -\nu\tau^{\nu}h_{\mu,\nu}[f](\tau) \tag{6}$$

or, setting $g = h_{\mu,\nu}[f]$ and changing τ into x,

$$\nu h_{\mu+1,\nu}[-x^{\nu}g(x)](\tau) = N_{\mu,\nu}h_{\mu,\nu}[g](\tau). \tag{7}$$

PROPOSITION 2.4

In general, for sufficiently well behaved $\phi(x)$ and non-negative i, j we can obtain from (6) *and* (7)

$$h_{\mu+i+j,\nu}N_{\mu+i+j,\nu}\ldots N_{\mu+i,\nu}[(-\nu x^{\nu})^{i}](\tau) = (-\nu\tau^{\nu})^{j}N_{\mu+i,\nu}\ldots N_{\mu,\nu}[\Phi(\tau)] \tag{8}$$

or taking the defining formula (4) *into consideration,*

$$h_{\mu+i+j,\nu}[x^{\nu(\mu+i+j)}(x^{1-2\nu}D)^{j}\phi(x)](\tau) = (-\nu\tau^{\nu})^{i+j}(\tau^{1-2\nu}D_{\tau})^{i}\Phi(\tau)]. \tag{9}$$

PROPOSITION 2.5

For any sufficiently smooth function $f(x)$ *on* $(0, \infty)$ *it can be shown that*

$$h_{\mu,\nu}[\Delta f(x)](\tau) = -\nu^2\tau^{2\nu}h_{\mu,\nu}[f](\tau) \tag{10}$$

provided that f *is* $h_{\mu,\nu}$*-transformable and that* $xf(x)$ *and* $xN_{\mu,\nu}f(x)$ *both tend to zero as* $x \to \infty$.

3. Spaces of fundamental and generalized functions

3.1 *Testing function spaces*

A complex valued function ϕ, defined and infinitely differentiable on $(0, \infty)$, is said to belong to the space $H_{\mu,\nu}(I)$ if and only if the numbers $\gamma_{m,k}^{\mu,\nu}(\phi)$ defined by

$$\gamma_{m,k}^{\mu,\nu}(\phi) = \sup_{0<x<\infty} \left| x^m (x^{1-2\nu}D)^k x^{-\nu\mu-\nu+1}\phi \right| \tag{11}$$

are finite for every pair m, k of non-negative integers where ν is real number and $\mu \geq -1/2$. $H_{\mu,\nu}(I)$ is a testing function space with the topology generated by the multinorm $\{\gamma_{m,k}^{\mu,\nu}(\phi)\}_{m,k=0}^{\infty}$ and we have

$$D(0,\infty) \subset H_{\mu,\nu}(0,\infty) \subset E(0,\infty), \tag{12}$$

where $D(0,\infty)$ and $E(0,\infty)$ denote respectively the restrictions of $D(R)$ and $E(R)$ to the positive real axis. Using (9) and following the same lines of [5], it can be readily shown that the $h_{\mu,\nu}$ transformation is a topological isomorphism of $H_{\mu,\nu}(0,\infty)$ onto itself.

3.2 *The space* $M(0, \infty)$ *of multipliers*

Denote by $M(0,\infty)$ the linear space of all infinitely smooth functions $\theta(x)$, $0 < x < \infty$ such that for each non-negative integer l there exists non-negative integer $l = l(i)$ for which

$$(1 + x^l)^{-1}(x^{1-2\nu}D)^i\theta(x) \tag{13}$$

is bounded on $(0,\infty)$. By using the generalized Leibnitz formula it can be shown that the map $\theta \to \theta\phi$ is an isomorphism of $H_{\mu,\nu}(0,\infty)$ for each $\theta \in M(0,\infty)$; $M(0,\infty)$ is the space of multipliers on $H_{\mu,\nu}(0,\infty)$.

3.3 *Duals of testing function spaces*

We denote $H_{\mu,\nu}^*(0,\infty)$ the space of all complex valued functions ψ, defined and infinitely smooth on $(0,\infty)$ which are of the form

$$\psi(x) = x\phi(x). \tag{14}$$

$H_{\mu,\nu}^*(0,\infty)$ is again a (complete) testing function space, with the topology generated by the sequence of multinorms

$$\gamma_{m,k}^{*,\mu,\nu}(\psi) = \gamma_{m,k}^{\mu,\nu}(x^{-1}\psi). \tag{15}$$

As usual we denote the dual of $H_{\mu,\nu}^*(0,\infty)$ by $H_{\mu,\nu}^{*\prime}(0,\infty)$.

For any $\psi(x) = x\phi(x) \in H^*_{\mu,\nu}(0,\infty)$, and any non-negative integer r, set

$$\varsigma^*_r(\psi) = \max_{0 \le m,k \le r} \gamma^{*,\mu,\nu}_{m,k}(\psi) = \max_{0 \le m,k \le r} \gamma^{\mu,\nu}_{m,k}(\phi) = \varsigma_r(\psi).$$

Then, for each $f \in H^*_{\mu,\nu}(0,\infty)$ there will exist constants c and r such that

$$\phi \in H_{\mu,\nu}(0,\infty) \Rightarrow |\langle f(x), x\phi(x)\rangle| \le C\varsigma_r(\phi). \tag{16}$$

In particular, let $f(x)$ be any locally integrable function on $(0,\infty)$ which is such that $xf(x) \in L^1(0,\infty)$ and $f(x)$ does not grow more rapidly than a polynomial when $x \to \infty$. Then $f(x)$ generates a regular generalized function in $H^*_{\mu,\nu}(0,\infty)$ by the formula

$$\langle f(x), x\phi(x)\rangle = \int_0^\infty xf(x)\phi(x)\mathrm{d}x. \tag{17}$$

Any generalized function in $H^*_{\mu,\nu}(0,\infty)$ not generated by the formula of the type (17) will be described as singular.

In general, the derivative of a generalized function in $H^*_{\mu,\nu}(0,\infty)$ (defined in the usual sense of Schwartz), is not in $H^*_{\mu,\nu}(0,\infty)$. However, in certain cases the result of applying a differential operator to a generalized function in $H^*_{\mu,\nu}(0,\infty)$ does yield a generalized function in $H^*_{\mu,\nu}(0,\infty)$. In particular, using for differential operators in a generalized sense the same notation as the one used for the corresponding operators which applied in a classical sense, we have the following results:

(i) $f \in E'(0,\infty) \subset H^{*\prime}_{\mu,\nu}(0,\infty) \Rightarrow Df \in H^{*\prime}_{\mu,\nu}(0,\infty)$;

(ii) $f \in H^{*\prime}_{\mu,\nu}(0,\infty) \Rightarrow (x^{1-2\nu}D)^k f \in H^{*\prime}_{\mu,\nu}(0,\infty)$;

(iii) $f \in H^{*\prime}_{\mu,\nu}(0,\infty) \Rightarrow \Delta^k f \in H^{*\prime}_{\mu,\nu}(0,\infty)$; for any non-negative integer k.

3.4 *Distributional generalized $h_{\mu,\nu}$-transform*

We can now define the generalized $h_{\mu,\nu}$-transform of any $f \in H^{*\prime}_{\mu,\nu}(0,\infty)$ by the analogue of the Parseval relation:

$$\langle f(x), x\phi(x)\rangle = \left\langle h_{\mu,\nu}[f](\tau), \tau\Phi(\tau)\right\rangle \tag{18}$$

and clearly we have that $f \in H^{*\prime}_{\mu,\nu}(0,\infty) \Rightarrow h_{\mu,\nu}[f] \in H^{*\prime}_{\mu,\nu}(0,\infty)$. Moreover, we can establish that

$$h_{\mu,\nu}[\Delta^k f(x)](\tau) = (-\nu^2\tau^{2\nu})^k h_{\mu,\nu}[f](\tau) \tag{19}$$

for any non-negative integer k.

The generalized $h_{\mu,\nu}$-transform of any distribution $\sigma \in E'(0,\infty)$, in the sense of (18), is a regular generalized function in $H^{*\prime}_{\mu,\nu}(0,\infty)$ generated by a smooth function $f(x)$ defined on $(0,\infty)$ by

$$f(x) = \left\langle \sigma(\tau), \tau^\nu J_\mu[(x\tau)^\nu]\right\rangle = \left\langle \sigma(\tau), \tau^\nu \Lambda(\tau) J_\mu[(x\tau)^\nu]\right\rangle, \tag{20}$$

where $\Lambda \in D(0,\infty)$ is such that $\Lambda(\tau) = 1$ on the support of σ. The function extend into the finite complex-plane as an entire function of exponential type which grows no faster than a polynomial on the positive real axis; it is easy to show that $f(x) \in M(0,\infty)$.

lassical generalized Hankel type convolution

INITION 4.1

us define $L^{p}_{\mu,\nu}[0,\infty)$, $1 \le p < \infty$, the space of Lebesgue measurable functions on o) such that

$$\left\|f\right\|_{\mu,\nu,p} = \left[\int_{0}^{\infty} x^{\mu+2\nu-1}\left|f(x)\right|^{p}\right]^{1/p} < \infty.$$

consider the kernel $D_{\mu,\nu}(x,\ y,\ z)$, $0 < x,\ y,\ z < \infty$ defined by

$$D_{\mu,\nu}(x,\ y,\ z) = \int_{0}^{\infty} \nu^{2}\tau^{2\nu} J_{\mu}[(x\tau)^{\nu}]J_{\mu}[(y\tau)^{\nu}]J_{\mu}[(z\tau)^{\nu}]\mathrm{d}\tau. \tag{21}$$

Properties of the kernel $D_{\mu,\nu}(x,\ y,\ z)$

owing Watson [10], Hirschmann [4] and Cholewinski [2] we can establish the following erties for (21):

or $0 < x,\ y < \infty$ and $0 \le \tau < \infty$, we have

$$\int_{0}^{\infty} z^{2\nu-1} J_{\mu}[(z\tau)^{\nu}]D_{\mu,\nu}(x,\ y,\ z)\mathrm{d}z = J_{\mu}[(x\tau)^{\nu}]J_{\mu}[(y\tau)^{\nu}]. \tag{22}$$

of.

$$\begin{aligned} D_{\mu,\nu}(x,\ y,\ z) &= \int_{0}^{\infty} \nu^{2}\tau^{2\nu} J_{\mu}[(x\tau)^{\nu}]J_{\mu}[(y\tau)^{\nu}]J_{\mu}[(z\tau)^{\nu}]\mathrm{d}z \\ &= \nu z^{1-\nu}\left[\nu z^{-1}\int_{0}^{\infty}(z\tau)^{\nu}[\tau^{-1+\nu}J_{\mu}[(x\tau)^{\nu}]J_{\mu}[(y\tau)^{\nu}]]J_{\mu}[(z\tau)^{\nu}\right]\mathrm{d}z \\ &= \nu z^{-1-\nu}h^{-1}_{\mu,\nu}\left[\tau^{-1+\nu}J_{\mu}[(x\tau)^{\nu}]J_{\mu}[(y\tau)^{\nu}]\right]. \end{aligned}$$

refore,

$$h^{-1}_{\mu,\nu}[(\nu)^{-1}z^{-1+\nu}D_{\mu,\nu}(x,\ y,\ z)] = \tau^{-1+\nu}J_{\mu}[(x\tau)^{\nu}]J_{\mu}[(y\tau)^{\nu}],$$

$$\nu\tau^{-1}\int_{0}^{\infty}(z\tau)^{\nu}J_{\mu}[(z\tau)^{\nu}](\nu)^{-1}z^{-1+\nu}D_{\mu,\nu}(x,\ y,\ z)\mathrm{d}z = \tau^{-1+\nu}J_{\mu}[(x\tau)^{\nu}]J_{\mu}[(y\tau)^{\nu}]$$

hence the result. In particular, taking $\tau = 0$, gives

$$\int_{0}^{\infty} z^{2\nu-1}D_{\mu,\nu}(x,\ y,\ z)\mathrm{d}z = 1, \tag{23}$$

that is, for which $x, y > 0$, $D_{\mu,\nu}(x, y, z)$ belongs to $L^1_{0,\nu}(0, \infty)$.
(iii) $0 < x, y, z < \infty$, $D_{\mu,\nu}(x, y, z) \geq 0$, and
(iv) $D_{\mu,\nu}(x, y, z) = D_{\mu,\nu}(z, x, y) = D_{\mu,\nu}(y, z, x) = \ldots.$

DEFINITION 4.3

We define the classical $h_{\mu,\nu}$-convolution, for any two function $f(x)$ and $g(x)$, $0 < x < \infty$ as

$$f * g(x) = \nu \int_0^\infty \int_0^\infty (yz)^\nu f(y) g(z) D_{\mu,\nu}(x, y, z) \mathrm{d}y \mathrm{d}z \tag{24}$$

whenever the integral exists. We observe the following properties:
(i) *Commutativity*: For any $x \in I$, $(f * g)(x) = (g * f)(x)$. Proof is obvious from the relation (24).
(ii) *Associativity*: For any $t \in I$, $(f * g) * h(t) = f * (g * h)(t)$,

$$(f * g) * h = \left[\nu \int_0^\infty \int_0^\infty (yz)^\nu f(y) g(z) D_{\mu,\nu}(x, y, z) \mathrm{d}y \mathrm{d}z\right] * h(t)$$

$$= \nu^2 \int_0^\infty \int_0^\infty (xs)^\nu \left[\int_0^\infty \int_0^\infty (yz)^\nu f(y) g(z) D_{\mu,\nu}(x, y, z) \mathrm{d}y \mathrm{d}z\right] h(s) D_{\mu,\nu}(x, s, t) \mathrm{d}x \mathrm{d}s.$$

Since the integral exists due to equation (24), by changing the order of integration and $D_{\mu,\nu}(x, y, z) D_{\mu,\nu}(x, s, t) = D_{\mu,\nu}(z, s, x) D_{\mu,\nu}(x, y, t)$ we have the result. While f and g are such that both $h_{\mu,\nu}(f)$ and $h_{\mu,\nu}(g)$ exists, we have the convolution product properties.
(iii)

$$h_{\mu,\nu}[x^{1+\nu}(f * g)(x)](\tau) = \tau^{1-\nu} h_{\mu,\nu}(f) h_{\mu,\nu}(g). \tag{25}$$

Proof.

$$\text{LHS} = \nu \tau^{-1} \int_0^\infty (x\tau)^\nu J_\mu[(x\tau)^\nu] x^{-1+\nu} (f * g)(x) \mathrm{d}x$$

$$= \nu \tau^{-1+\nu} \int_0^\infty \nu \left[\int_0^\infty \int_0^\infty (yz)^\nu f(y) g(z) D_{\mu,\nu}(x, y, z) \mathrm{d}z \mathrm{d}y\right] x^{2\nu-1} J_\mu[(xz)^\nu] \mathrm{d}x.$$

Changing the order of integration we get

$$= \nu^2 \tau^{-1+\nu} \int_0^\infty \int_0^\infty (yz)^\nu f(y) g(z) \left[\int_0^\infty x^{2\nu-1} J_\mu[(x\tau)^\nu] D_{\mu,\nu}(x, y, z) \mathrm{d}x\right] \mathrm{d}y \mathrm{d}z.$$

Using (22) we get

$$= \nu^2 \tau^{-1+\nu} \int_0^\infty \int_0^\infty (yz)^\nu f(y) g(z) J_\mu[(y\tau)^\nu] J_\mu[(z\tau)^\nu] \mathrm{d}y \mathrm{d}z$$

$$= \tau^{-1-\nu}\left[\nu\tau^{-1+\nu}\int_0^\infty (y\tau)^\nu J_\mu[(y\tau)^\nu]f(y)\mathrm{d}y\right]\left[\nu\tau^{-1}\int_0^\infty (z\tau)^\nu J_\mu[(z\tau)^\nu]g(z)\mathrm{d}z\right]$$

$$= \tau^{1-\nu}h_{\mu,\nu}[f](\tau)h_{\mu,\nu}[g](\tau).$$

4.4 $h_{\mu,\nu}$-*translation*

If the $h_{\mu,\nu}$-convolution $f * g$ exists, then using Fubinis theorem we can write it in the form

$$f * g(x) = \nu\int_0^\infty y^\nu f(y)\left[\int_0^\infty z^\nu g(z)D_{\mu,\nu}(x, y, z)\mathrm{d}z\right]\mathrm{d}y = \nu\int_0^\infty y^\nu f(y)g(x\circ y)\mathrm{d}y, \quad (26)$$

where we write

$$g(x\circ y) = \left[\int_0^\infty z^\nu g(z)D_{\mu,\nu}(x, y, z)\mathrm{d}z\right] \quad (27)$$

with $x \circ y$ denoting the $h_{\mu,\nu}$-translation on the positive real line. (The analogue of the translation consider for the definition of the usual convolution *.)

The function $g(x \circ y)$ will be called the $h_{\mu,\nu}$ translate of $g(x)$; provided $g(x)$ is locally bounded on $0 < x < \infty$, $g(x \circ y)$ is well-defined and continuous on $(0, \infty) \times (0, \infty)$, (Nussbaum [7]). The $h_{\mu,\nu}$-translation is a particular case of the translations of Delsarte [3], subsequently studied by Braaksma [1].

Theorem 4.5. *If* $g \in L^1_{0,\nu}(0,\infty) \cap L^\infty(0,\infty)$ *and* $a \in [0,\infty)$, *then a simple calculation using Fubinis theorem shows that*

$$h_{\mu,\nu}[x^{-\nu}g(x\circ a)](\tau) = J_\mu[(a\tau)^\nu]h_{\mu,\nu}[g(z)](\tau). \quad (28)$$

Proof.

$$\text{LHS} = \nu\tau^{-1}\int_0^\infty (x\tau)^\nu J_\mu[(x\tau)^\nu]x^{-\nu}g(x\circ a)\mathrm{d}x$$

$$= \nu\tau^{-1+\nu}\int_0^\infty J_\mu[(x\tau)^\nu]\int_0^\infty z^\nu g(z)D_{\mu,\nu}(x, a, z)\mathrm{d}z\mathrm{d}x.$$

Using Fubinis theorem,

$$= \nu\tau^{-1+\nu}\int_0^\infty z^\nu g(z)\mathrm{d}z\int_0^\infty J_\mu[(x\tau)^\nu]D_{\mu,\nu}(x, a, z)\mathrm{d}x.$$

Using (22) we get

$$= \nu\tau^{-1+\nu}\int_0^\infty z^\nu g(z)J_\mu[(z\tau)^\nu]J_\mu[(a\tau)^\nu]\mathrm{d}z$$

$$= J_\mu[(a\tau)^\nu]\{\nu z^{-1}\int_0^\infty (z\tau)^\nu J_\mu[(z\tau)^\nu]g(z)\mathrm{d}z\}$$
$$= J_\mu[(a\tau)^\nu]h_{\mu,\nu}[g(z](\tau).$$

5. Generalized Hankel type convolution of generalized functions

5.1

For fixed $x, y \in (0,\infty)$ then the function $D_{\mu,\nu}(x,y,z), 0 < z < \infty$ defines a regular generalized function in $H^*_{\mu,\nu}(0,\infty)$ which we denote by $D_{\mu,\nu}(x \circ y, z)$. In fact for fixed $x, y \in (0,\infty)$ and $\phi \in H_{\mu,\nu}(0,\infty)$ we have that

$$\langle D_{\mu,\nu}(x\circ y, z), z^\nu\phi(z)\rangle = \langle D_{\mu,\nu}(x,y,z), z^\nu\phi(z)\rangle$$
$$= \int_0^\infty z^\nu\phi(z)D_{\mu,\nu}(x,y,z)\mathrm{d}z = \phi(x\circ y) \tag{29}$$

and since

$$|\phi(x\circ y)| = \int_0^\infty \left|z^{1-\nu}z^{2\nu-1}D_{\mu,\nu}(x,y,z)\phi(z)\right|\mathrm{d}z$$
$$\leq \int_0^\infty \left|z^{1-\nu}\phi(z)\right|z^{2\nu-1}D_{\mu,\nu}(x,y,z)\mathrm{d}z$$
$$\leq \gamma_{0,0}^{0,\nu}(\phi)\int_0^\infty z^{2\nu-1}D_{\mu,\nu}(x,y,z)\mathrm{d}z$$
$$\leq \gamma_{0,0}^{0,\nu}(\phi), \quad \text{(by (23))} \tag{30}$$

then $D_{\mu,\nu}(x\circ y, z), 0 < z < \infty$ truly generates a continuous linear functional on $H^*_{\mu,\nu}(0,\infty)$ through (29). Moreover, since

$$\begin{aligned}\phi(x\circ y) = \langle D_{\mu,\nu}(x,y,z), z^\nu\phi(z)\rangle &= \langle z^{-1+\nu}D_{\mu,\nu}(x,y,z), z\phi(z)\rangle\\ &= \langle h_{\mu,\nu}[z^{-1+\nu}D_{\mu,\nu}(x,y,z)], \tau\Phi(\tau)\rangle\\ &= \langle \nu\tau^{-1+\nu}J_\mu[(x\tau)^\nu]J_\mu[(y\tau)^\nu], \tau\Phi(\tau)\rangle\\ &= \langle J_\mu[(x\tau)^\nu]J_\mu[(y\tau)^\nu], \tau^\nu\Phi(\tau)\rangle,\end{aligned}$$

then we can write

$$h_{\mu,\nu}[D_{\mu,\nu}(x\circ y, z)] = J_\mu[(x\tau)^\nu]J_\mu[(y\tau)^\nu],\ 0 < x, y < \infty \tag{31}$$

in the sense of $H^{*\prime}_{\mu,\nu}(0,\infty)$ and even in the classical sense.

We now show that, for every fixed $y > 0$, the following implication

$$\phi(x) \in H_{\mu,\nu}(0,\infty) \Rightarrow x^{-1+\nu}\phi(x \circ y) \in H_{\mu,\nu}(0,\infty) \tag{32}$$

holds.

In fact, since $\phi(x) \in H_{\mu,\nu}(0,\infty)$, then, $\Phi = h_{\mu,\nu}[\phi] \in H_{\mu,\nu}(0,\infty)$. On the other hand,

$$h_{\mu,\nu}[x^{-1+\nu}\phi(x \circ y)] = J_\mu[(y\tau)^\nu]\Phi(\tau);$$

but $J_\mu[(y\tau)^\nu] \in M_\tau(0,\infty)$ and so $J_\mu[(y\tau)^\nu]\Phi(\tau) \in H_{\mu,\nu}(0,\infty)$.

Hence, since $h_{\mu,\nu}$ transformation is an automorphism on $H_{\mu,\nu}(0,\infty)$, the function of x given by $h^{-1}_{\mu,\nu}[J_\mu[(y\tau)^\nu]\Phi(\tau)] = x^{-1+\nu}\phi(x \circ y$ also belongs to $H_{\mu,\nu}(0,\infty)$.

5.2 *Delsarte translation*

For any $\phi(x) \in H_{\mu,\nu}(0,\infty)$ and $0 < x, y < \infty$, following from well-known property of the Delsarte translation, we also have that

$$\Delta^m_{1,x}\phi(x \circ y) = \Delta^m_{1,y}\phi(x \circ y) \tag{33}$$

for any non-negative integer m, by

$$\Delta^m_{1,x} J_\mu[(x\tau)^\nu] = (-\nu^2\tau^{2\nu})^m J_\mu[(x\tau)^\nu].$$

PROPOSITION 5.3

If $\phi_1, \phi_2 \in H_{\mu,\nu}(0,\infty)$ and m is a non-negative integer, then
(i) $\phi_1 * \phi_2$ *exists for all* $0 < x < \infty$;
(ii) $x^{-1+\nu}\phi_1 * \phi_2 \in H_{\mu,\nu}(0,\infty)$;
(iii) $\Delta^m_1[\phi_1 * \phi_2] = (\Delta^m_1\phi_1) * \phi_2 = [\phi_1 * (\Delta^m_1\phi_2)$ (34)
(iv) $\phi_1 * \phi_2(x) = \phi_2 * \phi_1(x)$.

Proof. In fact (34)(i) follows since $\phi_1, \phi_2 \in L^p_{0,\nu}(0,\infty)$ for any p such that $1 \le p < \infty$; (34)(ii) is justified by the fact that the function

$$h_{\mu,\nu}[x^{-1+\nu}(\phi_1 * \phi_2)] = \tau^{1-\nu}\Phi_1(\tau).\Phi_2(\tau)$$

belongs to $H_{\mu,\nu}(0,\infty)$ and similarly for its $h^{-1}_{\mu,\nu}$-transform; (34)(iii) follows from (33) and differentiation under integral sign. Note finally that for any $\phi_1, \phi_2 \in H_{\mu,\nu}(0,\infty)$,

$$\phi_1 * \phi_2(x) = \nu\langle\phi_1(y), y^\nu\phi_2(x \circ y)\rangle = \nu\int_0^\infty y^\nu\phi_1(y)\phi_2(x \circ y)\mathrm{d}y$$

$$= \nu\int_0^\infty y^\nu\phi_1(y)\int_0^\infty z^\nu\phi_2(z)D_{\mu,\nu}(x,y,z)\mathrm{d}z\mathrm{d}y$$

Since $\phi_1, \phi_2 \in L^p_{0,\nu}(0,\infty)$ we can make use of Fubini's theorem to get

$$= \nu\int_0^\infty z^\nu\phi_2(z)\int_0^\infty y^\nu\phi_1(y)D_{\mu,\nu}(x,y,z)\mathrm{d}y\mathrm{d}z$$

$$= \nu \int_0^\infty z^\nu \phi_2(y)\phi_1(x \circ z)\mathrm{d}z = \nu \langle \phi_2, z^\nu \phi_1(x \circ z)\rangle = \phi_2 * \phi_1(x).$$

This proves (34)(iv).

5.4

If $\lambda \in H_{\mu,\nu}(0,\infty)$, then for each fixed $x \in (0,\infty)$; we have

$$x^{-1+\nu}\lambda(x \circ y) \in H_{\mu,\nu}(0,\infty)$$

it follows that for any $\sigma \in E'(0,\infty)$ the convolution $\sigma * \lambda(x)$ is well-defined by

$$x^{-1+\nu}\sigma * \lambda(x) = \langle \sigma(y), \nu y^\nu x^{-1+\nu}\lambda(x \circ y)\rangle. \tag{35}$$

Further,

$$\begin{aligned} h_{\mu,\nu}[x^{-1+\nu}\sigma * \lambda(x)](\tau) &= \nu\langle \sigma(y), y^\nu h_{\mu,\nu}[x^{-1+\nu}\lambda(x \circ y)]\rangle \\ &= \nu\langle \sigma(y), y^\nu J_\mu[(y\tau)^\nu]\Lambda(\tau)\rangle \\ &= \tau^{1-\nu}\langle \sigma(y), \nu\tau^{-1}(y\tau)^\nu J_\mu[(y\tau)^\nu]\rangle\Lambda(\tau) \\ &= \tau^{1-\nu}h_{\mu,\nu}[\sigma](\tau)\Lambda(\tau), \end{aligned} \tag{36}$$

where $\Lambda(\tau) = h_{\mu,\nu}[\lambda]$.

Now $h_{\mu,\nu}[x^{-1+\nu}\sigma*\lambda(x)](\tau) \in H_{\mu,\nu}(0,\infty)$, and therefore $x^{-1+\nu}\sigma*\lambda(x) \in H_{\mu,\nu}(0,\infty)$. Hence $x^{-1+\nu}\sigma * \lambda(x)$ generates a regular generalized function in $H^{*\prime}_{\mu,\nu}(0,\infty)$, and for any $\phi \in H_{\mu,\nu}(0,\infty)$ we get

$$\begin{aligned} &\langle x^{-1+\nu}\sigma * \lambda(x), x\phi(x)\rangle = \langle h_{\mu,\nu}[x^{-1+\nu}\sigma * \lambda(x)](\tau), \tau\Phi(\tau)\rangle \\ &= \langle \tau^{1-\nu}h_{\mu,\nu}[\sigma](\tau)h_{\mu,\nu}[\lambda](\tau), \tau\Phi(\tau)\rangle \\ &= \langle h_{\mu,\nu}[\sigma](\tau), \tau^{2-\nu}h_{\mu,\nu}[\lambda](\tau)\Phi(\tau)\rangle \\ &= \langle h_{\mu,\nu}[\sigma](\tau), \tau h_{\mu,\nu}[x^{-1+\nu}\lambda * \Phi(x)](\tau)\rangle \\ &= \langle \sigma(x), x^\nu\lambda * \Phi(x)](\tau)\rangle = \langle \sigma(x), x^\nu\lambda * \Phi(x)\rangle. \end{aligned} \tag{37}$$

This could be taken as the definition of the generalized Hankel type convolution of generalized function (or generalized $h_{\mu,\nu}$-convolution), and this in turn allows another form analogous to the direct product definition of the generalized Hankel type ordinary convolution:

$$\begin{aligned} &\langle x^{-1+\nu}\sigma * \lambda(x), x\phi(x)\rangle = \langle \sigma(x), x^\nu\lambda * \phi(x)\rangle \\ &= \langle \sigma(x), x\langle \lambda(y), \nu y^\nu x^{-1+\nu}\phi(x \circ y)\rangle\rangle \\ &= \langle \sigma(x), \nu x^\nu\langle \lambda(y), y^\nu\phi(x \circ y)\rangle\rangle = \langle \sigma(x) \otimes \lambda(y), \nu(xy)^\nu\phi(x \circ y)\rangle. \end{aligned} \tag{38}$$

5.5

For $f \in H^{*\prime}_{\mu,\nu}(0,\infty)$ and $\lambda \in H_{\mu,\nu}(0,\infty)$. The convolution is again well-defined as a generalized function in $H^{*\prime}_{\mu,\nu}(0,\infty)$. By

$$\langle x^{-1+\nu}f * \lambda(x), x\phi(x)\rangle = \langle f(x), x^\nu\lambda * \phi(x)\rangle$$

e $x^{\nu}\lambda * \phi(x) \in H_{\mu,\nu}(0,\infty)$ by (34)(ii). Using (18), we get

$$\begin{aligned}&\langle h_{\mu,\nu}[x^{-1+\nu} f * \lambda(x)](\tau), \tau\Phi(\tau)\rangle = \langle x^{-1+\nu} f * \lambda(x), x\phi(x)\rangle\\ &= \langle f(x), x^{\nu}\lambda * \phi(x)\rangle = \langle f(x), x^{\nu}\lambda * \phi(x)\rangle\\ &= \langle h_{\mu,\nu}[f](\tau), \tau[\tau^{1-\nu}h_{\mu,\nu}[\lambda](\tau)\Phi(\tau)\rangle = \langle \tau^{1-\nu}h_{\mu,\nu}[f](\tau)\Lambda(\tau), \tau\Phi(\tau)\rangle.\end{aligned}$$

hat, in the sense of $H^{*\prime}_{\mu,\nu}(0,\infty)$,

$$h_{\mu,\nu}[x^{-1+\nu} f * \lambda(x)] = \tau^{1-\nu}h_{\mu,\nu}[f]h_{\mu,\nu}[\lambda]. \tag{39}$$

ally, let $f \in H^{*\prime}_{\mu,\nu}(0,\infty)$ and $\sigma \in E'(0,\infty)$. Since, for any $\phi \in H_{\mu,\nu}(0,\infty)$ we e $\sigma * \phi(x) \in H_{\mu,\nu}(0,\infty)$, it follows that $x^{-1+\nu} f * \sigma$ is well-defined as a generalized ction in $H^{*\prime}_{\mu,\nu}(0,\infty)$ by

$$\langle x^{-1+\nu} f * \sigma(x), x\phi(x)\rangle = \langle f(x), x^{\nu}\sigma * \phi(x)\rangle. \tag{40}$$

before, this may also be expressed in the form

$$\langle x^{-1+\nu} f * \sigma(x), x\phi(x)\rangle = \langle f(x) \otimes \sigma(x), \nu(xy)^{\nu}\phi(x \circ y)\rangle \tag{41}$$

, using (18) again, we can derive the analogue of (39)

$$h_{\mu,\nu}[x^{-1+\nu} f * \sigma] = \tau^{1-\nu}h_{\mu,\nu}[f]h_{\mu,\nu}[\sigma]. \tag{42}$$

te that $h_{\mu,\nu}[\sigma] \in M(0,\infty)$, so that the product in (42) makes sense in $H^{*\prime}_{\mu,\nu}(0,\infty)$.

Algebraic properties of the generalized $h_{\mu,\nu}$- convolution

already remarked, the classical $h_{\mu,\nu}$-convolution defined in $L^1_{0,\nu}(0,\infty)$ is commutative associative; however, it possesses no identity element. We consider in turn these perties with respect to generalized $h_{\mu,\nu}$-convolution.

Commutativity

$\sigma \in E'(0,\infty)$, $\lambda \in H_{\mu,\nu}(0,\infty)$. We have

$$\begin{aligned}&\langle x^{-1+\nu}\sigma * \lambda(x), x\phi(x)\rangle = \langle \sigma(x), x^{\nu}\lambda * \phi(x)\rangle\\ &= \langle h_{\mu,\nu}[\sigma](\tau), \tau h_{\mu,\nu}[x^{-1+\nu}\lambda * \phi(x)]\rangle\\ &= \langle h_{\mu,\nu}[\sigma](\tau), \tau[\tau^{1-\nu}\Lambda(\tau)\Phi(\tau)]\rangle\\ &= \langle \tau h_{\mu,\nu}[\sigma](\tau)\Lambda(\tau), \tau\Phi(\tau)\rangle\\ &= \langle h_{\mu,\nu}[x^{-1+\nu}\lambda * \sigma](\tau), \tau\Phi(\tau)\rangle\\ &= \langle x^{-1+\nu}\lambda * \sigma(x), x\phi(x)\rangle,\end{aligned}$$

ere the last manipulation make sense since $h_{\mu,\nu}[\sigma] \in M(0,\infty)$ and see (37) and Pinto's oer for other type proof .

(ii) If $f \in H^{*\prime}_{\mu,\nu}(0,\infty)$, $\lambda \in H_{\mu,\nu}(0,\infty)$ then

$$\begin{aligned}
&\langle x^{-1+\nu} f * \lambda(x), x\phi(x)\rangle = \langle f(x), x^{\nu}\lambda * \phi(x)\rangle \\
&= \langle f(x), x[x^{-1+\nu}\lambda * \phi(x)]\rangle \\
&= \langle h_{\mu,\nu}[f(x)](\tau), \tau h_{\mu,\nu}[x^{-1+\nu}\lambda * \phi(x)]\rangle \\
&= \langle h_{\mu,\nu}[f(x)](\tau), \tau[\tau^{1-\nu} h_{\mu,\nu}[\lambda] h_{\mu,\nu}[\phi]\rangle \\
&= \langle \tau^{1-\nu}\Lambda(\tau) h_{\mu,\nu}[f](\tau), \tau\Phi(\tau)\rangle \\
&= \langle h_{\mu,\nu}[x^{-1+\nu}\lambda * f](\tau), \tau\Phi(\tau)\rangle \\
&= \langle x^{-1+\nu}\lambda * f(x), x\phi(x)\rangle.
\end{aligned}$$

This is justified because every function in $H_{\mu,\nu}(0,\infty)$ is also a multiplier in $H^{*\prime}_{\mu,\nu}(0,\infty)$ whenever $x\phi(x) \in H_{\mu,\nu}(0,\infty)$.

(iii) If $f \in H^{*\prime}_{\mu,\nu}(0,\infty)$, $\sigma \in E'(0,\infty)$ then the same kind of argument gives

$$\begin{aligned}
&\langle x^{-1+\nu} f * \sigma(x), x\phi(x)\rangle = \langle f(x), x^{\nu}\sigma * \phi(x)\rangle \\
&= \langle f(x), x[x^{-1+\nu}\sigma * \phi(x)]\rangle \\
&= \langle h_{\mu,\nu}[f(x)](\tau), \tau h_{\mu,\nu}[x^{-1+\nu}\sigma * \phi(x)]\rangle \\
&= \langle h_{\mu,\nu}[f(x)](\tau), \tau[\tau^{1-\nu} h_{\mu,\nu}[\sigma] h_{\mu,\nu}[\phi]\rangle \\
&= \langle \tau^{1-\nu} h_{\mu,\nu}[f](\tau) h_{\mu,\nu}[\sigma](\tau), \tau\Phi(\tau)\rangle \\
&= \langle h_{\mu,\nu}[x^{-1+\nu}\sigma * f](\tau), \tau\Phi(\tau)\rangle \\
&= \langle x^{-1+\nu}\sigma * f(x), x\phi(x)\rangle.
\end{aligned}$$

But since $h_{\mu,\nu}[f]$ does not belong to $M(0,\infty)$, no general commutativity property can be deduced. If, in addition, we have $f \in E'(0,\infty)$, then $h_{\mu,\nu}[f] \in M(0,\infty)$, and the argument to establish commutativity proceed as before.

6.2 *Associativity*

(i) $\sigma \in E'(0,\infty)$, $\lambda_1, \lambda_2 \in H_{\mu,\nu}(0,\infty)$. We can establish the result

$$x^{-1+\nu}[x^{-1+\nu}\sigma * \lambda_1] * \lambda_2 = x^{-1+\nu}\sigma * [x^{-1+\nu}\lambda_1 * \lambda_2] \tag{43}$$

in the following sense, for any $\phi \in H_{\mu,\nu}(0,\infty)$.

$$\begin{aligned}
&\langle x^{-1+\nu}(x^{-1+\nu}\sigma * \lambda) * \lambda_2, x\phi(x)\rangle \\
&= \langle x^{-1+\nu}\sigma * \lambda_1, x^{\nu}\lambda_2 * \phi(x)\rangle \\
&= \langle x^{-1+\nu}\sigma * \lambda_1, x(x^{-1+\nu}\lambda_2 * \phi(x))\rangle \\
&= \langle \sigma(x), x^{\nu}\lambda_1 * [x^{-1+\nu}\lambda_2 * \phi(x)]\rangle \\
&= \langle \sigma(x), x[x^{-1+\nu}\lambda_1 * (x^{-1+\nu}\lambda_2 * \phi(x)])\rangle \\
&= \langle x^{-1+\nu}\sigma * [x^{1-\nu}\lambda_1 * \lambda_2(x)], x\phi(x)\rangle.
\end{aligned}$$

The equality $x^{\nu}\lambda_1 * (x^{-1+\nu}\lambda_2 * \phi) = ((x^{\nu}\lambda_1 * x^{-1+\nu}\lambda_2) * \phi)$ is justified by the fact that λ_1, λ_2 and ϕ belong to $L^1_{0,\nu}(0,\infty)$. (ii) $f \in H^{*\prime}_{\mu,\nu}(0,\infty)$, $\sigma \in E'(0,\infty)$, $\lambda \in H_{\mu,\nu}(0,\infty)$. We have that

$$x^{-1+\nu}[x^{-1+\nu} f * \sigma] * \lambda(x) = x^{-1+\nu} f * [x^{-1+\nu}\sigma * \lambda](x). \tag{44}$$

Proof.

$$\begin{aligned}
&\langle x^{-1+\nu}[x^{-1+\nu} f * \sigma] * \lambda(x), x\phi(x)\rangle \\
&= \langle x^{-1+\nu} f * \sigma(x), x^{\nu}\lambda * \phi(x)\rangle \\
&= \langle x^{-1+\nu} f * \sigma(x), x[x^{-1+\nu}\lambda * \phi(x)]\rangle \\
&= \langle f(x), x^{\nu}\sigma * [x^{-1+\nu}\lambda * \phi(x)]\rangle \\
&= \langle f(x), x[x^{-1+\nu}\sigma * [x^{-1+\nu}\lambda * \phi(x)]]\rangle \\
&= \langle x^{-1+\nu} f * [x^{-1+\nu}\sigma * \lambda(x)], x\phi(x)\rangle.
\end{aligned}$$

(iii) If $f \in H^{*\prime}_{\mu,\nu}(0,\infty), \sigma_1, \sigma_2 \in E'(0,\infty)$. We show, finally, that

$$x^{-1+\nu}[x^{-1+\nu} f * \sigma_1] * \sigma_2(x) = x^{-1+\nu} f * [x^{-1+\nu}\sigma_1 * \sigma_2](x). \tag{45}$$

Proof.

$$\begin{aligned}
&\langle x^{-1+\nu}[x^{-1+\nu} f * \sigma_1] * \sigma_2(x), x\phi(x)\rangle \\
&= \langle x^{-1+\nu} f * \sigma_1(x), x^{\nu}\sigma_2 * \phi(x)\rangle \\
&= \langle x^{-1+\nu} f * \sigma_1(x), x[x^{-1+\nu}\sigma_2 * \phi(x)\rangle \\
&= \langle f(x), x^{\nu}\sigma_1 * [x^{-1+\nu}\sigma_2 * \phi(x)]\rangle \\
&= \langle f(x), x[x^{-1+\nu}\sigma_1 * (x^{-1+\nu}\sigma_2 * \phi(x))]\rangle \\
&= \langle x^{-1+\nu} f * [x^{-1+\nu}\sigma_1 * \sigma_2(x)], x\phi(x)\rangle.
\end{aligned}$$

6.3 *Identity element*

For a, b strictly positive we know that $D_{\mu,\nu}(a,b,z)$ defines a regular generalized function $D_{\mu,\nu}(a,b,z)$ in $H^{*\prime}_{\mu,\nu}(0,\infty)$ If either of a, b takes the value zero then $D_{\mu,\nu}(a,b,z)$ is no longer defined as an ordinary function since

$$D_{\mu,\nu}(a,0,z) = \int_0^\infty \nu^2\tau^{2\nu-1} J_\mu[(a\tau)^\nu] J_\mu[(z\tau)^\nu]\mathrm{d}\tau, \; a > 0,$$

is only a formal identity because the integral fails to converge for any z.

Instead, for any fixed $a > 0$, we consider the integral

$$\int_0^R \nu^2\tau^{2\nu-1} J_\mu[(a\tau)^\nu] J_\mu[(z\tau)^\nu]\mathrm{d}\tau \tag{46}$$

which for each $R > 0$ is uniformly convergent on $0 < z < \infty$.

DEFINITION 6.4

Define the generalized function $D_{\mu,\nu}(a,z)$ in $H^{*\prime}_{\mu,\nu}(0,\infty)$ by

$$D_{\mu,\nu}(a,z) = \lim_{R\to\infty} \int_0^R \nu^2\tau^{2\nu} J_\mu[(a\tau)^\nu] J_\mu[(z\tau)^\nu]\mathrm{d}\tau$$

in the sense that for any $\phi \in H_{\mu,\nu}(0,\infty)$,

$$\langle D_{\mu,\nu}(a,z), z^{\nu}\phi(z)\rangle$$

$$= \lim_{R\to\infty}\left\langle \int_0^R \nu^2\tau^{2\nu-1}J_{\mu}[(a\tau)^{\nu}]J_{\mu}[(z\tau)^{\nu}]\mathrm{d}\tau\mathrm{d}z, z^{\nu}\phi(z)\right\rangle. \tag{47}$$

For each finite $R > 0$ the integral (44) defines a function which generates a regular generalized function in $H^{*\prime}_{\mu,\nu}(0,\infty)$ (Sneddon [9]), Therefore,

$$\left\langle \int_0^R \nu^2\tau^{2\nu-1}J_{\mu}[(a\tau)^{\nu}]J_{\mu}[(z\tau)^{\nu}]\mathrm{d}\tau, z^{\nu}\phi(z)\right\rangle = \int_0^{\infty} z^{\nu}\phi(z)\int_0^R \nu^2\tau^{2\nu-1}J_{\mu}$$
$$[(a\tau)^{\nu}]J_{\mu}[(z\tau)^{\nu}]\mathrm{d}\tau\mathrm{d}z$$

or by Fubini's theorem

$$\left\langle \int_0^R \nu^2\tau^{2\nu-1}J_{\mu}[(a\tau)^{\nu}]J_{\mu}[(z\tau)^{\nu}]\mathrm{d}\tau, z^{\nu}\phi(z)\right\rangle = \int_0^R \nu^2 J_{\mu}[(a\tau)^{\nu}]\int_0^{\infty} z^{\nu}\phi(z)J_{\mu}[(z\tau)^{\nu}]\mathrm{d}z\mathrm{d}\tau$$

$$= \nu\int_0^R \tau^{\nu}J_{\mu}[(a\tau)^{\nu}]\nu\tau^{-1}\int_0^{\infty}(z\tau)^{\nu}\phi(z)J_{\mu}[(a\tau)^{\nu}]\mathrm{d}z\mathrm{d}\tau$$

$$= \nu\int_0^R \tau^{\nu}J_{\mu}[(a\tau)^{\nu}]\Phi(\tau)\mathrm{d}\tau.$$

Thus

$$\langle D_{\mu,\nu}(a,z), z^{\nu}\phi(z)\rangle = \lim_{R\to\infty}\int_0^R \nu\tau^{\nu}J_{\mu}[(a\tau)^{\nu}]\Phi(\tau)\mathrm{d}\tau$$

$$= a^{1-\nu}\left[\nu a^{-1}\int_0^R (a\tau)^{\nu}\phi(z)J_{\mu}[(a\tau)^{\nu}]\Phi(\tau)\mathrm{d}\tau\right]$$

$$= \lim_{R\to\infty} a^{1-\nu}\left[\nu a^{-1}\int_0^R (a\tau)^{\nu}\phi(z)J_{\mu}[(a\tau)^{\nu}]\Phi(\tau)\mathrm{d}\tau\right]$$

$$= a^{1-\nu}\phi(a) \tag{48}$$

and so

$$\begin{aligned} |\langle D_{\mu,\nu}(a,z), z^{\nu}\phi(z)\rangle| &= \lim_{R\to\infty} a^{1-\nu}\left[\nu a^{-1}\int_0^R (a\tau)^{\nu}\phi(z)J_{\mu}[(a\tau)^{\nu}]\Phi(\tau)\mathrm{d}\tau\right] \\ &= a^{1-\nu}\phi(a) \\ &\le \gamma_{0,0}^{a,\nu}(\phi) \end{aligned}$$

h shows that $D_{\mu,\nu}(a,z) \in H^{*\prime}_{\mu,\nu}(0,\infty)$. Moreover, since

$$\langle D_{\mu,\nu}(a,z), z^{\nu}\phi(z)\rangle = a^{1-\nu}\phi(a) = \langle J_{\mu}[(a\tau)^{\nu}], \nu\tau^{\nu}\Phi(\tau)\rangle,$$

btain

$$h_{\mu,\nu}[D_{\mu,\nu}(a,z)](\tau) = J_{\mu}[(a\tau)^{\nu}]. \tag{49}$$

let $(a_n)_{n=1}^{\infty}$ be a monotone decreasing sequence of positive real numbers, tending to as $n \to \infty$, and consider the sequence of generalized functions $(D_{\mu,\nu}(a_n,z))_{n=1}^{\infty}$ in $,(0,\infty)$. Since $H^{*\prime}_{\mu,\nu}(0,\infty)$ is complete, this limit is again a generalized function in $,(0,\infty)$. For each n and any $\phi \in H_{\mu,\nu}(0,\infty)$,

$$\langle D_{\mu,\nu}(a_n,z), z^{\nu}\phi(z)\rangle = a^{1-\nu}\phi(a_n).$$

therefore we define the generalized function $D_{\mu,\nu}(z)$ by

$$\begin{aligned}\langle D_{\mu,\nu}(a_n,z), z^{\nu}\phi(z)\rangle &= \lim_{n\to\infty}\langle D_{\mu,\nu}(a_n,z), z^{\nu}\phi(z)\rangle \\ &= \lim_{n\to\infty} a_n^{1-\nu}\phi(a_n) = \nu\phi(0+)\end{aligned} \tag{50}$$

ependently of the particular sequence $(a_n)_{n=1}^{\infty}$ chosen). Moreover, since

$$\langle D_{\mu,\nu}(z), z^{\nu}\phi(z)\rangle = \nu\phi(0+) = \langle 1, \nu\tau^{\nu}\Phi(\tau)\rangle = \nu\langle 1, \nu\tau^{\nu}\Phi(\tau)\rangle,$$

ave

$$h_{\mu,\nu}[D_{\mu,\nu}(z)](\tau) = \nu.1 = \nu \tag{51}$$

equality being understood in the sense of $H^{*\prime}_{\mu,\nu}(0,\infty)$. The generalized function $_{\nu,z}(x) \in H^{*\prime}_{\mu,\nu}(0,\infty)$ is the required identity element with respect to the general- $h_{\mu,\nu}$-convolution. In fact, it is easy to show that $D_{\mu,\nu,z}(x) \in E'(0,\infty)$ and therefore ny $f \in H^{*\prime}_{\mu,\nu}(0,\infty)$ and every $\phi \in H_{\mu,\nu}(0,\infty)$, by using the results in (40), (41) and we obtain

$$\begin{aligned}&\left\langle f * \frac{D_{\mu,\nu}(x)}{\nu}, x^{\nu}\phi(x)\right\rangle \\ &= \left\langle f(x) \otimes \frac{D_{\mu,\nu}(y)}{\nu}, \nu(xy)^{\nu}\phi(x \circ y)\right\rangle \\ &= \langle f(x), x^{\nu}\langle D_{\mu,\nu}(y), y^{\nu}\phi(x \circ y)\rangle\rangle \\ &= \langle f(x), x^{\nu}\phi(x)\rangle\end{aligned}$$

ch shows that

$$f * \frac{D_{\mu,\nu}(x)}{\nu} = f(x) \tag{52}$$

e sense of $H^{*\prime}_{\mu,\nu}(0,\infty)$, as asserted.

)ifferentiability properties of the $h_{\mu,\nu}$-convolution

conclude with a brief remark on the differentiability properties of the generalized $h_{\mu,\nu}$- volution. Let k be any nonnegative integer $f \in H^{*\prime}_{\mu,\nu}(0,\infty)$ and $\lambda \in H_{\mu,\nu}(0,\infty)$.

Then, since for any $\phi \in H_{\mu,\nu}(0,\infty)$,

$$\begin{aligned} A &\doteq \langle \Delta^k[f * \lambda(x)], x^\nu \phi(x)\rangle \\ &= \int_0^\infty (x^{\nu-1}\Delta_1^k x^{1-\nu})(f*\lambda)x^\nu\phi(x)\mathrm{d}x \\ &= \int_0^\infty x^{2\nu-1}\phi(x)\Delta_1^k x^{1-\nu}(f*\lambda)\mathrm{d}x \\ &= \nu\int_0^\infty x^{2\nu-1}\phi(x)\Delta_1^k x^{1-\nu}\left[\int_0^\infty\int_0^\infty (yz)^\nu f(y)\lambda(z)D_{\mu,z}(x,y,z)\mathrm{d}y\mathrm{d}z\right]\mathrm{d}x. \end{aligned}$$

Differentiating under the integral sign and using Fubini's theorem we get

$$\begin{aligned} A &= \nu\int_0^\infty x^{2\nu-1}\phi(x)\left[\int_0^\infty y^\nu f(y)\left\{\int_0^\infty z^\nu\lambda(z)\Delta_1^k x^{1-\nu}D_{\mu,\nu}(x,y,z)\mathrm{d}z\right\}\mathrm{d}y\right]\mathrm{d}x \\ &= \nu\int_0^\infty x^\nu\phi(x)\int_0^\infty y^\nu f(y)\int_0^\infty z^\nu\lambda(z)[x^{\nu-1}\Delta_1^k x^{1-\nu}D_{\mu,\nu}(x,y,z)]\mathrm{d}z\mathrm{d}y\mathrm{d}x \\ &= \int_0^\infty x^\nu\phi(x)f*[x^{\nu-1}\Delta_1^k x^{1-\nu}]\lambda(x)\mathrm{d}x \\ &= \langle f*[x^{\nu-1}\Delta_1^k x^{1-\nu}]\lambda(x), x^\nu\phi(x)\rangle = B. \end{aligned}$$

Similarly

$$\begin{aligned} B &= \langle [x^{\nu-1}\Delta_1^k x^{1-\nu}]f*\lambda(x), x^\nu\phi(x)\rangle \\ &\Rightarrow \Delta^k[f*\lambda] = f*[\Delta^k\lambda] = [\Delta^k f]*\lambda \qquad (53) \end{aligned}$$

in the sense of $H_{\mu,\nu}^{*\prime}(0,\infty)$.

If now $f \in H_{\mu,\nu}^{*\prime}(0,\infty)$ and $\sigma \in E'(0,\infty)$, then by the same kind of argument, and using (53), we derive the double equality.

$$\Delta^k[f*\sigma] = f*[\Delta^k\sigma] = [\Delta^k f]*\sigma \qquad (54)$$

in the sense of $H_{\mu,\nu}^{*\prime}(0,\infty)$.

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onlinear elliptic differential equations with multivalued onlinearities

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Abstract. In this paper we study nonlinear elliptic boundary value problems with monotone and nonmonotone multivalued nonlinearities. First we consider the case of monotone nonlinearities. In the first result we assume that the multivalued nonlinearity is defined on all $\mathbb{R}$. Assuming the existence of an upper and of a lower solution, we prove the existence of a solution between them. Also for a special version of the problem, we prove the existence of extremal solutions in the order interval formed by the upper and lower solutions. Then we drop the requirement that the monotone nonlinearity is defined on all of $\mathbb{R}$. This case is important because it covers variational inequalities. Using the theory of operators of monotone type we show that the problem has a solution. Finally in the last part we consider an eigenvalue problem with a nonmonotone multivalued nonlinearity. Using the critical point theory for nonsmooth locally Lipschitz functionals we prove the existence of at least two nontrivial solutions (multiplicity theorem).

Keywords. Upper solution; lower solution; order interval; truncation function; pseudomonotone operator; coercive operator; extremal solution; Yosida approximation; nonsmooth Palais–Smale condition; critical point; eigenvalue problem.

Introduction

 this paper we employ the method of upper and lower solutions, the theory of nonlinear erators of monotone type and the critical point theory for nonsmooth functionals in der to solve certain nonlinear elliptic boundary value problems, involving discontinuous nlinearities of both monotone and nonmonotone type.

Most of the works so far have treated semilinear probems. Only Deuel–Hess [12], al with a fully nonlinear equation, but their forcing term on the right hand side is a aratheodory function. Deuel–Hess use the method of upper and lower solutions, in order to ow that problem has a solution located in the order interval formed by the upper and lower lutions. More recently Dancer–Sweers [11] considered a semilinear elliptic problem, ith a Caratheodory forcing term, which is independent of the gradient of the solution and ey proved the existence of extremal solutions in the order interval (i.e the existence of a aximal and of a minimal solution there). Semilinear elliptic problems with discontinuities ve been studied by Chang [8] and Costa–Goncalves [10], who used critical point theory r nondifferentiable functionals, by Ambrosetti–Turner [4] and Ambrosetti–Badiale [5], who used the dual variational principle of Clarke [9] and by Stuart [23] and Carl–Heikkila [7], who used monotonicity techniques. In Carl–Heikkila [7], we encounter differential inclusions but they assume that the monotone term $\beta(\cdot)$ corresponding to the discontinuous nonlinearity, is defined everywhere i.e ($\mathrm{dom}\beta = \mathbb{R}$), while here we have a result where $\mathrm{dom}\beta \neq \mathbb{R}$, a case of special importance since it incorporates variational inequalities. We also consider the case where the term $\beta(\cdot)$ is nonmonotone, which corresponds to problems in mechanics, in which the constitutive laws are nonmonotone and multivalued and so are described by the subdifferential of nonsmooth and nonconvex potential functions (hemivariational inequalities).

2. Preliminaries

Let X be a reflexive Banach and X^* its topological dual. In what follows by $(\cdot, \cdot)$ we denote the duality brackets of the pair (X, X^*). A map $A : A \to 2^{X^*}$ is said to be monotone, if for all $[x_1, x_1^*]$, $[x_2, x_2^*] \in \mathrm{Gr}A$, we have $(x_2^* - x_1^*, x_2 - x_1) \geq 0$. The set $D = \{x \in X : A(x) \neq \emptyset\}$ is called the 'domain of A'. We say that $A(\cdot)$ is maximal monotone, if its graph is maximal with respect to inclusion among the graphs of all monotone maps from X into X^*. It follows from this definition that $A(\cdot)$ is maximal monotone if and only if $(v^* - x^*, v - x) \geq 0$ for all $[x, x^*] \in \mathrm{Gr}A$, implies $[v, v^*] \in \mathrm{Gr}A$. For a maximal monotone map $A(\cdot)$, for every $x \in D$, $A(x)$ is nonempty, closed and convex. Moreover, $\mathrm{Gr}A \subseteq X \times X^*$ is demiclosed, i.e. if $x_n \to x$ in X and $x_n^* \xrightarrow{w} x^*$ in X^* or if $x_n \xrightarrow{w} x$ in X and $x_n^* \to x^*$ in X^*, then $[x, x^*] \in \mathrm{Gr}A$. A single-valued $A : X \to X^*$ with domain all of X, is said to be hemicontinuous if for all $x, y, z \in X$, the map $\lambda \to (A(x + \lambda y), z)$ is continuous from $[0, 1]$ into $\mathbb{R}$ (i.e. for all $x, y \in X$, the map $\lambda \to A(x + \lambda y)$ is continuous from $[0, 1]$ into X^* furnished with the weak topology). A monotone hemicontinuous operator is maximal monotone. A map $A : X \to 2^{X^*}$ is said to be 'pseudomonotone', if for all $x \in X$, $A(x)$ is nonempty, closed and convex, for every sequence $\{[x_n, x_n^*]\}_{n \geq 1} \subseteq \mathrm{Gr}A$ such that $x_n \xrightarrow{w} x$ in X, $x_n^* \xrightarrow{w} x^*$ in X^* and $\limsup(x_n^*, x_n - x) \leq 0$, we have that for each $y \in X$, there corresponds a $y^*(y) \in A(x)$ such that $(y^*(y), x - y) \leq \liminf(x^*, x_n - y)$ and finally A is upper semicontinuous (as a set-valued map) from every finite dimensional subspace of X into X^* endowed with the weak topology. Note that this requirement is automatically satisfied if $A(\cdot)$ is bounded, i.e. maps bounded sets into bounded sets. A map $A : X \to 2^{X^*}$ with nonempty, closed and convex values, is said to be generalized pseudomonotone if for any sequence $\{[x_n, x_n^*]\}_{n \geq 1} \subseteq \mathrm{Gr}A$ such that $x_n \xrightarrow{w} x$ in X, $x_n^* \xrightarrow{w} x^*$ in X^* and $\limsup(x_n^*, x_n - x) \leq 0$, we have $[x, x^*] \in \mathrm{Gr}A$ and $(x_n^*, x_n) \to (x^*, x)$ (generalized pseudomonotonicity). The sum of two pseudomonotone maps is pseudomonotone and a maximal monotone map with domain $D = X$, is pseudomonotone. A pseudomonotone map which is also coercive (i.e $\frac{\inf[(x^*, x) : x^* \in A(x)]}{\|x\|} \to \infty$ as $\|x\| \to \infty$) is surjective.

A function $\varphi : X \to \widehat{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ is said to be proper, if it is not identically $+\infty$, i.e $\mathrm{dom}\varphi = \{x \in X : \varphi(x) < +\infty\}$ (the effective domain of φ) is nonempty. By $\Gamma_0(X)$ we denote the space of all proper, convex and lower semicontinuous functions. Given a proper, convex function $\varphi(\cdot)$, its subdifferential $\partial\varphi : X \to 2^{X^*}$ is defined by

$$\partial\varphi(x) = \{x^* \in X^* : (x^*, y - x) \leq \varphi(y) - \varphi(x) \text{ for all } y \in \mathrm{dom}\varphi\}.$$

If $\varphi \in \Gamma_0(X)$, then $\partial\varphi(\cdot)$ is maximal monotone (in fact cyclically maximal monotone). Finally recall that a $\varphi \in \Gamma_0(X)$ is locally Lipschitz in the interior of its effective domain.

Next let $\varphi : X \to \mathbb{R}$ be locally Lipschitz. For such a function we can define the generalized directional derivative of φ at $x \in X$ in the direction $h \in X$, as follows

$$\varphi^0(x; h) = \limsup_{x' \to x, \lambda \downarrow 0} \frac{\varphi(x' + \lambda h) - \varphi(x')}{\lambda}.$$

It is easy to see that $\varphi^0(x; \cdot)$ is sublinear and continuous and so by the Hahn–Banach theorem we can define the nonempty, weakly compact and convex set

$$\partial\varphi(x) = \{x^* \in X^* : (x^*, h) \leq \varphi^0(x; h) \text{ for all } h \in X\}.$$

The set $\partial\varphi(x)$ is called the (generalized) subdifferential of φ at x (see Clarke [9]). If φ is also convex, then this subdifferential coincides with the subdifferential of φ in the sense of convex analysis defined earlier. Moreover, in this case $\varphi^0(x; h) = \lim_{\lambda \downarrow 0} \frac{\varphi(x+\lambda h)-\varphi(x)}{\lambda}$ $= \varphi'(x; h)$ (the directional derivative of φ at x in the direction h). A function φ for which $\varphi^0(x; \cdot) = \varphi'(x; \cdot)$ is said to be regular at x. Finally recall that if x is a local extremum of φ, then $0 \in \partial\varphi(x)$. More generally a point $x \in X$ for which we have $0 \in \partial\varphi(x)$, is said to be a critical point of φ. For further details on operators of monotone type and subdifferentials, we refer to Hu-Papageorigiou [16] and Zeidler [25].

3. Existence results with monotone nonlinearities

Let $Z \subseteq \mathbb{R}^N$ be a bounded domain with a C^1-boundary Γ. In what follows by $A_1(\cdot)$ we denote the nonlinear, second order differential operator in divergence form defined by $A_1(x)(\cdot) = -\sum_{k=1}^N D_k a_k(\cdot, x(\cdot), Dx(\cdot))$. In this section we study the following boundary value problem:

$$\left\{ \begin{array}{l} A_1(x)(z) + a_0(z, x(z), Dx(z)) + \beta(z, x(z)) \ni g(x(z)) \text{ in } Z \\ x_{|\Gamma} = 0 \end{array} \right\}. \tag{1}$$

First using the method of upper and lower solutions, we establish the existence of (weak) solutions for problem (1), when $\text{dom}\beta = \mathbb{R}$. Let us start by introducing the hypotheses on the coefficient functions $a_k(z, x, y)$, $k \in \{1, 2, \dots, N\}$, and on the multifunction $\beta(r)$.

H(α_k): $a_k : E \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$, $k \in \{1, 2, \dots, N\}$, are functions such that

(i) for all $x \in \mathbb{R}$ and all $y \in \mathbb{R}^N$, $z \to a_k(z, x, y)$ is measurable;

(ii) for almost all $z \in Z$, $(x, y) \to a_k(z, x, y)$ is continuous;

(iii) for almost all $z \in Z$, all $x \in \mathbb{R}$ and all $y \in \mathbb{R}^N$, we have

$$|a_k(z, x, y)| \leq \gamma(z) + c(|x|^{p-1} + \|y\|^{p-1})$$

with $\gamma \in L^q(Z)$, $c > 0$, $2 \leq p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$;

(iv) for almost all $z \in Z$, all $x \in \mathbb{R}$ and all $y, y' \in \mathbb{R}^N$, $y \neq y'$, we have

$$\sum_{k=1}^N (a_k(z, x, y) - a_k(z, x, y'))(y_k - y'_k) > 0;$$

(v) for almost all $z \in Z$, all $x \in \mathbb{R}$ and all $y \in \mathbb{R}^N$, we have

$$\sum_{k=1}^{N} a_k(z, x, y) y_k \geq c_1 \|y\|^p - \gamma_1(z)$$

with $c_1 > 0$, $\gamma_1 \in L^1(Z)$.

Remark. By virtue of these hypotheses, we can define the semilinear form

$$\widehat{a} : W_0^{1,p}(Z) \times W_0^{1,p}(Z) \to \mathbb{R}$$

by setting

$$\widehat{a}(x, v) = \int_Z \sum_{k=1}^{n} a_k(z, x(z), Dx(z)) D_k v(z) \mathrm{d}z.$$

H(β): $\beta : Z \times \mathbb{R} \to 2^{\mathbb{R}}$ is a graph measurable multifunction such that for all $z \in Z$, $\beta(z, \cdot)$ is maximal monotone, $\mathrm{dom}\beta(z, \cdot) = \mathbb{R}$, $0 \in \beta(z, 0)$ and $|\beta(z, x)| = \max[|v| : v \in \beta(z, x)] \leq k(z) + \eta|x|^{p-1}$ a.e on Z with $k \in L^q(Z)$, $\eta > 0$.

Remark. It is well-known (see for example [16], example III.4.28(a), p. 348 and theorem III.5.6, p. 362), that for all $z \in Z$, $\beta(z, x) = \partial j(z, x)$ with $j(z, x)$ a jointly measurable function such that $j(z, \cdot)$ is convex and continuous (in fact locally Lipschitz). If $\beta^0(z, x) = \mathrm{proj}(0; \beta(z, x))$ (= the unique element of $\beta(z, x)$ with the smallest absolute value), then $x \to \beta^0(z, x)$ is nondecreasing and for every $(z, x) \in Z \times \mathbb{R}$, we have $\beta(z, x) = [\beta^0(z, x^-), \beta^0(z, x^+)]$. Moreover, $j(z, x) = j(z, 0) + \int_0^x \beta^0(z, s) \mathrm{d}s$. Since $j(z, \cdot)$ is unique up to an additive constant, we can always have $j(z, 0) = 0$. Since by hypothesis $0 \in \beta(z, 0)$, we infer that for all $z \in Z$ and all $x \in \mathbb{R}$, $j(z, x) \geq 0$. In what follows $\beta_-(z, x) = \beta^0(z, x^-)$ and $\beta_+(z, x) = \beta^0(z, x^+)$. So $\beta(z, x) = [\beta_-(z, x), \beta_+(z, x)]$. Evidently we have for almost all $z \in Z$ and all $x \in \mathbb{R}$, $|\beta_-(z, x)|, |\beta_+(z, x)| \leq k(z) + \eta|x|^{p-1}$.

To introduce the hypotheses on the rest of the data of (1), we need the following definitions.

DEFINITION

A function $\varphi \in W^{1,p}(Z)$ is said to be an 'upper solution' of (1) if there exists $x_1^* \in L^q(Z)$ such that $x_1^*(z) \in \beta(z, \varphi(z))$ a.e. on Z and

$$\widehat{a}(\varphi, v) + \int_Z a_0(z, \varphi, D\varphi) v(z) \mathrm{d}z + \int_Z x_1^*(z) v(z) \mathrm{d}z \geq \int_Z g(\varphi(z)) v(z) \mathrm{d}z$$

for all $v \in W_0^{1,p} \cap L^p(Z)_+$ and $\varphi_{|\Gamma} \geq 0$.

DEFINITION

A function $\psi \in W^{1,p}(Z)$ is said to be a 'lower solution' of (1), if there exists $x_0^* \in L^q(Z)$ such that $x_0^*(z) \in \beta(z, \psi(z))$ a.e. on Z and

$$\widehat{a}(\psi, v) + \int_Z a_0(z, \psi, D\psi) v(z) \mathrm{d}z + \int_Z x_0^*(z) v(z) \mathrm{d}z \leq \int_Z g(\psi(z)) v(z) \mathrm{d}z$$

for all $v \in W_0^{1,p} \cap L^p(Z)_+$ and $\psi_{|\Gamma} \leq 0$.

We can continue with the hypotheses on the data of (1):

H_0: There exist an upper solution $\varphi \in W^{1,p}(Z)$ and a lower solution $\psi \in W^{1,p}(Z)$ such that $\psi(z) \leq 0 \leq \varphi(z)$ a.e. on Z and for all $y \in L^p(Z)$ such that $\psi(z) \leq y(z) \leq \varphi(z)$ a.e. on Z we have $g(y(\cdot)) \in L^q(Z)$. Moreover, $g(\cdot)$ is nondecreasing.

$H(\alpha_0)$: $a_0 : Z \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$, is a function such that

(i) for all $x \in \mathbb{R}$ and all $y \in \mathbb{R}^N$, $z \to a_0(z, x, y)$ is measurable;

(ii) for almost all $z \in Z$, $(x, y) \to a_0(z, x, y)$ is continuous;

(iii) for almost all $z \in Z$ and all $x \in [\psi(z), \varphi(z)]$, we have

$$|a_0(z, x, y)| \leq \gamma_2(z) + c_2\|y\|^{p-1}$$

with $\gamma_2 \in L^q(Z)$, $c_2 > 0$.

DEFINITION

By a '(weak) solution' of (1), we mean a function $x \in W_0^{1,p}(Z)$ such that there exists $f \in L^q(Z)$ with $f(z) \in \beta(z, x(z))$ a.e. on Z and

$$\widehat{a}(x, v) + \int_Z a_0(z, x, Dx)v(z)\mathrm{d}z + \int_Z f(z)v(z)\mathrm{d}z = \int_Z g(x(z))v(z)\mathrm{d}z$$

for all $v \in W_0^{1,p}(Z)$.

Let $K = [\psi, \varphi] = \{y \in W^{1,p}(Z) : \psi(z) \leq y(z) \leq \varphi(z) \text{ a.e. on } Z\}$. Our approach will involve truncation and penalization techniques. So we introduce the following two functions:

$\tau : W^{1,p}(Z) \to W^{1,p}(Z)$ (the truncation function) defined by

$$\tau(x)(z) = \begin{cases} \varphi(z) & \text{if } \varphi(z) \leq x(z) \\ x(z) & \text{if } \psi(z) \leq x(z) \leq \varphi(z) \\ \psi(z) & \text{if } x(z) \leq \psi(z) \end{cases}$$

and $u : Z \times \mathbb{R} \to \mathbb{R}$ (the penalty function) defined by

$$u(z, x) = \begin{cases} (x - \varphi(z))^{p-1} & \text{if } \varphi(z) \leq x \\ 0 & \text{if } \psi(z) \leq x \leq \varphi(z) \\ -(\psi(z) - x)^{p-1} & \text{if } x(z) \leq \psi(z). \end{cases}$$

It is easy to check that the following are true (see also Deuel–Hess [12]).

Lemma 1. (a) *The truncation function map $\tau : W^{1,p}(Z) \to W^{1,p}(Z)$ is bounded and continuous.* (b) *The penalty function $u(z, x)$ is a Caratheodory function such that*

$$\int_Z u(z, x(z))x(z)\mathrm{d}z \geq c_3\|x\|_p^p - c_4$$

for all $x \in L^p(Z)$ and some $c_3, c_4 > 0$.

To solve (1), we first investigate the following auxiliary problem, with $y \in K$:

$$\left\{ \begin{array}{l} A_2(x)(z) + a_0(z, \tau(x)(z), D\tau(x)(z)) + \beta(z, x(z)) + \rho u(z, x(z)) \ni g(y(z)) \text{ on } Z \\ x_{|\Gamma} = 0, \rho > 0 \end{array} \right\}. \quad (2)$$

Here $A_2(x)$ is the nonlinear, second order differential operator in divergence form, defined by

$$A_2(x)(z) = -\sum_{k=1}^{N} D_k a_k(z, \tau(x), Dx).$$

In the next proposition we establish the nonemptiness of the solution set $S(y) \subseteq W_0^{1,p}(Z)$ of (2) for all $y \in K$.

PROPOSITION 2

If hypotheses $\mathrm{H}(a_k), \mathrm{H}(\beta), \mathrm{H}_0, \mathrm{H}(a_0)$ *hold and* $y \in K$, *then the solution set* $S(y) \subseteq W_0^{1,p}(Z)$ *of* (2) *is nonempty for* $\rho > 0$ *large.*

Proof. Let $\theta : W_0^{1,p}(Z) \times W_0^{1,p}(Z) \to \mathbb{R}$ be the semilinear Dirichlet form defined by

$$\theta(x, y) = \int_Z \sum_{k=1}^{N} a_k(z, \tau(x), Dx) D_k y(z) \mathrm{d}z.$$

By virtue of hypotheses $\mathrm{H}(a_k)$, this Dirichlet form defines a nonlinear operator $\widehat{A}_1 : W_0^{1,p}(Z) \to W^{-1,q}(Z)$ by $\langle \widehat{A}_1(x), y \rangle = \theta(x, y)$ (here by $\langle \cdot, \cdot \rangle$ we denote the duality brackets of the pair $(W_0^{1,p}(Z), W^{-1,q}(Z))$). Also let $\widehat{a}_0 : W_0^{1,p}(Z) \to L^q(Z)$ be defined by $\widehat{a}_0(x)(z) = a_0(z, \tau(x)(z), D\tau(x)(z))$. This is continuous and bounded (see hypothesis $\mathrm{H}(a_0)$).

Claim 1. The operator $\widehat{A}_2 = \widehat{A}_1 + \widehat{a}_0 : W_0^{1,p}(Z) \to W^{-1,q}(Z)$ is pseudomonotone.

To this end, let $x_n \xrightarrow{w} x$ in $W_0^{1,p}(Z)$ as $n \to \infty$ and assume that $\limsup \langle \widehat{A}_2(x_n), x_n - x \rangle \leq 0$. Then $\limsup \langle \widehat{A}_1(x_0) + \widehat{a}_0(x_n), x_n - x \rangle \leq 0$. From the Sobolev embedding theorem, we have $x_n \to x$ in $L^p(Z)$ and so $\langle \widehat{a}_0(x_n), x_n - x \rangle = (\widehat{a}_0(x_n), x_n - x)_{pq} \to 0$ (by $(\cdot, \cdot)_{pq}$ we denote the duality brackets of $(L^p(Z), L^q(Z))$. Therefore we obtain $\limsup \langle \widehat{A}_1(x_n), x_n - x \rangle \leq 0$.

We have

$$\begin{aligned} \langle \widehat{A}_1(x_n), x_n - x \rangle &= \int_Z \sum_{k=1}^{N} a_k(z, \tau(x_n), Dx_n)(D_k x_n - Dx) \mathrm{d}z \\ &= \int_Z \sum_{k=1}^{N} (a_k(z, \tau(x_n), Dx_n) - a_k(z, \tau(x_n), Dx))(D_k x_n - D_k x) \mathrm{d}z \\ &\quad + \int_Z \sum_{k=1}^{N} a_k(z, \tau(x_n), Dx)(D_k x_n - D_k x) \mathrm{d}z \end{aligned}$$

$$\geq \int_Z \sum_{k=1}^N a_k(z,\tau(x_n),Dx)(D_kx_n - D_kx)\mathrm{d}z \text{ (hypothesis H}(a_k)\text{(iv))}.$$

Since $x_n \xrightarrow{w} x$ in $W_0^{1,p}(Z)$, we have $x_n \to x$ in $L^p(Z)$ and then directly from the definition of the truncation map τ, we have $\tau(x_n) \to \tau(x)$ in $L^p(Z)$.

Therefore

$$\int_Z \sum_{k=1}^N a_k(z,\tau(x_n),Dx)(D_kx_n - D_kx)\mathrm{d}z \to 0 \text{ as } n \to \infty.$$

On the other hand we already know that $\limsup\langle A_1(x_n), x_n - x\rangle \leq 0$. Hence $\langle A_1(x_n), x_n - x\rangle \to 0$ as $n \to \infty$. From this it follows that

$$\int_Z \sum_{k=1}^N (a_k(z,\tau(x_n),Dx_n) - a_k(z,\tau(x_n),Dx))(D_kx_n - D_kx)\mathrm{d}z \to 0 \text{ as } n \to \infty.$$

Then invoking Lemma 6 of Landes [17], we infer that $D_kx_n(z) \to D_kx(z)$ a.e on Z for all $k \in \{1,2,\ldots,N\}$. So using Lemma 3.2 of Leray–Lions [18], we have that $\widehat{A}_1(x_n) \xrightarrow{w} \widehat{A}_1(x)$ in $W^{-1,q}(Z)$. We have already established earlier that $\langle\widehat{A}_1(x_n), x_n - x\rangle \to 0$. Since $\langle\widehat{A}_1(x_n), x\rangle \to \langle\widehat{A}_1(x), x\rangle$, we obtain that $\langle\widehat{A}_1(x_n), x_n\rangle \to \langle\widehat{A}_1(x), x\rangle$. Also $\langle\widehat{a}_0(x_n), x_n\rangle = (\widehat{a}_0(x_n), x_n)_{pq}$. But again by Lemma 3.2, Leray–Lions [18], we have that $\widehat{a}_0(x_n) \xrightarrow{w} \widehat{a}_0(x)$ in $L^q(Z)$. Since $x_n \to x$ in $L^p(Z)$ (by the Sobolev imbedding theorem, we have that $\langle\widehat{a}_0(x_n), x_n\rangle = (\widehat{a}_0(x_n), x_n)_{pq} \to (\widehat{a}_0(x), x)_{pq} = \langle\widehat{a}_0(x), x\rangle$. Therefore finally we have $\widehat{A}_2(x_n) \xrightarrow{w} \widehat{A}_2(x)$ in $W^{-1,q}(Z)$ and $\langle\widehat{A}_2(x_n), x_n\rangle \to \langle\widehat{A}_2(x), x\rangle$, which proves that $\widehat{A}_2$ is a generalized pseudomonotone. But $\widehat{A}_2$ is everywhere defined, single-valued and bounded. So from Proposition III.6.11, p. 366 of Hu–Papageorgiou [16], it follows that $\widehat{A}_2$ is pseudomonotone. This proves the claim.

Next let $U : W_0^{1,p}(Z) \to L^q(Z)$ be defined by $U(x)(z) = u(z, x(z))$. From the compact embedding of $W_0^{1,p}(Z)$ in $L^p(Z)$ and Lemma 1, we infer that $U(\cdot)$ is completely continuous (i.e. sequentially continuous from $W_0^{1,p}(Z)$ with the weak topology into $L^q(Z)$ with strong topology). Therefore $\widehat{A} = \widehat{A}_2 + \rho U : W_0^{1,p}(Z) \to W^{-1,q}(Z)$ is pseudomonotone.

From Lebourg's subdifferential mean value theorem (see Clarke [9], theorem 2.3.7, p. 41), we have that for almost all $z \in Z$ and all $x \in \mathbb{R}$, $|j(z,x)| \leq k(z)|x| + \eta|x|^p$. Thus if we define $\widehat{G} : L^p(Z) \to \mathbb{R}$ by $\widehat{G}(x) = \int_Z j(z,x(z))\mathrm{d}z$, we have that $\widehat{G}(\cdot)$ is continuous (in fact locally Lipschitz) and convex. Let $G = \widehat{G}|_{W_0^{1,p}}(Z)$. Then from Lemma 2.1 of Chang [8], we have that for all $x \in W_0^{1,p}(Z)$, $\partial G(x) = \partial\widehat{G}(x) \subseteq L^q(Z)$.

Then the auxiliary boundary value problem is equivalent to the following abstract operator inclusion

$$\widehat{A}(x) + \partial G(x) \ni \widehat{g}(y)$$

with $\widehat{g}(y)(\cdot) = g(y(\cdot)) \in L^q(Z)$ (see hypothesis H_0).

Claim 2. $x \to \widehat{A}(x) + \partial G(x)$ is coercive form $W_0^{1,p}(Z)$ into $W^{-1,q}(Z)$, for $\rho > 0$ large.

To this end, we have

$$\langle \widehat{A}(x), x\rangle = \langle \widehat{A}_1(x) + \widehat{a}_0(x) + \rho U(x), x\rangle.$$

From hypothesis $H(a_k)(v)$, we have

$$\langle \widehat{A}_1(x), x\rangle \geq c_1 \|Dx\|_p^p - \|\gamma_1\|_1 \geq c_5\|x\|_{1,p} - c_6, \text{ with } c_5, c_6. > 0. \tag{3}$$

Also from hypothesis $H(a_0)$ (iii), we have

$$\langle \widehat{a}_0(x), x\rangle \geq -c_7\|x\|_p\|x\|_{1,p}^{p-1} - c_8\|x\|_p \text{ for some } c_7, c_8 > 0. \tag{4}$$

From Young's inequality with $\epsilon > 0$, we obtain

$$\|x\|_p\|x\|_{1,p}^{p-1} \leq \frac{1}{\epsilon^p p}\|x\|_p^p + \frac{\epsilon^q}{q}\|x\|_{1,p}^p$$

and so using that in (4), we have

$$\langle \widehat{a}_0(x), x\rangle \geq -c_7\frac{1}{\epsilon^p p}\|x\|_p^p - c_7\frac{\epsilon^q}{q}\|x\|_{1,p}^p - c_8\|x\|_p. \tag{5}$$

Finally from lemma 1, we have

$$\langle \rho U(x), x\rangle \geq c_9\rho\|x\|_p^p - c_{10} \text{ for some } c_9, c_{10} > 0. \tag{6}$$

From (3), (5) and (6) it follows that

$$\langle \widehat{A}(x), x\rangle \geq \left(c_5 - c_7\frac{\epsilon^q}{q}\right)\|x\|_{1,p}^p + \left(c_9\rho - c_7\frac{1}{\epsilon^p p}\right)\|x\|_p^p - c_8\|x\|_p - c_6. \tag{7}$$

Choose $\epsilon > 0$ so that $c_5 > c_7\frac{\epsilon^q}{q}$. Then with $\epsilon > 0$ fixed this way choose $\rho > 0$ so that $c_2\rho > c_7\frac{1}{\epsilon^p p}$. From (7) it follows that $\widehat{A}$ is coercive.

Moreover, since by hypothesis $H(\beta)$ we have that $0 \in \beta(z, 0)$, it follows that $0 \in \partial G(0)$ and so $\langle x^*, x\rangle \geq 0$ for all $x^* \in \partial G(x)$. Thus $\widehat{A} + \partial G$ is coercive and this proves the claim.

Finally because $\partial G(\cdot)$ is maximal monotone and $\operatorname{dom}\partial G = X$, we have that $\partial G(\cdot)$ is pseudomonotone. So $\widehat{A} + \partial G$ is pseudomonotone (Claim 1) and coercive (Claim 2). Apply Corollary III.6.30, p. 372, of Hu–Papageorgiou [16] to conclude that $\widehat{A} + \partial G$ is surjective. So there exists $x \in W_0^{1,p}(Z)$ such that $\widehat{A}(x) + \partial G(x) \ni \widehat{g}(y)$

Having this auxiliary result, we can now prove the first existence theorem concerning our original problem (1).

Theorem 3. *If hypotheses* $H(a_k), H(a_0), H_0$ *and* $H(\beta)$ *hold, then problem* (1) *has a nonempty solution set.*

Proof. We consider the solution multifunction $S : K \to 2^{W_0^{1,p}(Z)}$ for the auxiliary problem (2), i.e for every $y \in K$, $S(y) \subseteq W_0^{1,p}(Z)$ is the solution set of (2). From Proposition 2, we know that $S(\cdot)$ has nonempty values.

Claim 1. $S(K) \subseteq K$.

Let $y \in K$ and let $x \in S(y)$. We have

$$\langle \widehat{A}_2(x), v\rangle + \langle x^*, v\rangle + \rho\langle U(x), v\rangle = \langle \widehat{g}(y), v\rangle$$

for some $x^* \in \partial G(x)$ and all $v \in W_0^{1,p}(Z)$. Since $\psi \in W^{1,p}(Z)$ is a lower solution, by definition we have

$$\widehat{a}(\psi, v) + \int_Z a_0(z, \psi, D\psi)v(z)\mathrm{d}z + \langle x_1^*, v\rangle \leq \langle \widehat{g}(\psi), v\rangle \text{ for all } v \in W_0^{1,p}(Z) \cap L^p(Z)_+$$

and for some $x_1^* \in L^q(Z)$ with $x_1^*(z) \in \beta(z, \psi(z))$ a.e on Z.

Let $v = (\psi - x)^+ \in W_0^{1,p}(Z) \cap L^p(Z)_+$ (see for example Gilbarg–Trudinger [13], Lemma 7.6, p. 145). From the definition of the convex subdifferential, we have

$$\langle x^*, (\psi - x)^+\rangle \leq G(x + (\psi - x)^+) - G(x)$$

and

$$\langle x_1^*, (\psi - x)^+\rangle \geq G(\psi) - G(\psi - (\psi - x)^+).$$

Using these two inequalities, we obtain

$$\begin{aligned} &-\langle \widehat{A}_2(x), (\psi - x)^+\rangle - G(x + (\psi - x)^+) + G(x)\\ &-\rho\langle U(x), (\psi - x)^+\rangle \leq -\langle \widehat{g}(y), (\psi - x)^+\rangle \end{aligned} \tag{8}$$

and

$$\begin{aligned} &\widehat{a}(\psi, (\psi - x)^+) + \int_Z a_0(z, \psi, D\psi)(\psi - x)^+\mathrm{d}z + G(\psi)\\ &-G(\psi - (\psi - x)^+) \leq \langle \widehat{g}(\psi), (\psi - x)^+\rangle. \end{aligned} \tag{9}$$

Note that $G(x) + G(\psi) - G(x + (\psi - x)^+) - G(\psi - (\psi - x)^+) = 0$. So adding (8) and (9), we obtain

$$\begin{aligned} &\widehat{a}(\psi, (\psi - x)^+) + \int_Z a_0(z, \psi, D\psi)(\psi - x)^+\mathrm{d}z - \langle \widehat{A}_2(x), (\psi - x)^+\rangle\\ &-\rho\langle U(x), (\psi - x)^+\rangle \leq \langle \widehat{g}(\psi) - \widehat{g}(y), (\psi - x)^+\rangle. \end{aligned} \tag{10}$$

First we estimate the quantity

$$\widehat{a}(\psi, (\psi - x)^+) + \int_Z a_0(z, \psi, D\psi)(\psi - x)^+\mathrm{d}z - \langle \widehat{A}_2(x), (\psi - x)^+\rangle.$$

We have

$$\begin{aligned} &\widehat{a}(\psi, (\psi - x)^+) - \langle \widehat{A}_2(x), (\psi - x)^+\rangle + \int_Z a_0(z, \psi, D\psi)(\psi - x)^+\mathrm{d}z\\ &\quad = \int_Z \sum_{k=1}^N (a_k(z, \psi, D\psi) - a_k(z, \tau(x), Dx))D_k(\psi - x)^+(z)\mathrm{d}z\\ &\quad + \int_Z (a_0(z, \psi, D\psi) - a_0(z, \tau(x), D\tau(x))(\psi - x)^+\mathrm{d}z. \end{aligned}$$

Since

$$D_k(\psi - x)^+(z) = \begin{cases} D_k(\psi - x)(z) & \text{if} \quad x(z) < \psi(z)\\ 0 & \text{if} \quad x(z) \geq \psi(z) \end{cases}$$

(see Gilbarg–Trudinger [13]), we have

$$\int_Z \sum_{k=1}^{N}(a_k(z,\psi,D\psi)-a_k(z,\tau(x),Dx))D_k(\psi-x)^+(z)\mathrm{d}z$$
$$=\int_{\{\psi>x\}}\sum_{k=1}^{N}(a_k(z,\psi,D\psi)-a_k(z,\psi,Dx))D_k(\psi-x)(z)\mathrm{d}z\geq 0$$

(see hypothesis H(a_k)) (iv)). Also because

$$D\tau(x)(z)=\begin{cases} D\varphi(z) & \text{if } \varphi(z)<x(z)\\ Dx(z) & \text{if } \varphi(z)\leq x(z)\leq \phi(z),\\ D\psi(z) & \text{if } x(z)<\psi(z)\end{cases}$$

we have

$$\int_Z (a_0(z,\psi,D\psi)-a_0(z,\tau(x),D\tau(x))(\psi-x)^+(z)\mathrm{d}z$$
$$=\int_{\{\psi>x\}}(a_0(z,\psi,D\psi)-a_0(z,\psi,D\psi))(\psi-x)(z)\mathrm{d}z=0.$$

Therefore finally we can write that

$$\widehat{a}(\psi,(\psi-x)^+)+\int_Z a_0(z,\psi,D\psi)(\psi-x)^+\mathrm{d}z-\left\langle\widehat{A}_2,(\psi-x)^+\right\rangle\geq 0. \quad (11)$$

Because $g(\cdot)$ is nondecreasing (see hypothesis H_0) and $y\in K$, we have

$$\langle\widehat{g}(\psi)-\widehat{g}(y),(\psi-x)^+\rangle=\int_Z(g(\psi(z))-g(y(z)))(\psi-x)^+(z)\mathrm{d}z\leq 0. \quad (12)$$

Using (11) and (12) in (10), we obtain

$$\rho\langle U(x),(\psi-x)^+\rangle\geq 0$$
$$\Longrightarrow \rho\int_Z -(\psi-x)^{p-1}(z)(\psi-x)^+(z)\mathrm{d}z=-\rho\int_Z[(\psi-x)^+(z)]^p\mathrm{d}z\geq 0$$
$$\Longrightarrow \|(\psi-x)^+\|_p=0 \text{ i.e } \psi\leq x.$$

Similarly we show that $x\leq\varphi$, hence $x\in K$. This proves the claim.

Claim 2. If $y_1\leq x_1\in S(y_1)$ and $y_1\leq y_2\in K$, then there exists $x_2\in S(y_2)$ such that $x_1\leq x_2$.

Since $x_1\in S(y_1)\subseteq K$, we have for some $f_1\in L^q(Z)$ with $f_1(z)\in\beta(z,x_1(z))$ a.e on Z,

$$\widehat{a}(x_1,v)+\int_Z a_0(z,x_1,Dx_1)v(z)\mathrm{d}z+\int_Z f_1(z)v(z)\mathrm{d}z=\int_Z g(y_1(z))v(z)\mathrm{d}z$$

for all $v\in W_0^{1,p}(Z)$,

$$\Longrightarrow \widehat{a}(x_1,v)+\int_Z a_0(z,x_1,Dx_1)v(z)\mathrm{d}z+\int_Z f_1(z)v(z)\mathrm{d}z\leq\int_Z g(y_2(z))v(z)\mathrm{d}z$$

all $v \in W_0^{1,p}(Z) \cap L^p(Z)_+$, since $g(\cdot)$ is nondecreasing and $y_1 \le y_2$. Thus $x_1 \in$ ${}^{,p}(Z)$ is a lower solution of the problem

$$\left\{\begin{array}{l} A_1(x)(z) + a_0(z, x, Dx) + \beta(z, x(z)) \ni g(y_2(z)) \\ x_{|\Gamma} = 0 \end{array}\right\}. \tag{13}$$

argument similar to that of Claim 1, gives us a solution $x_2 \in W_0^{1,p}(Z)$ of (13) such that $\le x_2 \le \varphi$. Note that $\varphi \in W^{1,p}(Z)$ remains an upper solution of (13), since $y_2 \in K$ and $(y_2(z)) \le g(\varphi(z))$ a.e on Z. This proves the claim.

im 3. For every $y \in K$, $S(y) \subseteq W_0^{1,p}(Z)$ is weakly closed.

o this end, let $x_n \in S(y), n \ge 1$, and assume that $x_n \xrightarrow{w} x$ in $W_0^{1,p}(Z)$. By definition have

$$\widehat{A}(x_n) + x_n^* = \widehat{g}(y), n \ge 1, \text{ with } x_n^* \in \partial G(x_n)$$
$$\Longrightarrow \big\langle \widehat{A}(x_n), x_n - x \big\rangle = (\widehat{g}(y), x_n - x)_{pq} - \langle x_n^*, x_n - x \rangle.$$

m the compact embedding of $W_0^{1,p}(Z)$ into $L^p(Z)$, we have that $x_n \to x$ in $L^p(Z)$ and $\widehat{g}(y), x_n - x)_{pq} \to 0$. Also $\{x_n^*\}_{n\ge 1} \subseteq L^q(Z)$ is bounded (see the proof of Proposition 2) so $\langle x_n^*, x_n - x\rangle = (x_n^*, x_n - x)_{pq} \to 0$. Therefore

$$\lim \big\langle \widehat{A}(x_n), x_n - x \big\rangle = 0 \Longrightarrow \widehat{A}(x_n) \xrightarrow{w} \widehat{A}(x) \text{ in } W^{-1,q}(Z)$$

ce $\widehat{A}$ is bounded, pseudomonotone).

lso we may assume that $x_n^* \xrightarrow{w} x^*$ in $L^q(Z)$. Since $[x_n, x_n^*] \in \mathrm{Gr}\partial G = \mathrm{Gr}\partial\widehat{G} \cap$ ${}_0^{1,p}(Z) \times L^q(Z))$ (see the proof of Proposition 2 and Chang [8], Lemma 2.1) and $\mathrm{Gr}\partial\widehat{G}$ emiclosed, we conclude that $x^* \in \partial G(x)$. Thus finally we have

$$\widehat{A}(x) + x^* = \widehat{g}(y), \text{ with } x^* \in \partial G(x),$$

$x \in S(y)$, which proves the claim.

laims 1, 2 and 3 and that fact that $W^{1,p}(Z)$ is separable, permit the application of position 2.4 of Heikkila–Hu [15], which gives $x \in S(x)$ (fixed point of $S(\cdot)$). Evidently is a weak solution of problem (1). □

nark. In fact with a little additional effort, we can show that the result is still valid, nstead we assume that there exists $M \ge 0$ such that $x \to g(x) + Mx$ is nonde- asing. However, to simplify our presentation we have decided to proceed with the nger hypothesis that $g(\cdot)$ is nondecreasing. Moreover, it is clear from our proof, that if $Z \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is defined by $a(z, x, y) = (a_k(z, x, y))_{k=1}^N$ and $x \in W_0^{1,p}(Z)$ is a ution of (1), then $-\mathrm{div}\,a(z, x, Dx) \in L^q(Z)$ and

$$\left\{\begin{array}{ll} -\mathrm{div}\,a(z, x(z), Dx(z)) + a_0(z, x(z), Dx(z)) + f(z) = g(x(z)) \text{ a.e on } & Z \\ x_{|\Gamma} = 0 & \end{array}\right\},$$

n $f \in L^q(Z)$, $f(z) \in \beta(z, x(z))$ a.e on Z (i.e x is a strong solution).

or a particular version of problem (1), we can show the existence of extremal solutions he order interval; K, i.e of solutions x_l, x_u in K such that for every solution $x \in K$, we e $x_l \le x \le x_u$.

So let $A_3x(z) = -\sum_{k=1}^{N} D_k a_k(z, Dx)$ (second order nonlinear differential operator in divergence form) and consider the following boundary value problem

$$\left\{ \begin{array}{l} A_3(x)(z) + a_0(z, x(z)) + \beta(z, x(z)) \ni g(x(z)) \text{ on } Z \\ x_{|\Gamma} = 0 \end{array} \right\}. \tag{14}$$

The hypotheses on the functions a_k and a_0 are the following:

H(α_k))′: $a_k : Z \times \mathbb{R}^N \to \mathbb{R},\ k \in \{1, 2, \ldots, N\}$, are functions such that

(i) for all $y \in \mathbb{R}^N$, $z \to a_k(z, y)$ is measurable;

(ii) for almost all $z \in Z$, $y \to a_k(z, y)$ is continuous;

(iii) for almost all $z \in Z$, and all $y \in \mathbb{R}^N$, we have

$$|a_k(z, y)| \leq \gamma(z) + c\|y\|^{p-1}$$

with $\gamma \in L^q(Z), c > 0, 2 \leq p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$;

(iv) for almost all $z \in Z$ and all $y, y' \in \mathbb{R}^N$, $y \neq y'$, we have

$$\sum_{k=1}^{N} (a_k(z, y) - a_k(z, y'))(y_k - y'_k) > 0;$$

(v) for almost all $z \in Z$ and all $y \in \mathbb{R}^N$, we have

$$\sum_{k=1}^{N} a_k(z, y) y_k \geq c_1 \|y\|^p - \gamma_1(z)$$

with $c_1 > 0, \gamma_1 \in L^1(Z)$.

H(α_0)′: $a_0 : Z \times \mathbb{R} \to \mathbb{R}$, is a function such that

(i) for all $x \in \mathbb{R}$ $z \to a_0(z, x)$ is measurable;

(ii) for almost all $z \in Z, x \to a_0(z, x)$ is continuous, nondecreasing;

(iii) for almost all $z \in Z$ and all $x \in [\psi(z), \varphi(z)]$, we have $\|a_0(z, x)\| \leq \gamma_2(z)$ with $\gamma_2 \in L^q(Z)$.

Then we can prove the following result.

PROPOSITION 4

If hypotheses H(a_k)′, H(a_0)′, H(β) *and* H_0 *hold, then problem* (14) *has extremal solutions in the order interval* K.

Proof. Hypotheses H(a_k)′ and H(a_0)′, imply that the map $S : K \to K$ is actually single-valued. Also we claim that it is increasing with respect to the induced partial order on K. Indeed let $y_1, y_2 \in K$, $y_1 \leq y_2$ and let $x_1 = S(y_1), x_2 = S(y_2)$. We have

$$\widehat{A}(x_1) + x_1^* = \widehat{g}(y_1)$$

d

$$\widehat{A}(x_2) + x_2^* = \widehat{g}(y_2)$$

ith $x_i^* \in \partial G(x_i), i = 1, 2$.

Using $(x_1 - x_2)^+ \in W_0^{1,p}(Z) \cap L^p(Z)_+$ as our test function, we have

$$\begin{aligned} \left\langle \widehat{A}(x_1) - \widehat{A}(x_2), (x_1 - x_2)^+ \right\rangle &= \langle x_1^* - x_2^*, (x_1 - x_2)^+ \rangle \\ &= \langle \widehat{g}(y_1) - \widehat{g}(y_2), (x_1 - x_2)^+ \rangle. \end{aligned} \tag{15}$$

virtue of hypotheses $\mathrm{H}(a_k)'$ and $\mathrm{H}(a_0)'$ (ii), we have

$$\left\langle \widehat{A}(x_1) - \widehat{A}(x_2), (x_1 - x_2)^+ \right\rangle \geq 0 \text{ (strictly if } x_1 \neq x_2). \tag{16}$$

lso from the monotonicity of the subdifferential, we have

$$\left\langle x_1^* - x_2^*, (x_1 - x_2)^+ \right\rangle = (x_1^* - x_2^*, (x_1 - x_2)^+)_{pq} \geq 0. \tag{17}$$

nally since by hypothesis H_0, $g(\cdot)$ is nondecreasing it follows that

$$\left\langle \widehat{g}(y_1) - \widehat{g}(y_2), (x_1 - x_2)^+ \right\rangle = (\widehat{g}(y_1) - \widehat{g}(y_2), (x_1 - x_2)^+)_{pq} \leq 0. \tag{18}$$

sing (16), (17) and (18) in (15), we infer that $(x_1 - x_2)^+ = 0$, hence $x_1 \leq x_2$. This proves e claim. Using Corollary 1.5 of Amann [2], we infer that $S(\cdot)$ has extremal fixed points K. Clearly these are the extremal solutions of (14) in K. □

Now we will consider a multivalued nonlinear elliptic problem, with a $\beta(\cdot)$ such that $\mathrm{m}\beta \neq \mathbb{R}$. This case is important because it covers variational inequalities.

So now we examine the following boundary value problem:

$$\left\{ \begin{array}{l} A_1(x)(z) + a_0(z, x(z)) + \beta(x(z)) \ni g(z) \text{ on } Z \\ x|_\Gamma = 0 \end{array} \right\}. \tag{19}$$

ur hypotheses on a_0 and β are the following:

$(a_0)''$: $a_0 : Z \times \mathbb{R} \to \mathbb{R}$, is a function such that

i) for all $x \in \mathbb{R}$, $z \to a_0(z, x)$ is measurable;

i) for almost all $z \in Z$, $x \to a_0(z, x)$ is continuous, nondecreasing;

i) for almost all $z \in Z$ and all $x \in \mathbb{R}$, we have $|a_0(z, x)| \leq \gamma_2(z) + c_2|x|$ with $\gamma_2 \in L^q(Z)$, $c_2 > 0$.

$(\beta)_1$: $\beta : \mathbb{R} \to 2^{\mathbb{R}}$, is a maximal monotone map with $0 \in \beta(0)$.

heorem 5. *If hypotheses* $\mathrm{H}(a_k)$, $\mathrm{H}(a_0)''$, $\mathrm{H}(\beta)_1$ *hold and* $g \in L^p(Z)$, *then the solution t of problem* (19) *is nonempty.*

roof. Recall $\beta = \partial j$ with $j \in \Gamma_0(\mathbb{R})$. Let $\beta_\varepsilon = \frac{1}{\varepsilon}(1 - (1 + \varepsilon\beta)^{-1})$, $\varepsilon > 0$, be the Yosida proximation of $\beta(\cdot)$ and consider the following approximation of problem (19):

$$\left\{ \begin{array}{l} \widehat{A}(x_1) - a_0(z, x(z)) + \beta_\varepsilon(x(z)) = g(z) \text{ on } Z \\ x|_\Gamma = 0 \end{array} \right\}. \tag{20}$$

As before let $\widehat{a} : W_0^{1,p}(Z) \times W_0^{1,p}(Z)$ be the semilinear form defined by

$$\widehat{a}(x, y) = \int_Z \sum_{k=1}^{N} a_k(z, x, Dx) D_k y(z) \mathrm{d}z$$

and let $\widehat{A}_1 : W_0^{1,p}(Z) \to W^{-1,q}(Z)$ be defined by

$$\langle \widehat{A}_1(x), y \rangle = \widehat{a}(x, y) \text{ for all } x, y \in W_0^{1,p}(Z).$$

Also let $\widehat{a}_0 : L^p(Z) \to L^q(Z)$ be the Nemitsky operator corresponding to a_0 i.e. $\widehat{a}_0(x)(\cdot) = a_0(\cdot, x(\cdot))$ (in fact note that by H$(a)''$ (iii) $\widehat{a}(x) \in L^p(Z) \subset L^q(Z)$ since $p \geq 2 \geq q$).

From Theorem 3,1 of Gossez–Mustonen [14] we know that $\widehat{A}_1$ is pseudomonotone, while exploiting the compact embedding of $W_0^{1,p}(Z)$ in $L^p(Z)$, we can easily see that $\widehat{a}_0|_{W_0^{1,p}}$ is completely continuous. Therefore $\widehat{A}_2 = \widehat{A}_1 + \widehat{a}_0$ is pseudomonotone.

Let $G_\varepsilon : W_0^{1,p}(Z) \to \mathbb{R}$ be the integral functional defined by $G_\varepsilon(x) = \int_Z j_\varepsilon(x(z))\mathrm{d}z$ with $j_\varepsilon(r)$ being the Moreau–Yosida regularization of $j(\cdot)$ (see for example Hu–Papageorgiou [16], Definition III.4.30, p. 349). We know that $G_\varepsilon(\cdot)$ is Gateaux differentiable and $\partial G_\varepsilon(x) = \partial j_\varepsilon(x(\cdot))$ (see Hu–Papageorgiou [16], Proposition III.4.32, p. 350). Then problem (20) is equivalent to the following operator equation

$$\widehat{A}_2(x) + \partial G_\varepsilon(x) = g. \tag{21}$$

Note that ∂G_ε is maximal monotone, with $\mathrm{dom}\,\partial G_\varepsilon = W_0^{1,p}(Z)$. Therefore ∂G_ε is pseudomonotone and hence so is $\widehat{A}_2 + \partial G_\varepsilon$. We will show that $\widehat{A}_2 + \partial G_\varepsilon$ is coercive. Since $0 = G_\varepsilon(0)$ and $\langle \partial G_\varepsilon(x), x \rangle \geq 0$, to establish the desired coercivity of $\widehat{A}_2 + \partial G_\varepsilon$, it suffices to show that $\widehat{A}_2$ is coercive. To this end we have

$$\langle \widehat{A}_2, x \rangle \geq c_1 \|Dx\|_p^p - \|\gamma_1\|_1 + \int_Z a_0(z, x(z)) x(z) \mathrm{d}a.$$

Since $a_0(z, \cdot)$ is nondecreasing (hypothesis H$(a)''$ (ii)) $(a_0(z, x(z)) - a_0(z, 0))x(z) \geq 0$ a.e on $\mathbb{R}$ and so

$$\begin{aligned}\int_Z a_0(z, x(z)) x(z) \mathrm{d}z &= \int_Z (a_0(z, x(z)) - a_0(z, 0)) x(z) \mathrm{d}z + \int_Z a_0(z, 0) x(z) \mathrm{d}z \\ &\geq \int_Z a_0(z, 0) x(z) \mathrm{d}z.\end{aligned}$$

Therefore is follows that

$$\begin{aligned}\langle \widehat{A}_2(x), x \rangle &\geq c_1 \|Dx\|_p^p - \|\gamma_1\|_1 + \int_Z a_0(z, 0) x(z) \mathrm{d}z \\ &\geq c_1 \|Dx\|_p^p - \|\gamma_1\|_1 - \|a_0(\cdot, 0)\|_q \|x\|_p,\end{aligned}$$

from which we infer the coercivity of $x \to (\widehat{A}_2 + \partial G_\varepsilon)(x)$. Thus Corollary III.6.30, p. 372, of Hu–Papageorgiou [16], implies that there exists $x_\varepsilon \in W_0^{1,p}(Z)$ which solves (21). Now let $\varepsilon_n \downarrow 0$ and set $x_n = x_{\varepsilon_n} n \geq 1$. We will derive some uniform bounds for the sequence $\{x_n\}_{n \geq 1} \subset W_0^{1,p}(Z)$. To this end, we have

$$\begin{aligned}&\int_Z \sum_{k=1}^{N} a_k(z, x_n, Dx_n) D_k x_n(z) \mathrm{d}z + \int_Z a_0(z, x_n) x_n(z) \mathrm{d}z + \int_Z \beta_{\varepsilon_n}(x_n) x_n(z) \mathrm{d}z \\ &\quad = \int_Z g(z) x_n(z) \mathrm{d}z\end{aligned}$$

$$\Longrightarrow c_1\|Dx_n\|_p^p - \|\gamma_1\|_1 - \|a_0(\cdot,0)\|_q\|x_n\|_p \leq \|g\|_q\|x_n\|_p$$

(since $\beta_\varepsilon(x_n(z))x_n(z) \geq 0$ a.e on Z).

From this inequality we deduce that $\{x_n\}_{n\geq 1} \subseteq W_0^{1,p}(Z)$ is bounded. Also note that $\eta_n(r) = |\beta_{\varepsilon_n}(r)|^{p-2}\beta_{\varepsilon_n}(r)$ is locally Lipschitz on $\mathbb{R}$ and $\eta_n(0) = 0$. So from Marcus–Mizel [20], we know that $\eta_n(x_n(\cdot)) \in W_0^{1,p}(Z)$, $n \geq 1$. Using this as our test function, we have

$$\begin{aligned}\int_Z \sum\nolimits_{k=1}^N a_k(z,x_n,Dx_n)D_k\eta_n(x_n)\mathrm{d}z \ &+ \int_Z a_0(z,x_n)\eta_n(x_n)\mathrm{d}z + \int_Z |\beta_{\varepsilon_n}(x_n)|^p\mathrm{d}z \\ &= \int_Z g(z)\eta_n(x_n(z))\mathrm{d}z. \qquad (22)\end{aligned}$$

Note that $D_k\eta_n(x_n(z)) = (p-1)|\beta_{\varepsilon_n}(x_n(z))|^{p-2}\beta'_{\varepsilon_n}(x_n(z))D_kx_n(z)$ a.e on Z (see Marcus–Mizel [20], and recall that $\beta_{\varepsilon_n}(\cdot)$ being Lipschitz is differentiable almost everywhere). Since $\beta_{\varepsilon_n}(\cdot)$ is nondecreasing, $(p-1)|\beta_{\varepsilon_n}(x_n(z))|^{p-2}\beta'_{\varepsilon_n}(x_n)) \geq 0$ a.e on Z. Thus using hypothesis H(a_k)(v), we have

$$\sum_{k=1}^N a_k(z,x_n,Dx_n)D_k\eta_n(x_n)\mathrm{d}z \geq -\|\gamma_1\|_1.$$

Moreover, from hypothesis H$(a_0)''$(iii) $a_0(\cdot,x_n(\cdot)) \in L^p(Z)$. In addition since $\beta_{\varepsilon_n}(\cdot)$ is $\frac{1}{\varepsilon_n}$-Lipschitz and $0 = \beta_{\varepsilon_n}(0)$, we have $|\beta_{\varepsilon_n}(r)| \leq \frac{1}{\varepsilon_n}|r|$, from which it follows that $|\beta_{\varepsilon_n}(x(\cdot))| \in L^q(Z)$. So by Holder's inequality, we have

$$\int_Z a_0(z,x_n(z))\eta_n(x(z))\mathrm{d}z \geq -\|\widehat{a}_0(x_n)\|_p\|\beta_{\varepsilon_n}(x_n)\|_p^{p-1}.$$

But since $\{x_n\}_{n\geq 1} \subseteq W_0^{1,p}(Z)$ is bounded, we have $\sup_{n\geq 1}\|\widehat{a}_0(x_n)\|_p \leq M_1$ (see hypothesis H$(a_0)''$(iii)). So we obtain

$$\int_Z a_0(z,x_n(z))\eta_n(x(z))\mathrm{d}z \geq -M_1\|\beta_{\varepsilon_n}(x_n)\|_p^{p-1}.$$

Returning to (22), we can write

$$\begin{aligned}&\|\beta_{\varepsilon_n}(x_n)\|_p^p - M_1\|\beta_{\varepsilon_n}(x_n)\|_p^{p-1} \leq \|g\|_p\|\beta_{\varepsilon_n}(x_n)\|_p^{p-1} + \|\gamma_1\|_1 \\ &\Longrightarrow \{\beta_{\varepsilon_n}(x_n)\}_{n\geq 1} \subseteq L^p(Z)\end{aligned}$$

is bounded, hence is bounded also in $L^2(Z)$. Hence by passing to a subsequence if necessary, we may assume that $x_n \overset{w}{\to} x$ in $W_0^{1,p}(Z)$ and $\beta_{\varepsilon_n}(x_n) \overset{w}{\to} v^*$ in $L^2(Z)$ as $n \to \infty$.

Also we have

$$\left\langle \widehat{A}(x_n), x_n - x_n\right\rangle + \langle\beta_{\varepsilon_n}(x_n), x_n - x\rangle = (g, x_n - x)_{pq}.$$

Exploiting the compact embedding of $W_0^{1,p}(Z)$ into $L^p(Z)$, we obtain

$$\begin{aligned}&\limsup\left\langle \widehat{A}_2(x_n), x_n - x\right\rangle \leq 0 \\ &\Rightarrow \widehat{A}_2(x_n) \overset{w}{\to} \widehat{A}_2(x) \text{ in } W^{-1,q}(Z)\end{aligned}$$

(recall that $\widehat{A}_2$ is pseudomonotone and bounded). Hence in the limit as $n \to \infty$ we have $\widehat{A}_2(x) + v^* = g$ in $W^{-1,q}(Z)$. Let $\widehat{\beta} : L^2(Z) \to 2^{L^2(Z)}$ be defined by

$$\widehat{\beta}(x) = \{u \in L^2(Z) : u(z) \in \beta(x(z)) \text{ a.e. on } Z\}.$$

We know that $\widehat{\beta}$ is maximal monotone (see Hu–Papageorgiou [16], p. 328). Using Proposition III.2.29, p. 325, of Hu–Papageorgiou [16], we have that $v^* \in \widehat{\beta}(x)$ and so $v^*(z) \in \beta(x(z))$. So $x \in W_0^{1,p}(Z)$ is a solution of [19]. □

4. Existence results with nonmonotone nonlinearities

In this section we examine a quasilinear elliptic problem with a multivalued nonmonotone nonlinearity. The problem that we study is a hemivariational inequality. Hemivariational inequalities are a new type of variational inequalities, where the convex subdifferential is replaced by the subdifferential in the sense of Clarke [9], of a locally Lipschitz function. Such inequalities are motivated by problems in mechanics, where the lack of convexity does not permit the use of the convex superpotential of Moreau [21]. Concrete applications to problems of mechanics and engineering can be found in the book of Panagiotopoulos [22]. Also our formulation incorporates the case of elliptic boundary value problems with discontinuous nonlinearities. Such problems have been studied (primarily for semilinear systems) by Ambrosetti–Badiale [5], Ambrosetti–Turner [3], [4], Badiale [6], Chang [8] and Stuart [23].

Let $Z \subseteq \mathbb{R}^N$ be a bounded domain with a C^1-boundary Γ. We start with a few remarks concerning the first eigenvalue of the negative p-Laplacian $-\Delta_p x = -\text{div}(\|Dx\|^{p-2}Dx)$, $2 \leq p < \infty$, with Dirichlet boundary conditions. We consider the following nonlinear eigenvalue problem:

$$\left\{ \begin{array}{l} -\text{div}(\|Dx(z)\|^{p-2}Dx(z)) = \lambda|x(z)|^{p-2}x(z) \text{ a.e. on } Z \\ x|_\Gamma = 0 \end{array} \right\}. \tag{23}$$

The least $\lambda \in \mathbb{R}$ for which (20) has a nontrivial solution is called the first eigenvalue of $-(\Delta_p, W_0^{1,p}(Z))$. From Lindqvist [19] we know that $\lambda_1 > 0$, is isolated and simple. Moreover, $\lambda_1 > 0$ is characterized via the Rayleigh quotient, namely

$$\lambda_1 = \min\left[\frac{\|Dx\|_p^p}{\|x\|_p^p} : x \in W_0^{1,p}(Z), x \neq 0\right].$$

This minimum is realized at the normalized first eigenfunction u_1, which we know that it is positive, i.e $u_1(z) > 0$ a.e on Z (note that by nonlinear elliptic regularity theory $u_1 \in C_{\text{loc}}^{1,\beta}(Z), 0 < \beta < 1$; see Tolksdorf [24]).

We consider the following nonlinear eigenvalue problem:

$$\left\{ \begin{array}{l} -\text{div}(\|Dx(z)\|^{p-2}Dx(z)) \in \lambda\partial j(z, x(z)) \text{ a.e on } Z \\ x|_\Gamma = 0, 2 \leq p < \infty, \lambda > 0 \end{array} \right\}. \tag{24}$$

Our approach to problem (24) will be variational, based on the critical point theory for nonsmooth locally Lipschitz functionals, due to Chang [8]. In this case the classical Palais–Smale condition (PS-condition for short) takes the following form. Let X be a Banach space and $f : X \to \mathbb{R}$ a locally Lipschitz function. We say that $f(\cdot)$ satisfies the nonsmooth

PS-condition, if any sequence $\{x_n\}_{n\geq 1} \subseteq X$ for which $\{f(x_n)\}_{n\geq 1}$ is bounded and $m(x_n) = \min\{\|x^*\| : x^* \in \partial f(x_n)\} \to 0$ as $n \to \infty$, has a strongly convergent subsequence. When $f \in C^1(X)$, we know that $\partial f(x_n) = \{f'(x_n)\}$ and so we see that the above definition of the PS-condition coincides with the classical one.

Our hypotheses on the function $j(z, r)$ in problem (24), are the following:

H(j): $j : Z \times \mathbb{R} \to \mathbb{R}$ is a function such that

(i) for all $x \in \mathbb{R} \to j(z, x)$ is measurable;

(ii) for almost all $z \in Z, x \to j(z, x)$ is locally Lipschitz;

(iii) for almost all $z \in Z$, all $x \in \mathbb{R}$ and all $v \in \partial j(z, x)$, we have

$$|v| \leq c_1(1 + |x|^{r-1})$$

with $c_1 > 0$, $1 \leq r < p$;

(iv) $j(\cdot, 0) \in L^\infty(Z)$, $\int_Z j(z, 0)\mathrm{d}z = 0$ and there exists $x_0 \in \mathbb{R}$ such that for almost all $z \in Z$, $j(z, x_0) > 0$;

(v) $\lim_{x\to 0}\sup \frac{pj(z,x)}{|x|^p} < 0$ uniformly for almost all $z \in Z$.

We will need the following nonsmooth variant of the classical 'Mountain Pass theorem'. The result is due to Chang [8].

Theorem 6. *If X is a reflexive Banach space, $V : X \to \mathbb{R}$ is a locally Lipschitz functional which satisfies the (PS)-condition and for some $r > 0$ and $y \in X$ with $\|y\| > r$ we have*

$$\max[V(0),\ V(y)] < \inf[V(x) : \|x\| = r].$$

Then there exists a nontrivial critical point $x \in X$ of V (i.e $0 \in \partial V(x)$) such that the critical value $c = V(x)$ is characterized by the following minimax principle

$$c = \inf_{\gamma\in\Gamma}\max_{0\leq r\leq 1} V(\gamma(\tau)),$$

where $\Gamma = \{\gamma \in C([0, 1], X) : \gamma(0) = 0, \gamma(1) = y\}$.

We have the following multiplicity result for problem (1).

Theorem 7. *If hypotheses* H(j) *hold, then there exists $\lambda_0 > 0$ such that for all $\lambda \geq \lambda_0$ problem* (24) *has at least two nontrivial solutions.*

Proof. For $\lambda > 0$, let $V_\lambda : W_0^{1,p}(Z) \to \mathbb{R}$ be defined by

$$V_\lambda(x) = \frac{1}{p}\|Dx\|_p^p - \lambda\int_Z j(z, x(z))\mathrm{d}z.$$

We know that V_λ is locally Lipschitz (see Clarke [9]).

Claim 1. V_λ satisfies the nonsmooth (PS)-condition. Let $\{x_n\}_{n\geq 1} \subseteq W_0^{1,p}(Z)$ be such that $|V_\lambda(x_n)| \leq M_1$ for all $n \geq 1$ and $m(x_n) \to 0$ as $n \to \infty$. Let $x_n^* \in \partial V_\lambda(x_n)$ such that

$m(x_n) = \|x_n^*\|$ for all $n \geq 1$. Its existence follows the fact that $\partial V_\lambda(x_n)$ is w-compact and the norm functional is weakly lower semicontinuous. We have

$$x_n^* = A(x_n) - \lambda v_n^*, n \geq 1.$$

Here $A : W_0^{1,p}(Z) \to W^{-1,q}(Z)$ is defined by

$$\langle A(x), y\rangle = \int_Z \|Dx\|^{p-2}(Dx, Dy)_{\mathbb{R}^N} \mathrm{d}z$$

for all $x, y \in W_0^{1,p}(Z)$ and $v_n^* \in \partial\psi(x_n)$ where $\psi(x) = \int_Z j(z, x(z))\mathrm{d}z$. It is easy to see that A is monotone, demicontinuous, thus maximal monotone.

From the Lebourg mean value theorem (see Clarke [9], Therorem 2.3.7, p. 41), we know that there exists $v^* \in \partial j(z, \eta x), 0 < \eta < 1$ such that $j(z, x) - j(z, 0) = v^* x$. Using this, together with hypothesis H(j)(iii) and the fact that $j(\cdot, 0) \in L^\infty(Z)$, we can write that for almost all $z \in Z$, and all $x \in \mathbb{R}$, we have $|j(z, x)| \leq \beta_1 + \beta_2|x|^r$ with $\beta_1, \beta_2 > 0$. Hence we have that

$$\begin{aligned} M_1 \geq V_\lambda(x_n) &= \frac{1}{p}\|Dx_n\|_p^p - \lambda \int_Z j(z, x_n(z))\mathrm{d}z \\ &\geq \frac{1}{p}\|Dx_n\|_p^p - \lambda\beta_1|Z| - \lambda\beta_3\|x_n\|_p^r \text{ for some } \beta_3 > 0. \end{aligned}$$

Here $|Z|$ denotes the Lebesgue measure of the domain $Z \subseteq \mathbb{R}^N$. Using Young's inequality with $\varepsilon > 0$, we have

$$\lambda\beta_3\|x_n\|_p^r \leq M_8 + \varepsilon\|x_n\|_p^p$$

for some $M_8 > 0$. Let $\varepsilon < \frac{\lambda_1}{p}$. We have

$$\begin{aligned} M_1 &\geq V_\lambda(x_n) \geq \frac{1}{p}\|Dx_n\|_p^p - \lambda\beta_1|Z| - M_\epsilon - \epsilon\|x_n\|_p^r \\ &\geq \left(\frac{1}{p} - \frac{\epsilon}{\lambda_1}\right)\|Dx_n\|_p^p - \lambda\beta_1|Z| - M_\varepsilon \text{ (Rayleigh quotient).} \end{aligned} \tag{25}$$

Since $\frac{1}{p} - \frac{\varepsilon}{\lambda_1} > 0$ (recall the choice of $\varepsilon > 0$), from the above inequality it follows that $\{x_n\}_{n\geq 1} \subseteq W_0^{1,p}(Z)$ is bounded. So we may assume that $x_n \xrightarrow{w} x$ in $W_0^{1,p}(Z)$ and so $x_n \to x$ in $L^p(Z)$ as $n \to \infty$. We have

$$\langle A(x_n), x_n - x\rangle = \lambda\langle v_n^*, x_n - x\rangle.$$

From Theorem 2.2 of Chang [8], we have that $\{v_n^*\}_{n\geq 1} \subseteq L^q(Z)$ and is bounded. So we have

$$\lim \langle A(x_n), x_n - x\rangle = \lim \lambda(v_n^*, x_n - x)_{pq}.$$

Since A is maximal monotone, we have $\langle A(x_n), x_n\rangle \to \langle A(x), x\rangle \Longrightarrow \|Dx_n\|_p \to \|Dx\|_p$.

Since $Dx_n \xrightarrow{w} Dx$ in $L^p(Z, \mathbb{R}^N)$ and $L^p(Z, \mathbb{R}^N)$ is uniformly convex, from the Kadec–Klee property (see Hu–Papageorgiou [16], Definition I.1.72 and Lemma I.1.74, p. 28) it follows that $Dx_n \to Dx$ in $L^p(Z, \mathbb{R}^N)$, hence $x_n \to x$ in $W_0^{1,p}(Z)$. This proves the claim.

From (25), we have that $V_\lambda(\cdot)$ is coercive. This combined with Claim 1, allow the use of Theorem 3.5 of Chang [8] which gives us $y_1 \in W_0^{1,p}(Z)$ such that $0 \in \partial V_\lambda(y_1)$ and

$$c_\lambda = \inf_{W_0^{1,p}(Z)} V_\lambda = V_\lambda(y_1).$$

From hypothesis H(j) (iv), for $\widehat{x} = x_0$, we have $\widehat{\psi}(\widehat{x}) > 0$ where $\widehat{\psi} : L^r(Z) \to \mathbb{R}$ is defined by $\widehat{\psi}(y) = \int_Z j(z, y(z))\mathrm{d}z$. Evidently $\widehat{\psi}$ is locally Lipschitz and $\widehat{\psi}|_{W_0^{1,p}(Z)} = \psi$. Since $W_0^{1,p}(Z)$ is embedded continuously and densely in $L^r(Z)$, from the continuity of $\widehat{\psi}$, it follows that we can find $x \in W_0^{1,p}(Z)$ such that $\widehat{\psi}(x) = \psi(x) > 0$. Then there exists $\lambda_0 > 0$ such that for $\lambda \geq \lambda_0$ we have $V_\lambda(y_1) = \frac{1}{p}\|Dy\|_p^p - \lambda\psi(y_1) < 0 = V_\lambda(0)$. So $y_1 \neq 0$.

Claim 2. There exists $r > 0$ such that $\inf[V_\lambda(x) : \|x\| = r] > 0$.

By virtue of hypotheses H(j)(v), we can find $\delta > 0$ such that for almost all $z \in Z$ and all $|x| \leq \delta$, we have for some $\gamma < 0$,

$$j(z, x) \leq \frac{\gamma |x|^p}{p}.$$

Also recall that $j(z, x) \leq \beta_1 + \beta_2 |x|^r$. Thus we can find $\beta_4 > 0$ large enough such that for almost all $z \in Z$ and all $x \in \mathbb{R}$, we have

$$j(z, x) \leq \frac{\gamma |x|^p}{p} + \beta_4 |x|^s \text{ with } p < s \leq p^* = \frac{Np}{N-p}.$$

Therefore, we can write that

$$V_\lambda(x) \geq \frac{1}{p}\left(1 - \frac{\lambda\gamma}{\lambda_1}\right)\|Dx\|_p^p - \lambda\beta_5\|Dx\|_p^s \text{ for some } \beta_5 > 0.$$

Note that $(1 - \frac{\lambda\gamma}{\lambda_1}) > 0$ (since $\gamma < 0$ and $0 < \lambda_0 \leq \lambda$, $\lambda_1 > 0$). Thus for every $\lambda \geq \lambda_0 > 0$ we can find $\|y\|_1 \rho > 0$ (depending in general on λ) such that $\inf[V_\lambda(x) : \|x\| = \rho] > 0$. Then $V_\lambda(y_1) < V_\lambda(0) < \inf[V_\lambda(x) : \|x\| = \rho]$ and so we can apply Theorem 6 and obtain $y_2 \neq 0$, $y_2 \neq y_1$ such that $0 \in \partial V_\lambda(y_2)$.

Now let $y = y_1$ or $y = y_2$. From $0 \in \partial v_\lambda(y)$ we have

$$A(y) = \lambda v^*$$

for some $v^* \in \partial\psi(y)$.

From Clarke [9] we know that $v^* \in L^q(Z)$ and $v^*(z) \in \partial j(z, y(z))$ a.e on Z. From the representation therorem for the elements in $W^{-1,q}(Z)$ (see Adams [1], Theorem 3.10, p. 50) we have that $\operatorname{div}(\|Dy\|^{p-1} Dy) \in W^{-1,q}(Z)$. So we have for all $u \in W_0^{1,p}(Z)$,

$$\begin{aligned}
&\langle A(y), u\rangle = \langle -\operatorname{div}(\|Dy\|^{p-2}Dy), u\rangle = \lambda(v^*, u)_{pq} \\
\Longrightarrow\quad & -\operatorname{div}(\|Dy(z)\|^{p-2}Dy(z) = \lambda v^*(z) \in \lambda\partial j(z, y(z)) \text{ a.e on} \\
\Longrightarrow\quad & y_1, y_2 \text{ are distinct, nontrivial solutions of (24).}
\end{aligned}$$

□

Remark. Our theorem extends Theorem 3.5 of Chang [8], who studies a semilinear problem and proves the existence of one solution for some $\lambda \in \mathbb{R}$. Moreover, in Chang $j(z,x) = \int_0^x h(z,s)\mathrm{d}s$. In addition our result extends Theorem 5.35 of Ambrosetti–Rabinowitz [2] to nonlinear problems with multivalued terms.

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